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LANDAU'S THEOREM AND MARDEN CONSTANT FOR HARMONIC ν-BLOCH MAPPINGS

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Abstract

Our main aim is to investigate the properties of harmonic ν-Bloch mappings. Firstly, we establish coefficient estimates and a Landau theorem for harmonic ν-Bloch mappings, which are generalizations of the corresponding results in Bonk *et al.* ['Distortion theorems for Bloch functions', *Pacific. J. Math.* **179** (1997), 241–262] and Chen *et al.* ['Bloch constants for planar harmonic mappings', *Proc. Amer. Math. Soc.* **128** (2000), 3231–3240]. Secondly, we obtain an improved Landau theorem for bounded harmonic mappings. Finally, we obtain a Marden constant for harmonic mappings.

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1. Introduction

We consider complex-valued harmonic mappings f defined in a simply connected domain $\Omega \subset \mathbb{C}$. The mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω (see [5] or [7]). It is well known that f is locally univalent and sense-preserving in Ω if and only if $|g'(z)| < |h'(z)|$ in Ω. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ and $\mathbb{D} = \mathbb{D}(0, 1)$. Throughout this paper we consider harmonic mappings in the open unit disk $\mathbb{D}$. For harmonic mappings f in $\mathbb{D}$, we use the standard notation

$$\Lambda_f(z) = \max_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = ||f_z(z)| - |f_{\overline{z}}(z)||,$$

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so that if f is sense-preserving, then the Jacobian J_f is given by

$$J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\overline{z}}|^2 := |h'|^2 - |g'|^2 > 0.$$

A harmonic mapping f is called a *harmonic Bloch mapping* if and only if

$$\sup_{z,w\in\mathbb{D},z\neq w} \frac{|f(z)-f(w)|}{\rho(z,w)} < +\infty,$$

where

$$\rho(z,w) = \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\overline{z}w}\right|}{1-\left|\frac{z-w}{1-\overline{z}w}\right|}\right) = \operatorname{arctanh}\left|\frac{z-w}{1-\overline{z}w}\right|$$

denotes the hyperbolic distance between z and w in $\mathbb{D}$. In the following, we denote the hyperbolic disk with center a and radius $r > 0$ by $\mathbb{D}_h(a,r) = \{z : \rho(a,z) < r\}$ and the hyperbolic circle by $\mathbb{S}_h(a,r) = \{z : \rho(a,z) = r\}$.

Obviously, for any $a \in \mathbb{D}$, the following are equivalent:

(1) $$\rho(a,z) = r;$$

(2) $$\left|\frac{z-a}{1-\overline{a}z}\right| = \tanh(r);$$

(3) $$\frac{|1-\overline{a}z|^2}{1-|z|^2} = \frac{1-|a|^2}{1-\tanh^2(r)}.$$

In [6] Colonna proved that

$$\sup_{z,w\in\mathbb{D},z\neq w} \frac{|f(z)-f(w)|}{\rho(z,w)} = \sup_{z\in\mathbb{D}}(1-|z|^2)\Lambda_f(z). \tag{1.1}$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $\mathcal{HB}_1$, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{\mathcal{HB}_1} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\Lambda_f(z).$$

For $\nu \in (0,\infty)$, a harmonic mapping f is called a *harmonic ν-Bloch mapping* if and only if $\|f\|_{\mathcal{HB}_\nu} < +\infty$, where

$$\|f\|_{\mathcal{HB}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\nu \Lambda_f(z). \tag{1.2}$$

Then the set of harmonic ν-Bloch mappings forms a ν-Banach space, denoted by $\mathcal{HB}_\nu$, with the norm given by (1.2).

Let $\mathcal{H}_0$ denote the class of all harmonic mappings f in $\mathbb{D}$ with $f(0) = 0$ and introduce the subclass

$$\mathcal{HB}_\nu^1(\alpha) = \{f \in \mathcal{H}_0 : f_z(0) = \alpha,\ f_{\overline{z}}(0) = 0 \text{ and } \|f\|_{\mathcal{HB}_\nu} \leq 1\},$$

where $\alpha \in (0,1]$.

Let χ denote the chordal metric on the extended complex plane $\mathbb{C}_\infty$. For geometric purposes we view $\mathbb{C}_\infty$ as the sphere in the three-dimensional space $\mathbb{R}^3$ with center at $(0, 0, 1/2)$ and radius $1/2$, and have

$$\chi(z, w) = \begin{cases} \dfrac{|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}} & \text{if } z, w \in \mathbb{C} \text{ with } z \neq w, \\ \dfrac{1}{(1+|z|^2)^{1/2}} & \text{if } w = \infty. \end{cases}$$

A meromorphically harmonic mapping f in $\mathbb{D}$ is called a *normal harmonic mapping* if $M(f) < \infty$, where

$$M(f) = \sup_{z, w \in \mathbb{D}, z \neq w} \frac{\chi(f(z), f(w))}{\rho(z, w)}.$$

By (1.1), we have

$$M(f) = \sup_{z \in \mathbb{D}} \left\{ \frac{(1-|z|^2)\Lambda_f(z)}{1+|f(z)|^2} \right\}.$$

Many authors have discussed the coefficient estimate, distortion theorem, and the existence of Landau–Bloch and Marden constants for analytic Bloch functions (see [1–3, 8–11]). But in the literature there are no analogous results for harmonic ν-Bloch mappings. In Section 2, we fill this gap by proving Theorems 2.1, 2.3, and 2.4, where Theorem 2.1 is an improvement of [2, Lemma 1] and Theorem 2.4 is a generalization of [3, Theorem 2 and Corollary 3]. In [4], the authors obtained Landau's theorem for bounded harmonic mappings by using an integral estimate. In Section 3, we use the variational method and subordination to get an improved version of Landau's theorem for harmonic mappings (see Theorem 3.1). Finally, we obtain a Marden constant for harmonic functions (see Theorem 4.1) which generalizes [8, Theorem 5].

2. Landau's theorem for harmonic Bloch mappings

Let η be an analytic function in $\mathbb{D}$ with $\eta(z) = \sum_{n=0}^{\infty} c_n z^n$. If $|\eta(z)| \leq 1$, then for each $n \geq 1$,

$$|c_0|^2 + |c_n| \leq 1. \tag{2.1}$$

We shall make use of this well-known estimate (see [12]).

THEOREM 2.1. *Let $f = h + \overline{g}$ be a harmonic mapping, where g and h are analytic in $\mathbb{D}$ with the expansions*

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2.2}$$

If $\lambda_f(0) = \alpha$ for some $\alpha \in (0, 1]$ and $\|f\|_{\mathcal{HB}_\nu} \leq M$ for $M > 0$, then

$$|a_n| + |b_n| \leq A_n(\alpha, \nu, M) = \inf_{0<r<1} \mu(r) \quad \text{for } n \geq 2, \tag{2.3}$$

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where

$$\mu(r)=\frac{M^2-\alpha^2(1-r^2)^{2\nu}}{nr^{n-1}(1-r^2)^{\nu}M}.$$

In particular, if $\nu=M=\alpha=1$, then

$$\mu(r)=\frac{2-r^2}{nr^{n-3}(1-r^2)}$$

and

$$|a_n|+|b_n|\le A_n, \tag{2.4}$$

where

$$A_n=A_n(1,1,1)=\begin{cases}0 & \text{for } n=2,\\ \frac{1}{3} & \text{for } n=3,\\ \frac{\sqrt{2}}{2}\left(\frac{3+\sqrt{17}}{(1+\sqrt{17})(\sqrt{5-\sqrt{17}})}\right)\approx 1.049\,889 & \text{for } n=4,\\ \mu\left(\sqrt{\frac{3n-7-\sqrt{n^2+6n-23}}{2(n-3)}}\right) & \text{for } n\ge 5.\end{cases}$$

The estimate of (2.4) is sharp when $n\in\{1,2,3\}$. The extreme function is

$$f(z)=\frac{3\sqrt{3}}{4}\left[\left(\frac{z+(\sqrt{3}/3)}{1+(\sqrt{3}/3)z}\right)^2-\frac{1}{3}\right] \tag{2.5}$$

or $\overline{f(z)}$.

PROOF. For a fixed $r\in(0,1)$, let $F(\zeta)=r^{-1}f(r\zeta)$. Then F has the form

$$F(\zeta)=\sum_{n=1}^{\infty}r^{n-1}a_n\zeta^n+\sum_{n=1}^{\infty}r^{n-1}\overline{b}_n\overline{\zeta}^n,\quad \zeta\in\mathbb{D}, \tag{2.6}$$

and therefore

$$\Lambda_F(\zeta)=\Lambda_f(r\zeta)\le m(r):=\frac{M}{(1-r^2)^{\nu}}. \tag{2.7}$$

For $\theta\in[0,2\pi)$ and $\zeta\in\mathbb{D}$, we set

$$T(\zeta)=\frac{F_{\zeta}(\zeta)+e^{i\theta}\overline{F_{\overline{\zeta}}(\zeta)}}{m(r)}.$$

Then, by (2.6), we see that

$$T(\zeta)=\frac{1}{m(r)}\sum_{n=1}^{\infty}n(a_n+e^{i\theta}b_n)r^{n-1}\zeta^{n-1}$$

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which is analytic in $\mathbb{D}$ and, by (2.7), $|T(\zeta)| \le 1$ in $\mathbb{D}$. Now, applying (2.1) to T gives

$$\frac{nr^{n-1}|a_n + e^{i\theta} b_n|}{m(r)} \le 1 - \frac{\lambda_f^2(0)}{m^2(r)} = 1 - \frac{\alpha^2}{m^2(r)} \quad \text{for } n \ge 2$$

and the arbitrariness of θ in $[0, 2\pi)$ implies that

$$|a_n| + |b_n| \le \frac{m^2(r) - \alpha^2}{nr^{n-1}m(r)} = \mu(r) \quad \text{for } n \ge 2, \tag{2.8}$$

where

$$\mu(r) = \frac{M^2 - \alpha^2(1-r^2)^{2\nu}}{nr^{n-1}(1-r^2)^{\nu} M}.$$

Thus, inequality (2.3) follows from (2.8).

In particular, for $\nu = M = \alpha = 1$, one has

$$\mu(r) = \frac{2 - r^2}{nr^{n-3}(1-r^2)}$$

which for $n = 2$ gives that $\inf_{0<r<1} \mu(r) = 0$.

Without loss of generality, for $n = 3$ and $\theta \in [0, 2\pi)$, let

$$f_\theta(z) = h(z) + e^{i\theta} g(z) = z + \sum_{n=2}^{\infty} (a_n + e^{i\theta} b_n) z^n.$$

Then we obtain that $|f'_\theta(z)|(1 - |z|^2) \le 1$ and therefore

$$|f'_\theta(z)| = |1 + 3(a_3 + e^{i\theta} b_3) z^2 + \cdots| \le \frac{1}{1 - |z|^2} = 1 + |z|^2 + \cdots .$$

In particular, this inequality gives that $|a_3 + e^{i\theta} b_3| \le 1/3$. Again, the arbitrariness of θ in $[0, 2\pi)$ implies the estimate

$$|a_3| + |b_3| \le 1/3.$$

For $1 \le n \le 3$, the extreme function is given by (2.5).

On the other hand, for $n \ge 4$, one sees that

$$\lim_{r \to 0+} \mu(r) = \lim_{r \to 1-} \mu(r) = +\infty.$$

This observation implies that the infimum of $\mu(r)$ must exist on (0, 1). Next, we compute that

$$\mu'(r) = -\frac{[(n-3)r^4 - (3n-7)r^2 + 2(n-3)]}{r^{n-2}(1-r^2)^2} = -\frac{(n-3)(r^2 - r_0^2)(r^2 - s_0^2)}{r^{n-2}(1-r^2)^2},$$

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where

$$r_0=\sqrt{\frac{3n-7-\sqrt{n^2+6n-23}}{2(n-3)}}<1 \quad \text{and} \quad s_0=\sqrt{\frac{3n-7+\sqrt{n^2+6n-23}}{2(n-3)}}>1,$$

and therefore we obtain that

$$\inf_{0<r<1}\mu(r)=\mu(r_0).$$

For $n\geq 4$, if we choose $r=\sqrt{(n-3)/(n-1)}$, then we deduce that

$$\mu\bigg(\sqrt{\frac{n-3}{n-1}}\bigg)=\frac{n+1}{2n}\bigg(1+\frac{2}{n-3}\bigg)^{(n-3)/2}<\frac{(n+1)e}{2n}$$

as $(1+1/x)^x$ in $(0,\infty)$ is monotonically increasing and tends to e as $x\to+\infty$. This observation gives that, for $n\geq 5$,

$$\mu(r_0)=\mu\bigg(\sqrt{\frac{3n-7-\sqrt{n^2+6n-23}}{2(n-3)}}\bigg)<\frac{(n+1)e}{2n}.$$

In particular, if $n=4$, then we can easily compute that

$$\mu(r_0)=\frac{2-r_0^2}{4r_0(1-r_0^2)}\bigg|_{n=4}=\frac{\sqrt{2}}{2}\bigg(\frac{3+\sqrt{17}}{(1+\sqrt{17})(\sqrt{5-\sqrt{17}})}\bigg)\approx 1.049\,889.$$

We observe that $(5/8)e$, the upper bound for $\mu(r_0)$, is approximately 1.698 926. The proof of this theorem is complete. □

REMARK 2.2. Theorem 2.1 is an improvement of [2, Lemma 1].

THEOREM 2.3. *Let f be a harmonic mapping with $f(0)=\lambda_f(0)-\alpha=0$ and $\|f\|_{\mathcal{HB}_\nu}\leq M$, where M and $\alpha\in(0,1]$ are constants. Then f is univalent in $\mathbb{D}_{\rho_0}$, where*

$$\rho_0=\varphi(r_0)=\max_{0<r<1}\varphi(r),\quad \varphi(r)=\frac{\alpha r(1-r^2)^\nu M}{\alpha M(1-r^2)^\nu-\alpha^2(1-r^2)^{2\nu}+M^2}.$$

Moreover, $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0=r_0\bigg[\alpha+\frac{M^2-\alpha^2(1-r_0^2)^{2\nu}}{M(1-r_0^2)^\nu}\log\frac{M^2-\alpha^2(1-r_0^2)^{2\nu}}{\alpha M(1-r_0^2)^\nu-\alpha^2(1-r_0^2)^{2\nu}+M^2}\bigg].$$

PROOF. As in Theorem 2.1, let $f=h+\overline{g}$, where g and h are analytic in $\mathbb{D}$ and have the form (2.2). Next, we fix $r\in(0,1)$ and consider $F(\zeta)=r^{-1}f(r\zeta)$ for $\zeta\in\mathbb{D}$ so that

$$F(\zeta)=\sum_{n=1}^{\infty}A_n\zeta^n+\sum_{n=1}^{\infty}\overline{B}_n\overline{\zeta}^n,$$

where $A_n = a_n r^{n-1}$ and $B_n = b_n r^{n-1}$. From the proof of Theorem 2.1, we obtain that

$$|A_n| + |B_n| \le r^{n-1}\mu(r) = \frac{m^2(r) - \alpha^2}{nm(r)}, \tag{2.9}$$

where $m(r)$ is given by (2.7).

To prove the univalence of F, we choose two distinct points $\zeta_1, \zeta_2 \in \mathbb{D}_{\rho_1(r)}$ and let $\zeta_1 - \zeta_2 = |\zeta_1 - \zeta_2| e^{i\theta}$, where

$$\rho_1(r) = \frac{\alpha}{\alpha + m(r) - \frac{\alpha^2}{m(r)}}.$$

Then (2.9) yields that

$$\begin{aligned}
|F(\zeta_2) - F(\zeta_1)| &= \left| \int_{[\zeta_1,\zeta_2]} F_\zeta(\zeta)\, d\zeta + F_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} \right| \\
&\ge \left| \int_{[\zeta_1,\zeta_2]} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta} \right| \\
&\quad - \left| \int_{[\zeta_1,\zeta_2]} (F_\zeta(\zeta) - F_\zeta(0))\, d\zeta + (F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0))\, d\overline{\zeta} \right| \\
&> |\zeta_1 - \zeta_2| \left[\lambda_F(0) - \sum_{n=2}^{\infty} (|A_n| + |B_n|) n \rho_1^{n-1}(r) \right] \\
&\ge |\zeta_1 - \zeta_2| \left[\alpha - \frac{m^2(r) - \alpha^2}{m(r)} \cdot \frac{\rho_1(r)}{1 - \rho_1(r)} \right] \\
&\ge 0.
\end{aligned}$$

Here in the last step we use the fact that

$$\alpha - \frac{m^2(r) - \alpha^2}{m(r)} \cdot \frac{\rho_1(r)}{1 - \rho_1(r)} = 0,$$

that is,

$$\rho_1(r) = \frac{\varphi(r)}{r}.$$

Thus, $F(\zeta_2) \ne F(\zeta_1)$. The univalence of F follows from the arbitrariness of ζ_1 and ζ_2. This implies that f is univalent in $\mathbb{D}_{r\rho_1(r)}$.

Now, for any $\zeta' = \rho_1(r) e^{i\theta} \in \partial\mathbb{D}_{\rho_1(r)}$, we easily obtain that

$$\begin{aligned}
|F(\zeta')| &\ge \alpha\rho_1(r) - \sum_{n=2}^{\infty} (|A_n| + |B_n|) \rho_1^n(r) \\
&\ge \alpha\rho_1(r) - \sum_{n=2}^{\infty} \frac{m^2(r) - \alpha^2}{nm(r)} \rho_1^n(r) \\
&= \alpha + \left(m(r) - \frac{\alpha^2}{m(r)} \right) \log \frac{m(r) - \frac{\alpha^2}{m(r)}}{\alpha + m(r) - \frac{\alpha^2}{m(r)}} = \frac{R_0}{r},
\end{aligned}$$

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where the last step is a consequence of the expression for ρ_1 given by $\rho_1(r) = r^{-1}\varphi(r)$ and the power series expansion of $-\log(1-x)$. Therefore, $f(\mathbb{D}_{r\rho_1(r)})$ contains a univalent disk of radius R_0. The proof of the theorem is complete. □

Next we consider a special case of Theorem 2.3 and obtain the following sharp estimate which is indeed a harmonic analog of [3, Theorem 2 and Corollary 3].

THEOREM 2.4. *Let $f \in \mathcal{HB}_\nu(\alpha)$. Then for z with $|z| < (a_0 + m_0(\alpha))/(1 + a_0 m_0(\alpha))$ and $a_0 = 1/\sqrt{1+2\nu}$, we have*

$$\Lambda_f(z) \geq \operatorname{Re}(f_z(z) + \overline{f_{\overline{z}}(z)}) \geq \frac{\alpha(m_0(\alpha) - |z|)}{m_0(\alpha)(1 - m_0(\alpha)|z|)^{2\nu+1}}. \tag{2.10}$$

The equalities occur if and only if $f(z) = e^{i\vartheta}F_\alpha(e^{-i\vartheta}z)$ for some $\vartheta \in [0, 2\pi)$, where

$$F_\alpha(z) = \frac{\alpha}{m_0(\alpha)} \int_0^z \frac{m_0(\alpha) - \zeta}{(1 - m_0(\alpha)\zeta)^{2\nu+1}}\, d\zeta$$

and $m_0(\alpha)$ satisfies

$$\sqrt{1+2\nu}\left(\frac{2\nu+1}{2\nu}\right)^\nu m_0(\alpha)(1 - m_0^2(\alpha))^\nu = \alpha.$$

Moreover, $f(\mathbb{D}_{m_0(\alpha)})$ contains a univalent disk of radius R_0, where

$$R_0 \geq \frac{\alpha}{m_0(\alpha)} \int_0^{m_0(\alpha)} \frac{(m_0(\alpha) - t)}{(1 - m_0(\alpha)t)^{2\nu+1}}\, dt. \tag{2.11}$$

The equality occurs if and only if $f(z) = e^{i\vartheta}F_\alpha(e^{-i\vartheta}z)$ for some $\vartheta \in [0, 2\pi)$.

PROOF. For $\theta \in [0, 2\pi)$, let

$$f_\theta(z) = \int_0^z f_\zeta(\zeta)\, d\zeta + e^{i\theta} \int_0^z \overline{f_{\overline{\zeta}}(\zeta)}\, d\zeta \quad \text{and} \quad \psi_\theta(z) = (1 - m_0(\alpha)z)^2 f_\theta'(z).$$

Define

$$G_\alpha(z) = \frac{\alpha}{m_0(\alpha)} \cdot \frac{m_0(\alpha) - z}{1 - m_0(\alpha)z},$$

where $m_0(\alpha)$ satisfies

$$\sqrt{1+2\nu}\left(\frac{2\nu+1}{2\nu}\right)^\nu m_0(\alpha)(1 - m_0^2(\alpha))^\nu = \alpha.$$

Fix $x \in (0, (a_0 + m_0(\alpha))/(1 + a_0 m_0(\alpha))]$, and let

$$\delta_x = \{z : \rho(0, z) = \rho(0, x)\} \subset \mathbb{D}_h(m_0(\alpha), \operatorname{arctanh} a_0)$$

be the hyperbolic circle. Since $\psi_\theta(z) \prec G_\alpha$ on $\mathbb{D}_h(m_0(\alpha), \operatorname{arctanh} a_0)$, ψ_θ maps the circle into the closed disk bounded by the circle $G_\alpha(\delta_x)$. We see that $G_\alpha(\delta_x)$ is a

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hyperbolic circle on $\mathbb{D}_t$ with center $G_\alpha(0)=\alpha$, where $t=\alpha a_0/m_0(\alpha)$. Also it is symmetric about the real axis $\mathbb{R}$. It is easy to see that G_α is decreasing on the interval

$$\mathbb{D}_h(m_0(\alpha), \operatorname{arctanh} a_0)\cap\mathbb{R}.$$

Hence $G_\alpha(x)$ has the property

$$G_\alpha(x)=\operatorname{Re} G_\alpha(x)=\min\{\operatorname{Re}[G_\alpha(z)] : z\in\delta(x)\}$$

which yields that

$$\Lambda_f(z)\geq \operatorname{Re}[f_\theta'(z)]=\operatorname{Re}(f_z(z)+e^{i\theta}\overline{f_{\overline{z}}(z)})\geq \frac{\alpha}{m_0(\alpha)}\cdot\frac{(m_0(\alpha)-|z|)}{(1-m_0(\alpha)|z|)^{2\nu+1}}$$

for z with $|z|\leq(1+m_0(\alpha)\sqrt{1+2\nu})/(\sqrt{1+2\nu}+m_0(\alpha))$. Finally, (2.10) is an immediate consequence of the arbitrariness of θ.

On the other hand, for any $\theta\in[0,2\pi)$, we have

$$\begin{aligned}|f_\theta(m_0(\alpha)e^{i\theta})| &= \left|\int_0^{m_0(\alpha)e^{i\theta}} f_\theta'(z)\,dz\right|\\ &\geq \frac{\alpha}{m_0(\alpha)}\int_0^{m_0(\alpha)}\frac{(m_0(\alpha)-t)}{(1-m_0(\alpha)t)^{2\nu+1}}\,dt.\end{aligned}$$

Therefore,

$$\min\{|f_\theta(z)| : |z|=m_0(\alpha)\}\geq\frac{\alpha}{m_0(\alpha)}\int_0^{m_0(\alpha)}\frac{(m_0(\alpha)-t)}{(1-m_0(\alpha)t)^{2\nu+1}}\,dt.$$

Again, the arbitrariness of θ implies that f is univalent on the disk $\mathbb{D}_{r_0}$ with $r_0=m_0(\alpha)$ and $f(\mathbb{D}_{r_0})$ contains a univalent disk of radius at least R_0, where R_0 is given by (2.11). The proof of the theorem is complete. □

REMARK 2.5. If $\nu=1$ and $f_{\overline{z}}\equiv 0$, then Theorem 2.4 coincides with [3, Theorem 2 and Corollary 3].

3. Landau's theorem for harmonic mappings

For the proof of Theorem 3.1, we need the following result due to Colonna [6]. However, we present here a simple proof as it uses only the subordination.

THEOREM A [6, Theorem 3]. *Assume that $f=h+\overline{g}$ is a harmonic mapping in $\mathbb{D}$ satisfying $f(\mathbb{D})\subset\mathbb{D}$, where g and h are analytic in $\mathbb{D}$. Then for any $z\in\mathbb{D}$,*

$$\|f\|_{\mathcal{HB}_1}\leq\frac{4}{\pi}+|f(0)|.$$

PROOF. For $\theta\in[0,2\pi)$, let $v_\theta(z)=\operatorname{Im}(e^{i\theta}f(z))$. Then

$$v_\theta(z)=\operatorname{Im}\big(e^{i\theta}h(z)+\overline{e^{-i\theta}g(z)}\big)=\operatorname{Im}\big(e^{i\theta}h(z)-e^{-i\theta}g(z)\big).$$

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Because $|v_\theta(z)| < 1$, it follows that

$$e^{i\theta}h(z) - e^{-i\theta}g(z) \prec K(z) = \lambda + \frac{2}{\pi}\log\left(\frac{1+z\xi}{1-z}\right), \tag{3.1}$$

where $\xi = e^{-i\pi \mathrm{Im}(\lambda)}$, $\lambda = e^{i\theta}h(0) - e^{-i\theta}g(0)$ and $\prec$ denotes the subordination [7, p. 27].

By the definition of the subordination, (3.1) implies that

$$e^{i\theta}h(z) - e^{-i\theta}g(z) = K(\omega_\theta(z)),$$

where $\omega_\theta(z)$ is an analytic function in $\mathbb{D}$ with $|\omega_\theta(z)| < 1$ for $z \in \mathbb{D}$. Thus, by the Schwarz–Pick lemma, we obtain that

$$\begin{aligned}(1-|z|^2)|e^{i\theta}h'(z) - e^{-i\theta}g'(z)| &= (1-|z|^2)|\omega_\theta'(z)||K'(\omega_\theta(z))| \\ &\leq (1-|\omega_\theta(z)|^2)|K'(\omega_\theta(z))| \\ &\leq \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|K'(\omega)| = \frac{4}{\pi}\end{aligned}$$

and therefore

$$\|f\|_{\mathcal{HB}_1} - |f(0)| = \max_{0\leq\theta<2\pi}(1-|z|^2)|h'(z) - e^{-2i\theta}g'(z)| \leq \frac{4}{\pi},$$

where the last inequality is a consequence of the arbitrariness of θ in $[0, 2\pi)$. □

We now recall a result due to Hengartner and Gauthier [4].

THEOREM B [4, Theorem 3]. *Let f be a harmonic mapping of $\mathbb{D}$ such that $f(0) = f_{\bar{z}}(0) = f_z(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$. Then f is univalent on a disk $\mathbb{D}_{\rho_0}$ with*

$$\rho_0 = \frac{\pi^2}{16mM},$$

and $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 = \frac{\pi^2}{32mM},$$

where $m = \inf_{r\in(0,1)}((3-r^2)/r(1-r^2)) \approx 6.850\,99$.

Our next result improves Theorem A in the following form.

THEOREM 3.1. *Let f be a harmonic mapping of $\mathbb{D}$ such that $f(0) = f_{\bar{z}}(0) = f_z(0) - 1 = 0$ and $|f(z)| < M$ in $\mathbb{D}$ for some $M > 1$. Then f is univalent on a disk $\mathbb{D}_\rho$ and $f(\mathbb{D}_\rho)$ contains a univalent disk of radius at least $R = (3\sqrt{3}M/\pi)\rho^2$, where ρ satisfies*

$$\frac{6\sqrt{3}M}{\pi}\rho(1-\rho^2) = 1.$$

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PROOF. For $z \in \mathbb{D}$, let $F(z) = -(3\sqrt{3}M/\pi)z^2$. Then $(1-|z|^2)|F'(z)| \le 4M/\pi$. Consider

$$G(z) = F \circ \phi_\rho(z) - F \circ \phi_\rho(0),$$

where $\phi_\rho(z) = (z-\rho)/(1-\rho z)$ and the constant $\rho \in (0, \sqrt{3}/3)$ satisfies the equation

$$\frac{6\sqrt{3}M}{\pi}\rho(1-\rho^2) = 1.$$

Further, we let

$$P(z) = (1-\rho z)^2 G'(z) = -\frac{6\sqrt{3}M(1-\rho^2)}{\pi}\phi_\rho(z).$$

Then the function P is a Möbius transformation and, for any $z \in \mathbb{S}_h(\rho, r)$,

$$|P(z)| = \frac{6\sqrt{3}M}{\pi}(1-\rho^2)\tanh(r).$$

Set $T(z) = (1-\rho z)^2 f'_\theta(z)$, where

$$f_\theta(z) = \int_0^z f_\zeta(\zeta)\, d\zeta + e^{i\theta}\int_0^z \overline{f_{\overline{\zeta}}(\zeta)}\, d\zeta$$

for $\theta \in [0, 2\pi)$. Then, for any $z \in \mathbb{S}_h(\rho, r)$, by Theorem A we have

$$|T(z)| \le \frac{4M}{\pi}\frac{|1-\rho z|^2}{1-|z|^2} = \frac{4M}{\pi}\left(\frac{1-\rho^2}{1-\tanh^2(r)}\right).$$

Fix $x \in (0, (1+\sqrt{3}\rho)/(\sqrt{3}+\rho)]$ and let

$$\delta_x = \{z : \rho(0,z) = \rho(0,x)\} \subset \mathbb{D}_h\left(\rho, \operatorname{arctanh}\frac{\sqrt{3}}{3}\right).$$

Since $T \prec P$ on $\mathbb{D}_h(\rho, \operatorname{arctanh}\sqrt{3}/3)$, T maps the circle into the closed disk bounded by the circle $P(\delta_x)$. We see that $P(\delta_x)$ is a hyperbolic circle on $\mathbb{D}_t$ with center $P(0) = 1$, where $t = (6M/\pi)(1-\rho^2)$. Also it is symmetric about the real axis $\mathbb{R}$. It is easy to see that P is decreasing on the interval $\mathbb{D}_h(\rho, \operatorname{arctanh}\sqrt{3}/3) \cap \mathbb{R}$. Hence $P(x)$ satisfies the property that

$$\operatorname{Re}[P(x)] = \min\{\operatorname{Re}[P(z)] : z \in \delta_x\},$$

which yields $\operatorname{Re}[T(x)] \ge \min\{|P(z)| : z \in \delta_x\} = P(x)$, whence

$$\operatorname{Re}[f'_\theta(z)] \ge \frac{6\sqrt{3}M}{\pi}\cdot\frac{(1-\rho^2)(\rho-|z|)}{(1-\rho|z|)^3}$$

for z with $|z| \le (1+\sqrt{3}\rho)/(\sqrt{3}+\rho)$. Hence

$$\min\{|f_\theta(z)| : |z| = \rho\} > \frac{3\sqrt{3}M}{\pi}\rho^2.$$

It follows that f is univalent on the disk $\mathbb{D}_\rho$ and $f(\mathbb{D}_\rho)$ contains a univalent disk of radius at least $R = (3\sqrt{3}M/\pi)\rho^2$. The proof of the theorem is complete. □

TABLE 1. Bloch-Landau constants for various M. The left-hand columns refer to Theorem B and the right-hand columns refer to Theorem 3.1.

M	ρ_0	R_0	M	ρ	R
1.1	0.0818528	0.0409264	1.1	0.302498	0.166483
1.3	0.0692601	0.03463	1.3	0.247744	0.131972
1.5	0.0600254	0.0300127	1.5	0.210916	0.110368
1.7	0.0529636	0.0264818	1.7	0.184059	0.0952566
1.9	0.0473885	0.0236942	1.9	0.163474	0.0839814
2	0.0450191	0.0225095	2	0.154864	0.0793347
2.5	0.0360152	0.0180076	2.5	0.12277	0.0623242
3	0.0300127	0.0150064	3	0.101822	0.0514442
3.5	0.0257252	0.0128626	3.5	0.0870306	0.0438474
4	0.0225095	0.0112548	4	0.0760142	0.038228
4.5	0.0200085	0.0100042	4.5	0.0674851	0.0338969
5	0.0180076	0.00900381	5	0.0606834	0.0304538

REMARK 3.2. We note that in the case of $M = 1$ in Theorem 3.1, we actually have $f(z) = z$ and $\rho = R = 1$. Now, if we set

$$H(x) = \frac{6\sqrt{3}M}{\pi} x(1 - x^2)$$

then, by a simple calculation, we deduce that $H(\pi^2/16mM) < 1$, where m is defined as in Theorem B. This implies that $\rho_0 = \pi^2/16mM < \rho$. Thus, Theorem 3.1 improves Theorem B. In Table 1, we have listed the improved values of the Bloch-Landau constants for various choices of M.

4. Marden constant

If f is a normal harmonic mapping and $a \in \mathbb{D}$, then set

$$s(a, f) = \sup\{r : f \text{ is univalent in the hyperbolic disk } \mathbb{D}_h(a, r)\}$$

and $s(f) = \sup\{s(a, f) : a \in \mathbb{D}\}$. The Marden constant for normal harmonic mappings f with $M(f) = m > 0$ is given by

$$M(m) = \inf\{s(f) : f \text{ is a normal harmonic function with } M(f) = m\},$$

where $M(f)$ is defined in the Introduction.

THEOREM 4.1. *Suppose that f is a normal harmonic mapping such that $f(0) = f_{\bar{z}}(0) = f_z(0) - m = 0$. Then*

$$M(m) \geq 2 \operatorname{arctanh}\left(\frac{1}{\sqrt{3(1 + m^2)}}\right).$$

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PROOF. By simple calculations,

$$[\arctan(|f(z)|)]_z = \frac{\overline{f(z)}f_z(z) + \overline{f_{\bar z}(z)}f(z)}{2(1+|f^2(z)|)|f(z)|}$$

and

$$[\arctan(|f(z)|)]_{\bar z} = \frac{\overline{f(z)}f_{\bar z}(z) + \overline{f_z(z)}f(z)}{2(1+|f^2(z)|)|f(z)|}.$$

These equalities give

$$\begin{aligned}\arctan(|f(z)|) &\le \int_{[0,z]} \frac{\Lambda_f(\zeta)}{1+|f(\zeta)|^2}|d\zeta| \\ &\le m\int_{[0,z]} \frac{|d\zeta|}{1-|\zeta|^2} \\ &= m \operatorname{arctanh}(|z|).\end{aligned}$$

This implies that f is harmonic in $\mathbb{D}_R$ with $R = \tanh(\pi/2m)$. Set $r = 1/\sqrt{1+m^2}$. Then $r < R$ and $F(z) = f(rz)/rm$ is harmonic in $\mathbb{D}$. Clearly, $F(0) = F_{\bar z}(0) = F_z(0) - 1 = 0$. Also, we find that

$$\begin{aligned}(1-|z|^2)\Lambda_F(z) &= \frac{(1-|z|^2)\Lambda_f(rz)}{m} \\ &\le \frac{1-|z|^2}{1-|rz|^2}[1+\tan^2(m \operatorname{arctanh}(|rz|))] \\ &\le 1.\end{aligned}$$

By Theorem 2.4, we obtain that F is schlicht in $\mathbb{D}_{r_0}$ with $r_0 = 1/\sqrt{3}$. Therefore,

$$M(m) \ge 2 \operatorname{arctanh}\left(\frac{1}{\sqrt{3(1+m^2)}}\right).$$

This concludes the proof. □

REMARK 4.2. If $f_{\bar z} \equiv 0$, then Theorem 4.1 coincides with [8, Theorem 5].

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