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NOETHER LATTICES REPRESENTABLE AS QUOTIENTS OF THE LATTICE OF MONOMIALLY GENERATED IDEALS OF POLYNOMIAL RINGS

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Noether lattices were introduced by R. P. Dilworth in [**5**] and constitute a natural abstraction of the lattice of ideals of a Noetherian ring. In his definitive work, Dilworth showed that a minimal prime of an element generated by n principal elements has rank $\leqq n$. Following standard ring theoretical terminology, a local Noether lattice with (unique) maximal element M is said to be *regular* if M has rank n and can be generated by n principal elements.

In [**3**], K. P. Bogart showed that a distributive regular local Noether lattice of Krull dimension n is isomorphic to RL_n, the sublattice of all ideals generated by monomials of any polynomial ring $k[x_1, \ldots, x_n]$ (k a field). In a later paper [**4**], Bogart extended his results on distributive regular local Noether lattices by showing that any distributive local Noether lattice is the image of a multiplicative map θ which preserves joins, and can in fact be thought of as the related congruence lattice.

This paper began with two related problems which occurred at about the same time. First: given Bogart's result above that every distributive local Noether lattice $\mathscr{L}$ is the image of a distributive regular local Noether lattice RL_n under a multiplicative map θ which preserves joins, what special properties does $\mathscr{L}$ have if θ is a lattice homomorphism? And, second: what are the special properties of the quotients RL_n/K, either in terms of internal properties or in terms of properties of the map θ, which distinguish them from the other distributive local Noether lattices? The first question led to a general investigation of what we have called r-homomorphisms, and yielded a generalized "Fundamental Theorem" for this class of homomorphisms. Applied to the original question, it shows that if θ is a lattice homomorphism, then $\mathscr{L}$ is, up to isomorphism, one of the quotients RL_n/K. Since the natural map $\pi_K : RL_n \rightarrow RL_n/K$ is a lattice homomorphism, the second problem is reduced to the problem of finding an internal characterization of the quotients RL_n/K. Here we discovered that the quotients RL_n/K are distinguished (among distributive local Noether lattices) by the property that the elements E_i of the minimal base of the maximal elements are (what we have called) *q-prime* (i.e., if F_1 and F_2 are principal elements such that $F_1F_2 \leqq E$, then $F_1 \leqq E$, $F_2 \leqq E$, or $F_1F_2 = 0$). A generalization of Bogart's result mentioned above is also obtained outside of the local case.

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It is convenient to introduce some terminology.

By a *homomorphism* (or *morphism*) between Noether lattices $\mathscr{L}$ and $\mathscr{L}'$ we will mean a multiplicative lattice homomorphism $\theta : \mathscr{L} \rightarrow \mathscr{L}'$. If θ is just a multiplicative map which preserves order, we will call θ an *O-morphism*. Similarly, if we abbreviate join, meet and residual division by J, M and R, respectively, we will call θ an *X-morphism* if θ is a multiplicative map which preserves the X-operation ($X = J, M, R$). (It is easy to see that for $X = J$, M, R, any X-morphism is an O-morphism.) If $\theta : \mathscr{L} \rightarrow \mathscr{L}'$ is a homomorphism, and if there exists a subset $\mathscr{G}$ of principal elements which generates $\mathscr{L}$ under joins such that $\theta(E)$ is principal, for every element $E \in \mathscr{G}$, then we call θ an *r-homomorphism*. We will also use the variations epimorphism and monomorphism, with or without further prefixes, when appropriate.

If $K \in \mathscr{L}$ we denote by π_K the natural map of $\mathscr{L}$ to $\mathscr{L}/K$ (i.e., $\pi_K(A) = A \vee K$). And if S is a submultiplicatively closed subset of $\mathscr{L}$ we denote by i_S the natural map of $\mathscr{L}$ to $\mathscr{L}_S$ (i.e., $i_S(A) = A_S$) (see [**2**, Section 2]). We note that, in our terminology, i_S is both an r-epimorphism and an R-epimorphism (an R-r-epimorphism), while π_K is a J-epimorphism. (If K is a distributive element, π_K is an M-morphism, but in general, π_K need not be either an R-morphism or an M-morphism, or may be an M-morphism and not an R-morphism; see Corollary 1.1.)

If θ is any O-morphism, we will denote by $\mathscr{K}(\theta)$ the join of all elements A such that $\theta(A) = \theta(O)$ and by $\mathscr{I}(\theta)$ the multiplicatively closed subset of all elements A such that $\theta(A) = (I)$.

It is easily seen that if $\theta : \mathscr{L} \rightarrow \mathscr{L}'$ is any O-morphism and if $\mathscr{I}(\theta) = S$, then $A_S \leqq B_S$ implies $\theta(A) \leqq \theta(B)$. Hence, naturally associated with any O-morphism θ is a map $\theta_S : \mathscr{L}_S \rightarrow \mathscr{L}'$ defined by $\theta_S(A_S) = \theta(A)$. Although discovered independently by the present authors, a slight variation of the map θ_S was first isolated and used by P. J. McCarthy to study what, in our setting, amounts to R-epimorphisms [**7**]. We record the principal properties of θ_S below without proof.

THEOREM 1. *Let $\theta : \mathscr{L} \rightarrow \mathscr{L}'$ be an O-morphism with $\mathscr{I}(\theta) = S$. Then*

(i) *θ_S is an O-morphism;*

(ii) *$\theta = \theta_S i_S$;*

(iii) *$\theta_S(X) = I$ if, and only if, $X = I$;*

(iv) *θ_S is a J-morphism (resp. M-morphism, R-morphism) if, and only if, θ is a J-morphism (resp. M-morphism, R-morphism);*

(v) *if θ is an R-morphism, then $\theta_S(A) \leqq \theta_S(B)$ if, and only if, $A \leqq B$. Hence $\theta(\mathscr{L})$ is isomorphic to $\mathscr{L}_S$ so that, in particular, $\theta(\mathscr{L})$ is a Noether lattice;*

(vi) *if θ is an R-epimorphism and $S = \{I\}$, then θ is an isomorphism;*

(vii) *if θ is an R-epimorphism, then θ is an M-J-morphism.*

COROLLARY 1.1. *If $\pi_K : \mathscr{L} \rightarrow \mathscr{L}/K$ is an R-morphism, and if $K \leqq J(\mathscr{L}) = \wedge \{M \mid M$ is maximal in $\mathscr{L}\}$, then $K = O$.*

COROLLARY 1.2. *If S is a submultiplicatively closed subset of $\mathscr{L}$ and if $\hat{S} = \mathscr{I}(S)$, then $\mathscr{L}_{\hat{S}} \cong \mathscr{L}_S$. Moreover, $\hat{S}$ is the largest multiplicatively closed subset of $\mathscr{L}$ such that $\hat{S} \supseteq S$ and $A_{\hat{S}} \mapsto A_S$ is an isomorphism.*

It is trivial that if $\theta : \mathscr{L} \to \mathscr{L}'$ is a J-morphism and $B \leqq \mathscr{K}(\theta)$, then the restriction of θ to $\mathscr{L}/B$ is a J-morphism. We denote the restriction of θ to $\mathscr{L}/B$ by θ_B. Of course, in general, θ_B will not be an isomorphism, even if $B = \mathscr{K}(\theta)$. However, (iii) of Theorem 1 allows us to restrict our attention to a special case.

THEOREM 2. *Let $\theta : \mathscr{L} \to \mathscr{L}'$ be a homomorphism such that $\mathscr{I}(\theta) = \{I\}$. If $\mathscr{K}(\theta) = K$, then the map $\theta_K : \mathscr{L}/K \to \mathscr{L}'$ is an r-monomorphism provided*

(i) *θ is an epimorphism; or, provided θ is an r-homomorphism and one of the following is satisfied:*

(ii) *$\mathscr{L}'$ is local;*

(iii) *O is prime in $\mathscr{L}'$;*

(iv) *if D and E are elements of $\mathscr{L}$ with $E \in \mathscr{G}$, then $\theta(D)\theta(E) = \theta(E)$ implies $E \leqq DE \vee K$.*

Proof. Clearly (ii) and (iii) imply (iv), since $\mathscr{I}(\theta) = \{I\}$. We show that if (i) holds then θ is an r-homomorphism satisfying (iv) and that if θ satisfies (iv), then θ_K is a monomorphism.

Hence, assume (i) holds and let D and E be elements of $\mathscr{L}$ with E principal. Then $\theta(D) \wedge \theta(E) = \theta(D \wedge E) = \theta((D : E)E) = \theta(D : E)\theta(E)$, so that $\theta(E)$ is weak meet principal, and therefore principal, in $\mathscr{L}'$, [**2**, Theorem 2.9]. If $\theta(D)\theta(E) = \theta(E)$, then $I = \theta(D) \vee (O : \theta(E))$. Choosing $C \in \mathscr{L}$ so that $\theta(C) = O : \theta(E)$, we get $I = \theta(D) \vee \theta(C) = \theta(D \vee C)$, so that $D \vee C = I$ and therefore $DE \vee CE = E$. Since $CE \leqq K$ by the choice of C, it follows that (iv) holds.

Now, assume that (iv) holds and that A and B are elements of $\mathscr{L}/K$ with $\theta(A) \leqq \theta(B)$. If E is any principal element in $\mathscr{G}$ such that $E \leqq A$, then $\theta(E) \leqq \theta(A) \leqq \theta(B)$, so that

$$\theta(E) = \theta(B) \wedge \theta(E) = \theta(B \wedge E) = \theta((B : E)E) = \theta(B : E)\theta(E).$$

Since A is the join of principal elements in $\mathscr{G}$, it follows that $A \leqq B \vee K = B$.

The following might well be called the fundamental theorem of r-homomorphisms.

THEOREM 3. *Let $\theta : \mathscr{L} \to \mathscr{L}'$ be an epimorphism. Let $S = \mathscr{I}(\theta)$ and $K = \mathscr{K}(\theta_S)$. Then the following diagram is commutative, all maps involved are r homorphisms and the map $(\theta_S)_K$ is an isomorphism.*

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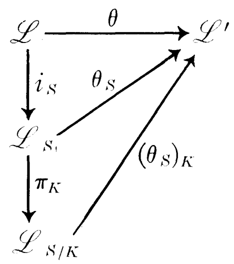

Proof. The results follow readily from Theorem 1 and Theorem 2.

We note that Theorem 2 can be used to obtain three alternative statements of Theorem 3 in which the conclusion is that $\mathscr{L}_S/K$ is isomorphic to the image in $\mathscr{L}'$ of θ. In particular, we observe that if $\theta : \mathscr{L} \rightarrow \mathscr{L}'$ is an r-homomorphism and if one of (ii), (iii), or (iv) of Theorem 2 is satisfied, then $\theta(\mathscr{L})$ is a sub-Noether lattice of $\mathscr{L}'$.

In [**4**], K. P. Bogart showed that if $\mathscr{L}$ is a distributive local Noether lattice with maximal element $\mathscr{M}$, then there exists a regular local Noether lattice RL_n and a J-epimorphism $\theta : RL_n \rightarrow \mathscr{L}$. If we denote the equivalence relation back induced on RL_n also by θ, then $RL_n/\theta \cong \mathscr{L}$. We extend this result to regular Noether lattices in general. ($\mathscr{L}$ is said to be *regular* if $\mathscr{L}_M$ is regular for each maximal element M of $\mathscr{L}$.)

THEOREM 4. *Let $\mathscr{L}$ be a distributive Noether lattice. Then there exists a regular Noether lattice domain $\mathscr{R}(\mathscr{L})$ and a J-epimorphism $\theta : \mathscr{R}(\mathscr{L}) \rightarrow \mathscr{L}$, which takes principal elements to principal elements, such that*

(i) *θ establishes a bijection between the maximal elements of $\mathscr{R}(\mathscr{L})$ and the maximal elements of $\mathscr{L}$, and*

(ii) $\mathscr{I}(\theta) = \{I\}$.

Proof. Let $\mathscr{F}$ be the family of maximal elements of $\mathscr{L}$. For each $M \in \mathscr{F}$, choose a finite set $p(M)$ of principal elements such that every prime $P \leqq M$ is the join of a subset of $p(M)$ (this is possible since there are only finitely many primes in $\mathscr{L}_M$). Let S be the multiplicative closure of $K = \bigcup_{M \in \mathscr{F}} p(M)$, and let $\mathscr{G}$ be the closure of S under joins, including O and I. Assume $\mathscr{G} \neq \mathscr{L}$ and let N be maximal in the complement of $\mathscr{G}$, so that N is not prime. Fix principal elements E, F such that $EF \leqq N$, $E \nleqq N$ and $F \nleqq N$. Then $N < N : E \neq I$, so $N : E \in \mathscr{G}$, by the maximality of N. Hence, we may choose $N_1, \ldots, N_k \in S$ with $N : E = N_1 \vee \ldots \vee N_k$. It follows that $N = \bigvee_{i=1}^k (N \wedge N_i) = \bigvee_{i=1}^k (N : N_i)N_i$. Now, $N_i \leqq N : E$ implies $E \leqq N : N_i$. Since also $N \leqq N : N_i$, it follows that $N : N_i \in \mathscr{G}$. But then $(N : N_i)N_i \in \mathscr{G}$, $i = 1, \ldots, k$, and therefore also $N \in \mathscr{G}$. Hence $\mathscr{G} = \mathscr{L}$.

Now, let X be the set of all subsets A of K such that $A \subseteq p(M)$ for some $M \in \mathscr{F}$. Then by Theorem 8 of [**1**], there exists a unique regular Noether lattice domain $\mathscr{R}(\mathscr{L})$ and a bijection θ from the set of principal primes of

$\mathscr{R}(\mathscr{L})$ onto K that extends to an isomorphism of posets $\hat{\theta}$: Spec $(\mathscr{R}(\mathscr{L})) \to X$ given by $\hat{\theta}(P) = \{\theta(P_1), \ldots, \theta(P_n)\}$, where $P = P_1 \vee \ldots \vee P_n$ is the unique decomposition of P as a join of nonzero principal primes. If we extend θ to a map of $\mathscr{R}(\mathscr{L})$ to $\mathscr{L}$ by taking products to products and joins to joins, then θ has the desired properties.

We note that above it is not sufficient to take $p(M)$ to be an arbitrary finite set of principal elements with join M (as it is in the local case). For example, $\mathscr{L} = RL_1 \oplus RL_1$ has two maximal elements, (M, I) and (I, M), both of which are principal. However neither (O, I) nor (I, O) is a join of powers of (M, I) and (I, M).

THEOREM 5. *Let* $\mathscr{L}$ *be a distributive Noether lattice. Then there exists a regular distributive Noether lattice domain* $\hat{\mathscr{L}}$ *and an r-epimorphism* $\theta : \hat{\mathscr{L}} \to \overline{\mathscr{L}}$ *if, and only if,* $\mathscr{L}$ *is isomorphic to a quotient* $\overline{\mathscr{L}}/K$ *of a distributive regular Noether lattice domain* $\overline{\mathscr{L}}$.

Proof. If $\mathscr{I}(\theta) = \hat{S}$, then $\hat{\mathscr{L}}_{\hat{S}}$ is a distributive regular Noether lattice domain [**1**]. By Theorem 3, $\mathscr{L} \cong \hat{\mathscr{L}}_{\hat{S}}/K$, where $K = \mathscr{K}(\theta_S)$.

Because of the additional structural knowledge of the local case, Theorem 5 can be strengthened considerably in the local case. If $X_1, \ldots, X_n$ is the minimal base of the maximal element of RL_n, we adopt the notation

$$RL_n = RL(X_1, \ldots, X_n).$$

The following theorem summarizes our results on distributive local Noether lattices and gives the internal characterization referred to in the introduction. Recall that an element E is q-prime if, for principal elements F_1, F_2, $F_1F_2 \leqq E$ implies $F_1 \leqq E$, $F_2 \leqq E$ or $F_1F_2 = 0$.

THEOREM 6. *Let* $(\mathscr{L}, M)$ *be a distributive local Noether lattice. Let* $E_1, \ldots, E_n$ *be the minimal base for the maximal element* M. *And let* $\theta : RL_n \to \mathscr{L}$ *be the unique J-epimorphism from* RL_n *to* $\mathscr{L}$ *satisfying* $\theta(X_i) = E_i$. *Then the following are equivalent:*

(i) E_i *is q-prime,* $i = 1, \ldots, n$;

(ii) θ *is an r-homomorphism;*

(iii) $\mathscr{L} \cong RL_n/K$, *where* $K = \mathscr{K}(\theta)$;

(iv) $\mathscr{L} \cong RL_m/K$, *for some* K;

(v) *if* E, F *are principal elements of* RL_n *with* $\theta(E) = \theta(F) \neq 0$, *then* $E = F$.

Proof. Theorem 3 shows that (ii) implies (iii). That (iii) implies (iv) is obvious. The verification that (ii) implies (i) is straightforward, using that the elements $X_i \in RL_m$ are prime and that principal elements in $\mathscr{L}$ are join-irreducible.

Assume that (i) holds and that $O \neq \theta(E) = \prod_1^n E_i^{s_i} = \prod_1^n E_i^{r_i} = \theta(F)$. If $s_i > 0$. But then $E_i^{s_i - 1} \prod_{j \neq i} E_j^{s_j} = E_i^{s_i - 1} \prod_{j \neq i} E_j^{r_j}$. That (i) implies (v) now follows by induction.

Now, assume that (v) holds and that A and B are elements of RL_n such that $\theta(A) = \theta(B)$. Let $A_1, \ldots, A_m$ be the (unique) minimal base for A and let $B_1, \ldots, B_r$ be the minimal base for B. Then for each t there exist u and v such that $\theta(A_t) \leqq \theta(B_u) \leqq \theta(A_v)$, whence $\theta(A_t) = \theta(E)\theta(A_v) = \theta(EA_v)$, for some principal element $E \in RL_n$. It follows that $\theta(A_t) = 0$ or $A_t = EA_v$. In the latter $E = I$ and $t = v$, so that $A_t = B_u$. Hence $A \leqq B \vee K$, where $K = \mathscr{K}(\theta)$. Similarly $B \leqq A \vee K$. Since $\theta(A) = \theta(A \vee K)$, we have that $\theta(A) = \theta(B)$ if, and only if, $A \vee K = B \vee K$. Since $(A \vee K) \wedge (B \vee K) = (A \wedge B) \vee K$, it follows that θ is an r-homomorphism. Hence (v) implies (ii), and the proof is complete.

It is obvious that if $\mathscr{L}'$ is isomorphic to a quotient $\mathscr{L}/K$ and $\mathscr{L}$ itself is isomorphic to a quotient of a distributive regular local Noether lattice, then $\mathscr{L}'$ is isomorphic to a quotient of a regular, local Noether lattice. The following proves the somewhat surprising result that any sub-Noether lattice of a quotient of a distributive regular local Noether lattice is isomorphic to a quotient of a distributive regular local Noether lattice.

THEOREM 7. *Let* $: \mathscr{L} \rightarrow \mathscr{L}'$ *be an r-monomorphism, where* $\mathscr{L}'$ *is isomorphic to a quotient of a distributive regular local Noether lattice. Then* $\mathscr{L} \cong RL_n/K$ *for some* n *and some* K.

Proof. Since $\theta(I)$ is idempotent, either $\theta(I) = I$ or $\theta(I) = 0$. In the latter case, $\mathscr{L} = \{0\}$. Similarly, $\theta(0)$ is idempotent, and therefore either $\mathscr{L} = \{0\}$ or $\theta(0) = 0$. We may assume $\theta(I) = I$, $\theta(0) = 0$, and $I \neq 0$. Let $E_1, \ldots, E_n$ be a minimal base for the maximal element M of $\mathscr{L}$, and let $E_1', \ldots, E_n'$ be a minimal base for the maximal element M' of $\mathscr{L}'$. We may assume that $\mathscr{L}' = RL_m/K$ and that $E_i' = X_i \vee K$. Note that in RL_m/K the intersection of a finite collection of principal elements is principal. Also,

$$\prod_{j=1}^{m} X_j^{e_j} \vee K = \bigwedge_{j=1} X^{e_j} \vee K, \text{ and}$$
$$\prod_{j=1}^{m} X_j^{e_j} \vee K \leqq \prod_{j=1}^{m} X_j^{f_j} \vee K$$

if, and only if, either

$$\prod_{j=1}^{m} X_j^{e_j} \leqq K \text{ or } e_j \geqq f_j \text{ for all } j = 1, \ldots, m.$$

Fix r and s, $1 \leqq r < s \leqq n$. Then

$$\theta(E_r) \wedge \theta(E_s) = \theta(E_r \wedge E_r) \leqq \theta(ME_s) = \bigvee_{i=1}^{n} \theta(E_iE_s),$$

so

$$\theta(E_r) \wedge \theta(E_s) \leqq \theta(E_iE_s), \text{ for some } i = 1, \ldots, n.$$

Set

$$\theta(E_i) = \prod_{j=1}^{m} X_j^{i_j} \vee K, \; i = 1, \ldots, n.$$

We assume that $r_j \geqq s_j$ for $1 \leqq j \leqq u$ and that $r_j < s_j$ for $j > u$. Then

$$\theta(E_r) \wedge \theta(E_s) = (\prod_{j=1}^m X_j^{r_j} \vee K) \wedge (\prod_{j=1}^m X_j^{s_j} \vee K)$$
$$= (\bigwedge_{j=1}^m X_j^{r_j} \vee K) \wedge (\bigwedge_{j=1}^m X_j^{s_j} \vee K) = (\bigwedge_{j=1}^u X_j^{r_j})$$
$$\wedge (\bigwedge_{j>u} X_j^{s_j}) \vee K \leqq \theta(E_i)\theta(E_s) = \prod_{j=1}^n X_j^{i_j+s_j} \vee K.$$

If $\theta(E_r) \wedge \theta(E_s) = 0$, then clearly

$$\theta(E_r) \wedge \theta(E_s) = \theta(E_r)\theta(E_s).$$

Otherwise, $r_j = i_j + s_j$ for $1 \leqq j \leqq u$ and $s_j = i_j + s_j$ for $j > u$. It follows that $i_j \leqq r_j$ for all j, and hence that $\theta(E_i) \leqq \theta(E_r)$. Since θ is an embedding and $E_1, \ldots, E_n$ is a minimal base for M, it follows that $i = r$, and therefore that

$$\theta(E_r) \wedge \theta(E_s) = \theta(E_r)\theta(E_s).$$

Hence $E_r \wedge E_s = E_rE_s$ for all $r \neq s$. But then $(E_r : E_s)E_s = E_rE_s$, so that $E_r : E_s = E_r \vee (0 : E_s)$. Since every principal element in $\mathscr{L}$ is a product of $E_1, \ldots, E_n$, it follows that E_r is q-prime for all r, and hence that $\mathscr{L}$ is a quotient of RL_n.

We note that $\mathscr{L} = [M^2, M^3] \cup \{I\}$ is naturally embedded in RL_n/M^3 (when M is the maximal element of RL_n) whereas for $n \geqq 2$, the number of elements in a minimal base for M^2 in $\mathscr{L}$ exceeds the number of elements in a minimal base for M in RL_n/M^3. However, if $\mathscr{L}'$ is taken to be a domain in Theorem 7, this cannot happen.

THEOREM 8. *Let* $(\mathscr{L}, M)$ *be a local Noether lattice and let* $\theta : \mathscr{L} \to RL_n$ *be an r-monomorphism. If* $E_1, \ldots, E_m$ *is a minimal base for the maximal element of* $\mathscr{L}$, *then* $\mathscr{L} \cong RL_m$ *for some* $m \leqq n$.

Proof. We may assume $\mathscr{L} \neq \{0\}$. Of necessity, $\mathscr{L}$ must be a domain, since RL_n is. By Theorem 7, $\mathscr{L}$ is isomorphic to RL_m/K, for some K, so since the only primes of RL_m are generated by subsets of the minimal base for the maximal element of RL_m, we may assume $\mathscr{L} = RL_m$. Let $X_1, \ldots, X_m$ be the minimal base for the maximal element of RL_m and let $Y_1, \ldots, Y_n$ be the minimal base for the maximal element of RL_n. If $\theta(Y_i)$ and $\theta(Y_j)$ have a common factor, say X_k, then there exist principal elements E_i and E_j in RL_n such that $\theta(Y_i) = X_kE_i$ and $\theta(Y_j) = X_kE_j$. If $i \neq j$, then

$$X_k^2E_iE_j = (X_kE_i)(X_kE_j) = \theta(Y_i)\theta(Y_j) = \theta(Y_i \wedge Y_j)$$
$$= \theta(Y_i) \wedge \theta(Y_j) = X_kE_i \wedge X_kE_j = X_k(E_i \wedge E_j) \geqq X_kE_iE_j,$$

which is a contradiction. A simple counting argument now shows that $m \leqq n$.

If $\mathscr{L}$ is any Noether lattice and $E_1, \ldots, E_n$ are principal elements, we denote by $RL(E_1, \ldots, E_n)$ the multiplicative lattice consisting of all joins of power products of $E_1, \ldots, E_n$.

It follows from the previous results that if $E_1, \ldots, E_n$ is a subset of the minimal base for the maximal element of RL_m/K, then $RL(E_1, \ldots, E_n)$ is a sub-Noether lattice of RL_m/K and is in fact isomorphic to a quotient of RL_n. Although the elements $E_1, \ldots, E_n$ do not necessarily form a prime sequence, this behavior is reminiscent of that described in [**6**], and the analogy is made even tighter by the fact that the elements $Q_i = E_1 \vee \ldots \vee E_i$ form a chain of q-prime elements of length n. These observations suggest natural generalizations of the definitions of prime sequence and regular. Specifically, if $\mathscr{L}$ is a Noether lattice, we call an ordered sequence $E_1, \ldots, E_n$ of nonzero principal elements (contained in the radical of $\mathscr{L}$) a *q-prime sequence* if it satisfies the conditions

(i) $(E_1 \vee \ldots \vee E_i) : E_{i+1} = E_1 \vee \ldots \vee E_i \vee (0 : E_{i+1})$, for all $i = 1, \ldots, n-1$, and

(ii) $(0 : E_i) \wedge (J_1 \vee J_2) = ((0 : E_i) \wedge J_1) \vee ((0 : E_i) \wedge J_2)$, for all $i = 1, \ldots, n$, and for all $J_1, J_2 \in RL(E_1, \ldots, E_n)$.

We call a local Noether lattice $(\mathscr{L}, M)$ *q-regular* if there exists a q-prime chain $Q_0 < Q_1 < \ldots < Q_d$, where d is the number of elements in a minimal base for M.

We note that since the elements $E_1, \ldots, E_n$ are principal, (i) is equivalent to

$$(E_1 \vee \ldots \vee E_i) \wedge E_{i+1} = (E_1 \vee \ldots \vee E_i)E_{i+1}$$

and (ii) is equivalent to

$$E_i(J_1 \wedge J_2) = E_iJ_1 \wedge E_iJ_2,$$

for all i and for all $J_1, J_2 \in RL(E_1, \ldots, E_n)$.

We begin by showing that, as for prime sequences, q-prime sequences are order independent.

LEMMA 9.1. *Let $E_1, \ldots, E_n$ be a q-prime sequence and $\varphi \in S_n$. Then $E_{\varphi(1)}, \ldots, E_{\varphi(n)}$ is a q-prime sequence.*

Proof. Since $E_2 \wedge E_1 = E_1 \wedge E_2 = E_1E_2$, it suffices to show that

$$(E_1 \vee \ldots \vee E_{i-1}) \wedge E_{i+1} = (E_1 \vee \ldots \vee E_{i-1})E_{i+1}$$

and that

$$(E_1 \vee \ldots \vee E_{i-1} \vee E_{i+1}) \wedge E_i = (E_1 \vee \ldots \vee E_{i-1} \vee E_{i+1})E_i.$$

for all $i \geqq 2$.

Now,

$$\begin{aligned}(E_1 \vee \ldots \vee E_{i-1}) \wedge E_{i+1} &= (E_1 \vee \ldots \vee E_{i-1}) \wedge E_{i+1} \\ &\wedge (E_1 \vee \ldots \vee E_i) = (E_iE_{i+1} \vee \ldots \vee E_{i-1}E_{i+1}) \\ &\vee ((E_1 \vee \ldots \vee E_{t-1}) \wedge E_i \wedge E_iE_{i+1}) \\ &= (E_1E_{t+1} \vee \ldots \vee E_{t-1}E_{i+1}) \vee ((E_1 \vee \ldots \vee E_{i-1}) \wedge E_{i+1})E_t,\end{aligned}$$

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so

$$(E_1 \vee \ldots \vee E_{i-1}) \wedge E_{i+1} = E_1E_{i+1} \vee \ldots \vee E_{i-1}E_{i+1} = (E_1 \vee \ldots \vee E_{i-1})E_{i+1},$$

by the Intersection Theorem.

Similarly,

$$(E_1 \vee \ldots \vee E_{i-1} \vee E_{i+1}) \wedge E_i = (E_1 \vee \ldots \vee E_{i-1} \vee E_{i+1}) \wedge (E_1 \vee \ldots \vee E_i) \wedge E_i = ((E_1 \vee \ldots \vee E_{i-1}) \vee ((E_1 \vee \ldots \vee E_i) \wedge E_{i+1})) \wedge E_i = (E_1 \vee \ldots \vee E_{i-1} \vee E_iE_{i+1}) \wedge E_i = ((E_1 \vee \ldots \vee E_{i-1}) \wedge E_i) \vee E_{i+1}E_i = (E_1 \vee \ldots \vee E_{i-1} \vee E_{i+1})E_i.$$

LEMMA 9.2. *Let* $E_1, \ldots, E_n$ *be a q-prime sequence and* $e_1, \ldots, e_n$ *nonnegative integers. Then*

$$\bigwedge_{j=1}^n E_i^{e_i} = \prod_1^n E_i^{e_i}.$$

Proof. Since for $r \neq s$, E_r, E_s is a q-prime sequence, we have

$$E_r^{i+1} \wedge E_s^{j+1} = E_r^{i+1} \wedge E_s^{j+1} \wedge E_r \wedge E_s = (E_r^{i+1} \wedge E_s^{j+1}) \wedge E_sE_r = E_r^{i+1} \wedge (E_s^{j+1} \wedge E_rE_s) = E_r^{i+1} \wedge ((E_s^j \wedge E_r)E_s) = E_r^{i+1} \wedge ((E_s^jE_r)E_s) = E_r^{i+1} \wedge (E_s^{j+1}E_r) = (E_r^i \wedge E_s^{j+1})E_r = (E_r^iE_s^{j+1})E_r = E_r^{i+1}E_s^{j+1},$$

by induction on the sum of the exponents. Hence

$$\bigwedge_{i=1}^n E_i^{e_i} = \bigwedge_{i=1}^{n-1} (E_i^{e_i} \wedge E_n^{e_n}) = \bigwedge_{i=1}^{n-1} E_i^{e_i}E_n^{e_n} = (\bigwedge_{i=1}^{n-1} E_i^{e_i})E_n^{e_n} = (\prod_{i=1}^{n-1} E_i^{e_i})E_n^{e_n} = \prod_{i=1}^n E_i^{e_i},$$

by induction on n.

LEMMA 9.3. *Let* $E_1, \ldots, E_n$ *be a q-prime sequence and let* J *be a join of power products of* $E_2, \ldots, E_n$. *Then* $E_1 \wedge J = E_1J$.

Proof. If no power product involved has length >1, then the result follows from Lemma 9.1. Hence, assume some power product involving E_n has length >1. Write $J = K \vee BE_n$, where K is the join of power products of $E_2, \ldots, E_{n-1}$.

By induction on the sum of the lengths of the power products of which J is the supremum, we have

$$E_1 \wedge J = E_1 \wedge ((E_1 \vee K) \wedge (K \vee BE_n)) = E_1 \wedge (K \vee ((E_1 \vee K) \wedge BE_n)) = E_1 \wedge (K \vee (((E_1 \vee K) \wedge E_n) \wedge BE_n)) = E_1 \wedge (K \vee (((E_1 \vee K)E_n) \wedge BE_n))$$

(by the inductive hypothesis, since E_n does not appear in $E_1 \vee K$ written as a join of power products)

$$= E_1 \wedge (K \vee ((E_1 \vee K) \wedge B)E_n)) = E_1 \wedge (K \vee ((E_1 \wedge B) \vee (K \wedge B))E_n))$$

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(since $E_1 \wedge (K \vee B) = E_1(K \vee B) = E_1K \vee E_1B = (E_1 \wedge K) \vee (E_1 \wedge B))$

$$= E_1 \wedge (K \vee E_1BE_n) = (E_1 \wedge K) \vee E_1BE_n = E_1K \vee E_1BE_n$$
$$= E_1(K \vee BE_n) = E_1J.$$

LEMMA 9.4. *Let $E_1, \ldots, E_n$ be a q-prime sequence in $\mathscr{L}$. Then $RL(E_1, \ldots, E_n)$ is a distributive sublattice of $\mathscr{L}$.*

Proof. If P and J_i are elements of $RL(E_1, \ldots, E_n)$, where P and J_i are power products, then $P \wedge J_i$ is an element of $RL(E_1, \ldots, E_n)$, by Lemma 9.2.

Hence, to show that

$$P \wedge (\bigvee_{i=1}^{s} J_i) = \bigvee_{i=1}^{s} (P \wedge J_i),$$

it suffices to consider the case $P = E_1^{r+1}$. Moreover, by Lemma 9.3, we may proceed by induction on r. Let $J_i = \prod_{j=1}^{n} E_j^{i_j}$ and assume $i_1 \geqq 1$ for $i = 1, \ldots, u$ and $i_1 = 0$ for $i > u$. Also, for $1 \leqq i \leqq u$, let

$$J_i' = E_1^{i_1-1} \prod_{j=2}^{n} E_j^{i_j}.$$

Then

$$E_1^{r+1} \wedge (\bigvee_{s=1}^{i} J_i) = E_1^{r+1} \wedge E_1 \wedge (\bigvee_{i=1}^{s} J_i) = E_1^{r+1} \wedge ((\bigvee_{i=1}^{u} J_i)$$
$$\vee (\bigvee_{i>u} E_1J_i)) = E_1(E_1^{r} \wedge ((\bigvee_{i=1}^{u} J_i') \vee (\bigvee_{i>u} J_i))$$
$$= E_1((\bigvee_{i=1}^{u} E_1^{r} \wedge J_i') \vee \bigvee_{i>u} (E_1^{r} \wedge J_i)) = \bigvee_{i=1}^{u} (E^{r+1} \wedge J_i).$$

The equation

$$(\bigvee_{i=1}^{u} P_i) \wedge (\bigvee_{j=1}^{s} J_j) = \vee_{i,j} (P_i \wedge J_j)$$

now follows by induction on u.

THEOREM 9. *Let $E_1, \ldots, E_n$ be a q-prime sequence in $\mathscr{L}$. Then*

$$RL(E_1, \ldots, E_n)$$

is a q-regular distributive Noether sublattice of $\mathscr{L}$.

Proof. Since $RL(E_1, \ldots, E_n)$ is a distributive sublattice of $\mathscr{L}$ by Lemma 9.4, and since every element of $RL(E_1, \ldots, E_n)$ is, by definition, a join of power products of $E_1, \ldots, E_n$, it suffices to show that the elements E_i are principal in $RL(E_1, \ldots, E_n)$.

By Lemma 9.3 and Lemma 9.4, it is immediate that $J \wedge E_i$ is a multiple of E_i, for every $J \in RL(E_1, \ldots, E_n)$. On the other hand, if $J \in RL(E_1, \ldots, E_n)$ and P is a power product of $E_1, \ldots, E_n$, then $PE_i \leqq JE_i$ implies

$$PE_i = PE_i \wedge JE_i = (P \wedge J)E_i,$$

so that (in $\mathscr{L}$)

$$P = (P \wedge J) \vee (P \wedge (0 : E_i)) = (P \wedge J) \vee (0 : PE_i)P.$$

It follows that either $PE_i = 0$ or that $P \leqq J$, whence $P \leqq J \vee (0 : E_i)$ in

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$RL(E_1, \ldots, E_n)$. Hence E_i is both weak meet principal and weak join principal, and therefore principal, in $RL(E_1, \ldots, E_n)$.

THEOREM 10. *Let* $(\mathscr{L}, M)$ *be a distributive q-regular local Noether lattice. If* $E_1, \ldots, E_n$ *is a minimal base for* M, *and if*

$$K = \vee \{X_1^{e_1} \ldots X_n^{e_n} | E_1^{e_1} \ldots E_n^{e_n} = 0\},$$

then $\mathscr{L} \cong RL_n/K$. Conversely, any quotient of RL_n is a distributive *q-regular* local Noether lattice.

Proof. Let $Q_0 < Q_1 < \ldots < Q_n$ be a q-prime chain in $\mathscr{L}$. It is easily seen that each of the elements Q_i is generated by a subset of $E_1, \ldots, E_n$ with i elements, so we may assume that $0 = Q_0$, and that $Q_i = E_1 \vee \ldots \vee E_i$. It follows that $E_1, \ldots, E_n$ is a q-prime sequence in $\mathscr{L}$, and hence by Lemma 9.1 that each of the elements E_i is q-prime. The isomorphism of $\mathscr{L}$ with RL_n/K now follows from Theorem 6.

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