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A HAHN-BANACH THEOREM IN SUBBASE CONVEXITY THEORY

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0. Introduction. In the last fifteen years, topology has shown up with an increasing interest in the use of closed subbases. Starting from Frink's internal characterization of complete regularity (Frink [**6**]), De Groot and Aarts used closed subbases to obtain Hausdorff compactifications of completely regular spaces, thus giving a characterization of the latter in terms of their subbases [**1**]. The main tool of that paper is the notion of a linked system, which naturally leads to the notions of supercompactness and superextensions [**7**]. After 1970, these two topics developed to indepedennt theories, with several deep results available at this moment. Most results up to 1976 are summarized in [**12**].

In supercompactness theory, Strok and Szymańsky's result of [**20**] that metric compacta have binary subbases settles a hard conjecture of De Groot. Simplified proofs of this result have been given in [**4**] and [**18**]. In [**2**], Bell and van Mill present a measure for the strength of compactness in terms of closed subbases (called compactness number) in such a way that supercompactness (compactness number 2) is the strongest. They prove the existence of compacta with any preassigned compactness degree, using elaborate combinatorial techniques.

In superextension theory, another hard conjecture (again, of De Groot) has been settled in [**10**], namely that the superextension of the unit interval is a Hilbert cube. Van Mill recently generalized his result to non-degenerate metric continua [**11**], thereby parallelling the efforts of Curtis, Schori and West in hyperspace theory ([**19**] and [**3**]). Besides a great portion of infinite dimensional topology, Van Mill's proofs also make an extensive use of subbase convexity theory.

This completely new theory started from investigations of [**13**] and [**5**], combined with the techniques and ideas exposed in the author's paper on the Lefschetz fixed point property of superextensions [**21**], and it fits in with the setting of abstract convexity theory [**9**].

One of the most exciting aspects of subbase convexity theory (developed mainly for normal binary subbases) is the great number of parallelisms with ordinary convexity theory in classes of (subspaces of) topological vectorspaces, e.g., the existence of retractions onto convex sets, the existence of "nearest point" mappings, and continuity of convex closure

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operators [**14**, 2.6, 3.4]. Even the notion of a linear map has been translated successfully within the framework of subbases ([**17**, 3.1] and [**15**, 1.2]). A noteworthy result is obtained in the former paper, linking up the theory with its very beginning: any two disjoint members of a normal subbase can be separated by a "subbase linear" map into [0, 1], a result which extends the classical Urysohn theorem from normal spaces to completely regular spaces.

One purpose of the present paper is to prove another parallel result, which is famous in vectorspace theory: The Hahn-Banach theorem (§ 4). Regarding only the geometric interpretation (separation of convex sets by linear maps) it turns out that an almost literal translation of the theorem in the language of subbase convexity is valid for "compact" convexities: compare our Theorem 11 with the Hahn-Banach theorem as formulated e.g. in [**8**, p. 57].

To obtain such a result we have also introduced a notion of convexity for arbitrary (non-closed) subsets, giving some extra parallellism with linear convexities (§ 3).

In Section 2, we show that certain hyperspace convexities satisfy the above mentioned "compactness" condition, thereby extending the class of "good" convexity structures.

Proposition 8′ in Section 2 was given to me by the referee.

1. Basic notions and results.

1. *Generalities*. All topological spaces are assumed to be T_1. Recall that a *T_1-subbase* $\mathscr{S}$ of a space X is a closed subbase such that for each $S \in \mathscr{S}$ and for each $x \in X - S$ there is an $S' \in \mathscr{S}$ with $x \in S' \subset X - S$, and that a *normal subbase* $\mathscr{S}$ is a closed subbase such that for any pair $S_1, S_2 \in \mathscr{S}$ of disjoint sets there exist $S_1', S_2' \in \mathscr{S}$ with

$$S_1 \subset S_1' - S_2'; \quad S_2 \subset S_2' - S_1'; \quad S_1' \cup S_2' = X.$$

In [**14**, 2.1], a *topological convexity* on a space X is defined to be a collection $\mathscr{C}$ of nonempty closed sets of X such that $X \in \mathscr{C}$, $\mathscr{C}$ is closed under nonempty intersection, and $\mathscr{C}$ is a closed subbase for the topology of X. Note that $\mathscr{C} \cup \{\emptyset\}$ is an *abstract convexity structure* in the sense of [**9**, p. 471]; we omit $\emptyset$ only to facilitate the use of hyperspaces. If $A \subset X$ is an arbitrary set, then

$$I_{\mathscr{C}}(A) = \bigcap \{C | A \subset C \in \mathscr{C}\}$$

will be called the *$\mathscr{C}$-convex closure* of A.

If $\mathscr{S}$ is a closed subbase of X, then $H(X, \mathscr{S})$ will denote the collection of all nonempty sets of type $\bigcap \mathscr{S}'$, with $\mathscr{S}' \subset \mathscr{S}$. This yields a topological convexity on X which is said to be *generated* (or *induced*) by $\mathscr{S}$. Members of $H(X, \mathscr{S})$ will also be called *$\mathscr{S}$-convex sets*, and we write $I_{\mathscr{S}}$ instead of $I_{H(X, \mathscr{S})}$.

This explicit description of a convexity in terms of a generating subbase occurs quite naturally in many circumstances (e.g. in case of superextensions, cf. [**21**, p. 39], or on hyperspaces, cf. Theorem 3 below), motivating the (informal) expression "subbase convexity theory."

We let $H(X)$ denote the set of all nonempty closed subsets of a space X. If $A_1, \ldots, A_n$ are nonempty subsets of X, then we write

$$\langle A_1, \ldots, A_n \rangle = \left\{ C \in H(X) | C \subset \bigcup_{i=1}^{n} A_i \text{ and } C \cap A_i \neq \emptyset \text{ for each } i \right\}.$$

Then $H(X)$ is topologized by taking as an open base the family of all sets of type $\langle O_1, \ldots, O_n \rangle$, where $n \geqq 1$ and $O_1, \ldots, O_n$ are open sets of X. The resulting space is the *hyperspace* of X. Note that a topological convexity $\mathscr{C}$ on X is a subset of $H(X)$. In the sequel, $\mathscr{C}$ will always be topologized as a subspace of $H(X)$ whenever topological considerations on $\mathscr{C}$ are involved.

A topological convexity $\mathscr{C}$ on X will be called *compact* if (i) $\mathscr{C}$ is normal T_1 as a subbase of X, and (ii) the subspace $\mathscr{C}$ of $H(X)$ is compact.

Notice that a space carrying a compact convexity is automatically compact and Hausdorff.

Compact convexities were studied in [**14**]. A main result of that paper is formulated in Theorem 2 below. The most interesting examples of compact convexities are the following ones:

(1) $\mathscr{C} = H(X)$ for a compact T_2 space X;

(2) $\mathscr{C}$ is the set of all closed linearly convex sets of a compact convex subspace of a locally convex topological vectorspace (cf. [**14**, Example 5.1]);

(3) If X is compact T_2, then the family

$$\{\langle C, X \rangle | \quad C \in H(X)\} \cup \{\langle C \rangle | \quad C \in H(X)\}$$

generates a compact convexity on $H(X)$ (cf. [**16**, Theorem 1.3]);

(4) $\mathscr{C}$ is a normal binary convexity (cf. [**14**, Theorem 3.8]). A convexity $\mathscr{C}$ is called *binary* if for each family $\mathscr{C}' \subset \mathscr{C}$ of which any two members meet, $\bigcap \mathscr{C}' \neq \emptyset$: cf. [**22**, p. 48]. Equivalently, $\mathscr{C}$ has *Helly number* 2 (cf. [**9**, p. 473]) and the underlying space is compact;

(5) If $\mathscr{C}_i$ is a compact convexity on X_i, $i = 1, 2$, then

$$\mathscr{C}_1 \times \mathscr{C}_2 = \{C_1 \times C_2 | \quad C_1 \in \mathscr{C}_1, C_2 \in \mathscr{C}_2\}$$

is a compact convexity on $X_1 \times X_2$ as one can easily verify.

The following result on compactness of convexities will be useful. If $\mathscr{C}$ is a convexity on X, then $\mathscr{H}(\mathscr{C})$ denotes the collection of all sets of type

$$\langle C, X \rangle \cap \mathscr{C} \text{ or } \langle C \rangle \cap \mathscr{C}, \quad C \in \mathscr{C}.$$

2. Theorem. *Let $\mathscr{C}$ be a normal T_1 convexity on the compact space X. Then the following assertions are equivalent:*

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(i) *$\mathscr{C}$ is compact;*

(ii) *the convex closure operator $I_{\mathscr{C}} : H(X) \to \mathscr{C}$ is a continuous retraction;*

(iii) *$\mathscr{H}(\mathscr{C})$ is a normal T_1 subbase for $\mathscr{C}$;*

See [**14**, Theorem 4.2]. Note the particular case $\mathscr{C} = H(X)$.

2. Compact convexities on hyperspaces. As a first application, we shall prove the following result, which is a considerable sharpening of a result in [**16**, Theorem 1.3], and which provides a new class of examples of compact convexities (notation of Theorem 2):

3. THEOREM. *Let $\mathscr{C}$ be a compact convexity on X. Then the convexity on $\mathscr{C} \subset H(X)$ induced by the closed subbase $\mathscr{H}(\mathscr{C})$ is also compact.*

Proof. The family $H(\mathscr{C}, \mathscr{H}(\mathscr{C}))$ is easily seen to be a normal T_1 subbase (the proof is identical to the one in [**16**] for the case $\mathscr{C} = H(X)$).

Let $\mathscr{A} \subset H(X)$. A set $B \in H(X)$ is called a *transversal set* of $\mathscr{A}$ (cf. [**16**]) if B meets all members of $\mathscr{A}$. The collection of all transversal sets of $\mathscr{A}$ will be denoted by $\perp(\mathscr{A})$. If $\mathscr{A} \subset \mathscr{C}$, then we shall write

$$\perp_{\mathscr{C}}(\mathscr{A}) = \perp(\mathscr{A}) \cap \mathscr{C}.$$

With this notation we obtain a diagram of functions which is easily seen to commute:

$$\begin{array}{ccc} H(\mathscr{C}) & \xrightarrow{\perp_{\mathscr{C}}} & H(\mathscr{C}) \\ {\scriptstyle \text{inclusion}}\downarrow & & \uparrow{\scriptstyle H(I_{\mathscr{C}})} \\ H(H(X)) & \xrightarrow{\perp} & H(H(X)) \end{array}$$

Here, $H(I_{\mathscr{C}})$ denotes the canonical extension of the map $I_{\mathscr{C}} : H(X) \to \mathscr{C}$ to the corresponding hyperspaces. (Note that for compact Y, Z and for a continuous $f : Y \to Z$, the map $H(f) : H(Y) \to H(Z)$ defined by $H(f)(A) = f(A)$, is continuous.) Notice that $H(I_{\mathscr{C}})$ is continuous since $\mathscr{C}$ is compact (Theorem 2). It has been proved in [**16**, Corollary 1.5] that the transversality map $\perp$ is continuous. Hence, the map $\perp_{\mathscr{C}}$ is continuous.

Let $\mathscr{A} \subset \mathscr{C}$ be closed and nonempty. By definition,

$$(*) \quad \begin{aligned} I_{\mathscr{H}(\mathscr{C})}(\mathscr{A}) &= \cap \{\langle C, X\rangle \mid \ C \in \perp_{\mathscr{C}}(\mathscr{A})\} \cap \langle I_{\mathscr{C}}(\cup \mathscr{A})\rangle \cap \mathscr{C} \\ &= \perp_{\mathscr{C}}\perp_{\mathscr{C}}(\mathscr{A}) \cap \langle I_{\mathscr{C}}(\cup \mathscr{A})\rangle. \end{aligned}$$

Using the latter formula, we are able to prove the compactness of the induced convexity in the following way.

Assume $\mathscr{A} \in H(\mathscr{C})$ is not $\mathscr{H}(\mathscr{C})$-convex. Then there is a

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$C \in I_{\mathscr{H}(\mathscr{C})}(\mathscr{A}) - \mathscr{A}$, i.e. a (closed) convex set C of X such that

$$C \in \perp_{\mathscr{C}} \perp_{\mathscr{C}}(\mathscr{A}); \quad C \subset I_{\mathscr{C}}(\bigcup \mathscr{A}); \quad C \notin \mathscr{A}.$$

Let $\mathscr{O}, \mathscr{R} \subset \mathscr{C}$ be disjoint open sets such that $C \in \mathscr{O}$ and $\mathscr{A} \subset \mathscr{R}$. Choose open sets $O_1, \ldots, O_n$ of X such that

$$C \in \langle O_1, \ldots, O_n \rangle \cap \mathscr{C} \subset \mathscr{O}.$$

As $C = I_{\mathscr{C}}(C)$ and as $I_{\mathscr{C}}$ is continuous, there exist open sets $P_1, \ldots, P_m$ of X such that $C \in \langle P_1, \ldots, P_m \rangle$, and such that for each $D \in \langle P_1, \ldots, P_m \rangle$,

$$I_{\mathscr{C}}(D) \in \langle O_1, \ldots, O_n \rangle.$$

We may assume moreover that each O_i includes some set P_j.

Since $C \subset I_{\mathscr{C}}(\bigcup \mathscr{A})$, we find that $I_{\mathscr{C}}(\bigcup \mathscr{A}) \cap P_j \neq \emptyset$ for each j. By the continuity of $I_{\mathscr{C}}$ and of the union operator, there is a neighborhood U of $\mathscr{A} \in H(\mathscr{C})$ such that for each $\mathscr{A}' \in U$ and for each j,

$$I_{\mathscr{C}}(\bigcup \mathscr{A}') \cap P_j \neq \emptyset.$$

By the continuity of $\perp_{\mathscr{C}} \circ \perp_{\mathscr{C}}$, there is another neighborhood V of $\mathscr{A} \in H(\mathscr{C})$ such that for each $\mathscr{A}' \in V$, $\perp_{\mathscr{C}} \circ \perp_{\mathscr{C}}(\mathscr{A}')$ meets $\langle P_1, \ldots, P_m \rangle$. Then

$$W = \bigcup \cap V \cap \langle \mathscr{R} \rangle$$

is a neighborhood of $\mathscr{A} \in H(\mathscr{C})$ which contains no $\mathscr{H}(\mathscr{C})$-convex set. Indeed, let $\mathscr{A}' \in W$. Then there is a set

$$C'' \in \perp_{\mathscr{C}} \perp_{\mathscr{C}}(\mathscr{A}') \cap \langle P_1, \ldots, P_m \rangle,$$

and there exist points

$$a_j' \in I_{\mathscr{C}}(\bigcup \mathscr{A}') \cap P_j, \quad j = 1, \ldots, m.$$

Then define

$$C' = I_{\mathscr{C}}(C'' \cup \{a_1', \ldots, a_m'\}) \cap I_{\mathscr{C}}(\bigcup \mathscr{A}').$$

If $B \in \mathscr{C}$ meets all members of $\mathscr{A}'$ then so does $I_{\mathscr{C}}(\bigcup \mathscr{A}') \cap B$, i.e.,

$$I_{\mathscr{C}}(\bigcup \mathscr{A}') \cap B \in \perp_{\mathscr{C}}(\mathscr{A}').$$

Since $C'' \in \perp_{\mathscr{C}} \perp_{\mathscr{C}}(\mathscr{A}')$, we have

$$\emptyset \neq C'' \cap I_{\mathscr{C}}(\bigcup \mathscr{A}') \cap B \subset C' \cap B,$$

from which it follows that $C' \in \perp_{\mathscr{C}} \perp_{\mathscr{C}}(\mathscr{A}')$. Also, $C' \in \langle I_{\mathscr{C}}(\bigcup \mathscr{A}') \rangle$, whence

$$C' \in I_{\mathscr{H}(\mathscr{C})}(\mathscr{A}').$$

On the other hand,

$$C'' \cup \{a_1', \ldots, a_m'\} \in \langle P_1, \ldots, P_m \rangle$$

whence

$$I_{\mathscr{C}}(C'' \cup \{a_1', \ldots, a_m'\}) \in \langle O_1, \ldots, O_n \rangle,$$

and, in particular, $C' \subset \bigcup_{i=1}^{n} O_i$. For each $i \in \{1, \ldots, n\}$ there is a P_j included in O_i, so

$$a_j' \in C' \cap P_j \subset C' \cap O_i.$$

This shows that $C' \in \langle O_1, \ldots, O_n \rangle$. It follows that $C' \notin \mathscr{R}$ and hence that $C' \notin \mathscr{A}'$. Therefore, $\mathscr{A}'$ is not $\mathscr{H}(\mathscr{C})$-convex.

3. More about subbase convexity. For the purpose of our Hahn-Banach theorem, we need to have a good notion of convexity for arbitrary subsets of a space. There are three reasonable candidates, of which the third one will turn out to be the best one in the present circumstances:

4. *Definitions.* Let $\mathscr{C}$ be a convexity on X, and let $A \subset X$ be arbitrary. Then

(i) A is *n-convex relative to* $\mathscr{C}$ $(n \geqq 2)$ if for each $F \subset A$ with $|F| \leqq n$, $I_{\mathscr{C}}(F) \subset A$;

(ii) A is *weakly convex relative to* $\mathscr{C}$ if A is n-convex for each $n \geqq 2$;

(iii) A is *convex relative to* $\mathscr{C}$ if for each compact set $B \subset A$, $I_{\mathscr{C}}(B) \subset$ A.

The notions of weak convexity and of 2-convexity have already occurred implicitely in the literature on the subject of convexity (cf. [**14**] Theorems 2.4 and 4.2, which form a part of Theorem 5 below).

To end with, we say that a set $A \subset X$ is (*weakly*) *biconvex relative to* $\mathscr{C}$ if both A and $X - A$ are (weakly) convex relative to $\mathscr{C}$. These biconvex sets will play the same role as the "half-spaces" in vectorspace theory. Their use is essential in our proof of the Hahn-Banach theorem.

Among the above defined notions there exist several relations:

5. THEOREM. *Let $\mathscr{C}$ be a normal T_1 convexity on X;*

(i) *if $\mathscr{C}$ is a compact convexity, then a closed set of X is weakly convex if and only if it is convex;*

(ii) *if $\mathscr{C}$ is a binary convexity* (*cf. Section* 1) *then a subset of X is 2-convex if and only if it is weakly convex;*

(iii) *if $\mathscr{C}$ is a binary convexity, then an open or closed subset of X is 2-convex if and only if it is convex.*

Proof. (i) is a reformulation of Theorem 4.2 of [**14**]. For a proof of (ii) and of (iii) we need the following auxiliary result:

5′. LEMMA. *Let $\mathscr{C}$ be a normal binary convexity and let $A_1, \ldots, A_n \in \mathscr{C}$. Then*

$$I_{\mathscr{C}}(\textstyle\bigcup_{i=1}^{n} A_i) = \bigcup \{I_{\mathscr{C}}\{a_1, \ldots, a_n\} \mid a_i \in A_i, \quad i = 1, \ldots, n\} \quad (*).$$

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(Note that this formula is equally valid for a linear convexity on a compact convex set in a vectorspace). Recall [**14**, Theorem 3.4] that there is a "nearest point map"

$$p : X \times \mathscr{C} \rightarrow X$$

sending a pair (x, C) to the unique point of C with the property that, whenever $x \in D \in \mathscr{C}$ and $D \cap C \neq \emptyset$, we have $p(x, C) \in D$ (cf. [**14**]).

If $x \in I_{\mathscr{C}}\{a_1, \ldots, a_n\}$, where $a_i \in A_i$, $i = 1, \ldots, n$, then obviously

$$x \in I_{\mathscr{C}}\left(\bigcup_{i=1}^{n} A_i\right).$$

If the latter is true, then define $a_i = p(x, A_i)$ for each i. If $x \notin I_{\mathscr{C}}\{a_1, \ldots, a_n\}$, then by the normality of $\mathscr{C}$ there exist $S_1, S_2 \in \mathscr{C}$ such that

$$x \in S_1 - S_2; \quad I_{\mathscr{C}}\{a_1, \ldots, a_n\} \subset S_2 - S_1; \quad S_1 \cup S_2 = X.$$

Now, $\bigcup_{i=1}^{n} A_i \not\subset S_2$, for otherwise $x \in S_2$. Therefore, $A_i \cap S_1 \neq \emptyset$ for some i. But $x \in S_1$, and hence $a_i \in S_1$ by the defining property of the nearest point map, a contradiction.

To prove (ii), let $A \subset X$ be 2-convex. We prove by induction on $n \geqq 2$ that A is n-convex for each n. Assume this has been proved for all $m \leqq n \geqq 2$, and let $F \subset A$ contain $n + 1$ points. Fixing $x_0 \in F$, we find that

$$I_{\mathscr{C}}(F) = I_{\mathscr{C}}(I_{\mathscr{C}}(F - \{x_0\}) \cup \{x_0\}) = \bigcup \{I_{\mathscr{C}}\{x, x_0\} | x \in I_{\mathscr{C}}(F - \{x_0\})\}$$

by the above formula (*). As $|F - \{x_0\}| = n$, we have $I_{\mathscr{C}}(F - \{x_0\}) \subset A$, and hence $I_{\mathscr{C}}\{x, x_0\} \subset A$ for each $x \in I_{\mathscr{C}}(F - \{x_0\})$, proving that $I_{\mathscr{C}}(F) \subset A$.

To prove (iii), we first notice that the equivalence of 2-convexity and of convexity for closed sets is essentially Theorem 2.4 of [**14**]. Let $O \subset X$ now be an open set which is 2-convex, and let $A \subset O$ be closed. $\mathscr{C}$ being a normal T_1 subbase on the compact space X, we can find for each $x \in A$ a convex closed neighborhood A_x contained in O. X being compact, a finite number $A_1, \ldots, A_n$ of these neighborhoods suffices to cover A. By (*) and by (ii), we find that

$$I_{\mathscr{C}}(A) \subset I_{\mathscr{C}}\left(\bigcup_{i=1}^{n} A_i\right) \subset O.$$

By (i) above, the expressions "$A \in \mathscr{C}$" and "A is nonempty $\mathscr{C}$-convex closed" are equivalent for a compact convexity $\mathscr{C}$.

Relative to normal binary subbases, the notions of weak convexity and of convexity are not equivalent for arbitrary subsets: let $X = [0, 1]^\alpha$ ($\alpha \geqq \omega_0$) be a Tychonov cube and let $\mathscr{T}$ be its canonical normal binary convexity, generated by the subbase of all sets of type

$$\pi_i^{-1}[0, t] \quad \text{or} \quad \pi_i^{-1}[t, 1], \quad t \in [0, 1], \quad i \in \alpha,$$

where $\pi_i : X \to [0, 1]$ is the ith projection. Let $A \subset X$ be the set of all points $(x_i)_{i\in\alpha}$ of X with $x_i \neq 0$ for an at most finite number of indices i. Then A is 2-convex relative to $\mathcal{T}$, and hence weakly convex. However, it is not convex: let $B \subset X$ be the set of all points $(x_i)_{i\in\alpha}$ in X with $x_i \neq 0$ for at most one index i. Then B is compact, $B \subset A$, and $I_{\mathcal{T}}(B) = X \not\subset A$.

It is an open problem whether convexity and weak convexity relative to a compact convexity are equivalent for open subsets.

6. Convexity preserving mappings. Let $\mathcal{C}$ and $\mathcal{D}$ be convexities on, respectively, X and Y, and let $f : X \to Y$ be a function. Then f is called a *convexity preserving* (c.p.) *map* if for each $D \in \mathcal{D}$ it is true that $f^{-1}(D) \in \mathcal{C} \cup \{\emptyset\}$ (cf. [**15**, p. 77]).

Notice that a cp map is automatically continuous. We have the following result:

7. THEOREM. *Let $\mathcal{C}$ and $\mathcal{D}$ be convexities on, respectively, the compact spaces X and Y, and let $f : X \to Y$ be a function;*

(i) *if f is a* cp *map, then for each (weakly) $\mathcal{D}$-convex set $B \subset Y$, the set $f^{-1}(B) \subset X$ is (weakly) $\mathcal{C}$-convex;*

(ii) *if $\mathcal{C}$ is a compact convexity, and if f is continuous, then f is a* cp *map if and only if for each finite set $F \subset X$,*

$$f(I_{\mathcal{C}}(F)) \subset I_{\mathcal{C}}(f(F)).$$

The proof is left to the reader. For (ii), use Theorem 5(i).

The following result will be useful in constructing (bi-) convex sets:

8. THEOREM. *Let $\mathcal{C}$ be a compact convexity on the space X. Then for each subset A of X, the mapping*

$$k_A : X \to \mathcal{C}$$

sending $x \in X$ to $I_{\mathcal{C}}(\{x\} \cup A)$ is convexity preserving.

Proof. The map k_A is continuous by the continuity of $I_{\mathcal{C}}$ and of the union operator. We can therefore apply Theorem 7(ii) for a simple proof that k_A is a cp map. Notice that if $A_1, \ldots, A_n \in \mathcal{C}$, then $I_{\mathcal{H}(\mathcal{C})}\{A_1, \ldots, A_n\}$ equals the family

$$\{C \in \mathcal{C} \mid \quad C \subset I_{\mathcal{C}}(\textstyle\bigcup_{i=1}^{n} A_i) \text{ and for each } a_i \in A_i, i = 1, \ldots, n,$$
$$C \cap I_{\mathcal{C}}\{a_1, \ldots, a_n\} \neq \emptyset\},$$

as one can easily verify. Let $x_1, \ldots, x_n \in X$ and let $x \in I_{\mathcal{C}}\{x_1, \ldots, x_n\}$. We have to show that

$$I_{\mathcal{C}}(\{x\} \cup A) \in I_{\mathcal{H}(\mathcal{C})}\{I_{\mathcal{C}}(\{x_1\} \cup A), \ldots, I_{\mathcal{C}}(\{x_n\} \cup A)\}.$$

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First, $I_{\mathscr{C}}(\{x\} \cup A)$ is obviously included in the set

$$I_{\mathscr{C}}\left(\bigcup_{i=1}^{n} I_{\mathscr{C}}(\{x_i\} \cup A)\right).$$

Assume next that there exist $a_1 \in I_{\mathscr{C}}(\{x_1\} \cup A), \ldots, a_n \in I_{\mathscr{C}}(\{x_n\} \cup A)$, such that

$$I_{\mathscr{C}}(\{x\} \cup A) \cap I_{\mathscr{C}}\{a_1, \ldots, a_n\} = \emptyset.$$

By normality of $\mathscr{C}$ (as a subbase), there exist $S_1, S_2 \in \mathscr{C}$ with

$$\{x\} \cup A \subset S_1 - S_2; \quad \{a_1, \ldots, a_n\} \subset S_2 - S_1; \quad S_1 \cup S_2 = X.$$

Then $\{x_1, \ldots, x_n\} \subset S_2$, for otherwise e.g. $x_i \in S_1$, whence

$$a_i \in I_{\mathscr{C}}(\{x_i\} \cup A) \subset S_1,$$

a contradiction. Therefore,

$$x \in I_{\mathscr{C}}\{x_1, \ldots, x_n\} \subset S_2,$$

again a contradiction.

The mappings $k_A, A \subset X$, are in one variable. Their cp property breaks down if more variables are considered:

8′. *Fact. If $\mathscr{C}$ is any convexity on any X with at least two points, then the map $f : X \times X \to \mathscr{C}$, defined by*

$$f(x, y) = I_{\mathscr{C}}\{x, y\} \quad (x, y \in X)$$

is not cp *relative to the product convexity $\mathscr{C} \times \mathscr{C}$ on $X \times X$ (cf.* 1(5)).

(With the above notation, this corresponds to the case $A = \emptyset$).

Proof. Let $a \neq b$ in X. Then

$$\mathscr{K} = \{C \in \mathscr{C} \mid \ a \in C\} = \langle\{a\}, X\rangle \cap \mathscr{C}$$

is a convex set of $\mathscr{C}$, and clearly

$$\{a\} \times X \cup X \times \{a\} \subset f^{-1}(\mathscr{K}).$$

If $f^{-1}(\mathscr{K})$ were convex relative to $\mathscr{C} \times \mathscr{C}$, then

$$X \times X = I_{\mathscr{C} \times \mathscr{C}}(\{a\} \times X \cup X \times \{a\}) \subset f^{-1}(\mathscr{K}).$$

However, $f(b, b) = \{b\} \notin \mathscr{K}$.

A final auxiliary result needed for the construction of biconvex sets is the following one:

9. Lemma. *Let $\mathscr{C}$ be a compact convexity on X, and let $O \subset X$ be convex and open. Then the sets $\langle O \rangle \cap \mathscr{C}$ and $\langle O, X \rangle \cap \mathscr{C}$ are $\mathscr{H}(\mathscr{C})$-convex and open.*

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Proof. $\langle O \rangle \cap \mathscr{C}$ and $\langle O, X \rangle \cap \mathscr{C}$ are open by definition. Let $\mathscr{A} \subset \mathscr{C}$ be closed—and hence compact.

If $\mathscr{A} \subset \langle O \rangle \cap \mathscr{C}$ then $\bigcup \mathscr{A} \subset O$, and $\bigcup \mathscr{A}$ is a compact set of X. Hence $I_{\mathscr{C}}(\bigcup \mathscr{A}) \subset O$, and it follows (using formula $(*)$ of Theorem 3) that

$$I_{\mathscr{H}(\mathscr{C})}(\mathscr{A}) \subset \langle I_{\mathscr{C}}(\bigcup \mathscr{A}) \rangle \cap \mathscr{C} \subset \langle O \rangle \cap \mathscr{C}.$$

If $\mathscr{A} \subset \langle O, X \rangle \cap \mathscr{C}$, then $A \cap O \neq \emptyset$ for each $A \in \mathscr{A}$. Choose an open set O_A of X for each $A \in \mathscr{A}$, such that $A \cap O_A \neq \emptyset$ and $O_A \subset \bar{O}_A \subset O$ (notice that X is compact T_2 and hence regular). $\mathscr{A} \subset \mathscr{C}$ being compact, there exist $A_1, \ldots, A_n \in \mathscr{A}$ such that

$$\mathscr{A} \subset \bigcup_{i=1}^{n} \langle O_{A_i}, X \rangle.$$

Hence

$$\bigcup_{i=1}^{n} \bar{O}_{A_i} \in \perp(\mathscr{A}) \quad \text{(notation of Theorem 3)},$$

and consequently,

$$B = I_{\mathscr{C}}\left(\bigcup_{i=1}^{n} \bar{O}_{A_i} \right) \in \perp_{\mathscr{C}}(\mathscr{A}).$$

O being convex, we find that $B \subset O$. Therefore, if $C \in \perp_{\mathscr{C}}\perp_{\mathscr{C}}(\mathscr{A})$, then $C \cap B \neq \emptyset$ and, consequently, $C \cap O \neq \emptyset$. Using formula $(*)$ of Theorem 3 again,

$$I_{\mathscr{H}(\mathscr{C})}(\mathscr{A}) \subset \perp_{\mathscr{C}}\perp_{\mathscr{C}}(\mathscr{A}) \subset \langle O, X \rangle \cap \mathscr{C}.$$

Notice that the converse of Lemma 9 is also true: If $\langle O \rangle \cap \mathscr{C}$ or $\langle O, X \rangle \cap \mathscr{C}$ are $\mathscr{H}(\mathscr{C})$-convex, then $O \subset X$ is convex ($O \subset X$ arbitrary). This follows from the fact that the canonical embedding of X in $\mathscr{C}$ is a cp map (cf. [**15**] Example 1.3(c)). It is not difficult to prove that $\langle O, X \rangle \cap \mathscr{C}$ is weakly convex if and only if O is weakly convex. However: if $\langle O \rangle \cap \mathscr{C}$ is weakly convex and if $O \subset X$ is open, then O is convex, by an argument similar to the one used in the proof of Theorem 5(iii).

4. The Hahn-Banach theorem.

10. Theorem (Biconvex Enlargements). *Let $\mathscr{C}$ be a compact convexity of X and let C, O be (weakly) convex disjoint sets of which C is closed and O is open. Then there is a (weakly) biconvex closed set D of X, such that $C \subset D$ and $D \cap O = \emptyset$.*

Notice that the closed sets C and D are convex in either case (cf. Theorem 5(i)).

Proof. We first deal with the weak convex case. Let

$$\mathscr{D} = \{D| \quad D \in \mathscr{C}, C \subset D, D \cap O = \emptyset\},$$

and let $\mathscr{D}' \subset \mathscr{D}$ be a chain with respect to inclusion. Then $\cup \mathscr{D}' \cap O = \emptyset$, and hence $\overline{\cup \mathscr{D}'} \cap O = \emptyset$. Since $\overline{\cup \mathscr{D}'}$ is the limit of the net $\mathscr{D}' \subset \mathscr{C}$, we find by compactness that $\overline{\cup \mathscr{D}'} \in \mathscr{C}$. Hence $\mathscr{D}$ has a maximal element D by Zorn's Lemma. For each finite set $\{x_1, \ldots, x_n\} \subset X$ we write

$$D(x_1, \ldots, x_n) = \{y| \quad I_{\mathscr{C}}\{x_1, \ldots, x_n, y\} \cap D \neq \emptyset\}.$$

Notice that $D(x_1, \ldots, x_n)$ is the inverse image of the $\mathscr{H}(\mathscr{C})$-convex set $\langle D, X \rangle$ under the cp mapping $k_{\{x_1, \ldots, x_n\}}$ (cf. Theorem 8). Hence, $D(x_1, \ldots, x_n)$ is convex and closed. Obviously, $D \subset D(x_1, \ldots, x_n)$ and $D(x_1, \ldots, x_n) = X$ if $x_i \in D$ for some i.

For each $n \geqq 1$ we prove the following statement by induction on the number p, where $O \leqq p \leqq n$:

$C_n(p)$. *If* $x_1, \ldots, x_n \in X - D$ *are such that* $x_1, \ldots, x_p \notin O$ *and* $x_{p+1}, \ldots, x_n \in O$, *then*

$$D(x_1, \ldots, x_n) = D.$$

Proof of $C_n(O)$. Let $x_1, \ldots, x_n \in O$ and assume that $D(x_1, \ldots, x_n) \neq D$. By the maximality of D there is a point $y \in D(x_1, \ldots, x_n) \cap 0$. Hence,

$$I_{\mathscr{C}}\{x_1, \ldots, x_n, y\} \cap D \neq \emptyset.$$

However, $\{x_1, \ldots, x_n, y\} \subset O$, contradicting that O is weakly convex.

Proof of $C_n(p-1) \Rightarrow C_n(p)$. Let $x_1, \ldots, x_n \in X - D$ where $x_1, \ldots, x_p \notin O$ and $x_{p+1}, \ldots, x_n \in O$. If $D(x_1, \ldots, x_n) \neq D$, then we can again choose a point

$$y \in D(x_1, \ldots, x_n) \cap O$$

by maximality of D. Hence

$$I_{\mathscr{C}}\{x_1, \ldots, x_n, y\} \cap D \neq \emptyset,$$

and it follows that

$$x_p \in D(x_1, \ldots, x_{p-1}, y, x_{p+1}, \ldots, x_n).$$

By $C_n(p-1)$, the latter set equals D, contradicting that $x_p \notin D$.

This completes the inductive proof of $C_n(p)$. If $n \geqq 1$ and if $x_1, \ldots, x_{n+1} \in X - D$, then $I_{\mathscr{C}}\{x_1, \ldots, x_{n+1}\} \subset X - D$, for otherwise

$$x_{n+1} \in D(x_1, \ldots, x_n) = D,$$

a contradiction. Hence, D is weakly biconvex.

In order to obtain the second part of the theorem, we have to proceed

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in a dual way. We briefly sketch the idea: Let

$$\mathscr{P} = \{P| \quad P \text{ is convex open, } O \subset P, C \cap P = \emptyset\}.$$

If $\mathscr{P}' \subset \mathscr{P}$ is a chain, then $\bigcup \mathscr{P}' \cap C = \emptyset$ and $\bigcup \mathscr{P}'$ is obviously convex and open again.

Let $P \in \mathscr{P}$ be a maximal element, and put

$$P(x_1, \ldots, x_n) = \{y|I_{\mathscr{C}}\{x_1, \ldots, x_n, y\} \cap P \neq \emptyset\}.$$

Combining Lemma 9 with Theorems 8 and 7(i), $P(x_1, \ldots, x_n)$ is a convex open set including P. Using the above induction technique, we find that the closed set $X - P$ is weakly convex, and hence convex.

A biconvex set can be regarded as a kind of a half-space, and the above theorem can already be interpreted as some sort of Hahn-Banach theorem. In order to obtain a fully parallel result with vector space theory, we now examine the existence of cp maps which separate convex sets. The range of these cp maps is the unit interval (not the real line), equipped with its linear structure, which corresponds to the (normal binary) convexity

$$\{[t_1, t_2]| \quad 0 \leqq t_1 \leqq t_2 \leqq 1\}.$$

11. THE HAHN-BANACH THEOREM. *Let $\mathscr{C}$ be a compact convexity on X.*

(i) *If C and D are disjoint closed convex sets of X, then there is a* cp *map $f : X \to [0, 1]$ which strictly separates C and D i.e.,*

$$C \subset f^{-1}[0, t) \quad \text{and} \quad D \subset f^{-1}(t, 1] \quad \text{for some } t \in [0, 1].$$

(ii) *If each biconvex closed set of X is a G_δ-set, and if C, O are disjoint convex sets of which C is closed and O is open, then there is a* cp *map $f : X \to [0, 1]$ which separates C and O, i.e.,*

$$C \subset f^{-1}(0) \text{ and } O \subset f^{-1}(0, 1].$$

Notice that (i) guarantees at the same time that there exist convex open sets in X (e.g. the inverse images of open intervals).

Proof. (i) is actually a reformulation of [**17**, Theorem 2.1] which states that if $\mathscr{S}$ is a normal T_1 subbase of X and if $S_0, S_1 \in \mathscr{S}$ are disjoint, then there is a map

$$f : X \to [0, 1]$$

such that $f(S_0) = 0$, $f(S_1) = 1$, and for each $t \in [0, 1]$, $f^{-1}[0, t]$ and $f^{-1}[t, 1]$ are (countable) intersections of members of $\mathscr{S}$. Hence, (i) is already valid for a normal T_1 convexity.

Before proving (ii), we want to point out that the G_δ-condition on biconvex closed sets is necessary: If $D \subset X$ is biconvex closed, then separating D and $X - D$ by a cp map as in (ii) yields that D is G_δ.

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For a proof of (ii) we need two preparatory lemmas:

12. LEMMA. *Let $\mathscr{C}$ be a normal T_1 convexity, and let C be a biconvex closed G_δ-set of the compact space X. Then there is a sequence $(K_n)_{n=1}^{\infty}$ of biconvex closed sets of X such that*

(1) $K_n \subset \operatorname{int} K_{n+1}$;

(2) $\bigcup_{n=0}^{\infty} K_n = X - C.$

Indeed, let $(A_n)_{n=0}^{\infty}$ be a sequence of closed sets of X such that

$$A_0 = \emptyset \quad \text{and} \quad \bigcup_{n=0}^{\infty} A_n = X - C.$$

By induction, we construct a sequence of biconvex closed sets $(K_n)_{n=0}^{\infty}$, satisfying (1), and also

(2′) $A_n \subset K_n \subset X - C$ for each $n \geqq 0$.

Put $K_0 = \emptyset$. If $K_0, \ldots, K_n$ are properly constructed, then $(A_{n+1} \cup K_n) \cap C = \emptyset$, whence by the convexity of $X - C$,

$$I_{\mathscr{C}}(A_{n+1} \cup K_n) \cap C = \emptyset.$$

Using (i) of Theorem 11, there is a $t \in [0, 1]$ and a cp map $f : X \to [0, 1]$ such that

$$I_{\mathscr{C}}(A_{n+1} \cup K_n) \subset \operatorname{int} f^{-1}[0, t]; \quad C \cap f^{-1}[0, t] = \emptyset.$$

Putting $K_{n+1} = f^{-1}[0, t]$ completes the induction, and Lemma 12 easily follows.

13. LEMMA. *Let $\mathscr{C}$ be a normal T_1 convexity on the compact space X and let C be a biconvex closed G_δ-set. For each open set 0 of X including C there is a biconvex closed set H of X such that $C \cap H = \emptyset$ and $0 \cup H = X$.*

Let $(K_n)_{n=0}^{\infty}$ be a sequence of biconvex closed sets as in Lemma 12. Then

$$\{O\} \cup \{\operatorname{int} K_n | \quad n \geqq 0\}$$

is an open cover of X which, by compactness, reduces to a finite subcover. The largest set K_n appearing in this cover yields the desired set H.

We now proceed with a proof of part (ii). By Theorem 10, there is a biconvex closed extension D of C not meeting O. Since D is a G_δ-set, we can apply Lemma 12 to obtain a sequence $K(1/2^n)_{n=0}^{\infty}$ of biconvex closed sets such that

$$K(1) = \emptyset; K(1/2^n) \subset \operatorname{int} K(1/2^{n+1}); X - D = \bigcup_{n=0}^{\infty} K(1/2^n).$$

Using Lemma 13, we can find for each $n \geqq 0$ a biconvex closed set $H(1/2^n)$, such that

$$H(1/2^{n+1}) \cap K(1/2^n) = \emptyset; \quad H(1/2^n) \cup K(1/2^n) = X.$$

Notice that $H(1) = X$ and that $H(1/2^{n+1}) \subset H(1/2^n)$ for each n. We also put $K(O) = X$ and $H(O) = \emptyset$.

Modifying an Urysohn-type procedure from [**17**, 2.1], we now construct for each dyadic rational $r \in [0, 1]$ two convex closed sets $H(r)$, $K(r)$, such that

(1) if $r < s$, then $H(r) \subset H(s)$, $K(r) \supset K(s)$, $H(r) \cap K(s) = \emptyset$;
(2) for each dyadic $t \in [0, 1]$, $H(t) \cup K(t) = X$.

These conditions are already satisfied if r, s and t are among the dyadics $0, 1$ or $1/2^n$. We proceed inductively as follows. Assume that $H(t)$ and $K(t)$ are properly constructed for each dyadic t with denominator $\leqq 2^n$. We then construct $H(t)$ and $K(t)$ simultaneously for all

$$t \in \{3/2^{n+1}, \ldots, (2^{n+1} - 1)/2^{n+1}\} \tag{$*$}$$

(all fractions are assumed to be irreducible). Let $r < s$ be two consecutive fractions with denominator $\leqq 2^n$, and let $t = \frac{1}{2}(r + s)$. If $r = 0$, then $t = 1/2^{n+1}$, and $H(t)$ and $K(t)$ have already been constructed. If $r > 0$, then t is in the set $(*)$. Using (1), $H(r) \cap K(s) = \emptyset$. As $\mathscr{C}$ is a normal subbase, there exist $S_1, S_2 \in \mathscr{C}$ such that

$$H(r) \subset S_1 - S_2; \quad K(s) \subset S_2 - S_1; \quad S_1 \cup S_2 = X.$$

We put $H(t) = S_1$ and $K(t) = S_2$. Using (2), it easily follows that

$$H(r) \subset H(t) \subset X - K(s) \subset H(s);$$
$$K(s) \subset K(t) \subset X - H(r) \subset K(r).$$

Hence, the new sets fit in with our inductive assumption of (1) and (2).

Having produced $H(t)$ and $K(t)$ in this way for all dyadic $t \in [0, 1]$, we define a mapping $f : X \to [0, 1]$ as follows:

$$f(x) = \inf \{r \mid x \in H(r)\}.$$

If $s < f(x)$ is dyadic, then $x \notin H(s)$, whence $x \in K(s)$ by (2). If $f(x) < s$ for a dyadic s, then choose a dyadic r with $f(x) < r < s$. Hence, $x \in H(r)$ and consequently $x \notin K(s)$ by (1). This shows that

$$f(x) = \sup \{s \mid x \in K(s)\}.$$

Using both formulas for $f(x)$, it is easily seen that for each real number $t \in [0, 1]$,

$$f^{-1}[0, t] = \bigcap \{H(r) \mid r > t, r \text{ dyadic}\}$$
$$f^{-1}[t, 1] = \bigcap \{K(s) \mid s < t, s \text{ dyadic}\},$$

showing that f is a cp mapping. Moreover, if $x \in X - D$, then $x \in K(1/2^n)$ for some $n > 0$, and hence $f(x) > 0$. It follows that $C \subset f^{-1}(0)$ and $O \subset f^{-1}(0, 1]$.

We finally present an example, showing that the compactness condition on a convexity cannot be removed from the second part of the Hahn-Banach theorem.

14. *Example.* Let $X = [0, 1]^2$ be the unit square with projections

$$\pi_1, \pi_2 : X \rightarrow [0, 1]$$

onto the first, resp. the second axis. Define

$$\mathscr{C} = \{X\} \cup \{A \in H(X) | \quad \pi_1^{-1}(t) \not\subset A \text{ and } \pi_2^{-1}(t) \not\subset A \text{ for each } t \in [0, 1]\}.$$

Obviously, $\mathscr{C}$ is a T_1 convexity on X, and $\mathscr{C} \subset H(X)$ is not a closed subset. We show that $\mathscr{C}$ is normal as a subbase of X:

If d denotes the usual metric on X, then we write

$$D(x, r) = \{y \in X | \quad d(x, y) \leqq r\} \qquad (r > 0).$$

Let $A, B \in \mathscr{C}$ be disjoint and nonempty. Then $A \cup B \in \mathscr{C} - \{X\}$ as one can easily verify. Let $t \in [0, 1]$, and choose $x(t) \in \pi_1^{-1}(t) - (A \cup B)$. Let $\epsilon(t) > 0$ be such that

$$D(x(t), \epsilon(t)) \cap (A \cup B) = \emptyset, \quad \epsilon(t) < 1/2.$$

Notice that $D(x(t), \epsilon(t)) \cup A \in \mathscr{C}$. Select a finite cover of $[0, 1]$ of type

$$\{[t_i - \epsilon_i, t_i + \epsilon_i] | \quad i = 1, \ldots, n\}, \qquad \epsilon_i = \epsilon(t_i),$$

and let x_i, $i = 1, \ldots, n$ be the corresponding points in $\pi_1^{-1}(t_i)$ respectively. The property

$$D(x_i, \epsilon_i) \cap (A \cup B) = \emptyset$$

remains valid if x_i is moved a little along $\pi_1^{-1}(t_i)$.

Hence we may assume that the second coordinates of the x_i are mutually different. It easily follows that

$$A' = [A \cup \cup_{i=1}^n [t_i - \epsilon_i, t_i + \epsilon_i] \times \{\pi_2(x_i)\}] \cap X$$

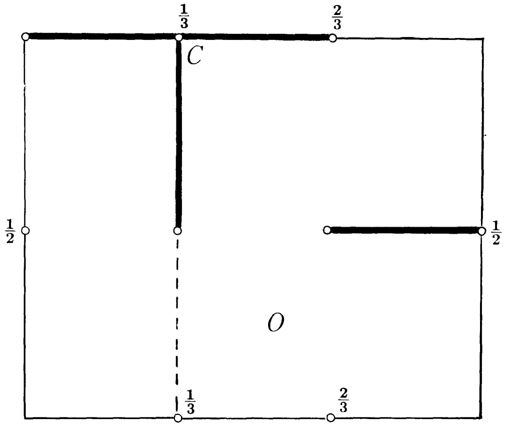
is in $\mathscr{C}$ again. Moreover, $A' \cap B = \emptyset$, and A' meets each vertical fiber of X. Similarly, A' can be extended to a set $A'' \in \mathscr{C}$ such that $A'' \cap B = \emptyset$, and A'' meets each horizontal (and vertical) fiber of X. Applying the same procedure on B, we obtain an extension B'' of B such that $B'' \in \mathscr{C}$, $B'' \cap A'' = \emptyset$, and B'' meets each fiber of X.

X being normal, there exist closed sets $S_1, S_2 \subset X$ such that

$$A'' \subset S_1 - S_2; \quad B'' \subset S_2 - S_1; \quad S_1 \cup S_2 = X.$$

As $S_1 \cap B'' = \emptyset$, no fiber of X can be included in S_1, whence $S_1 \in \mathscr{C}$. Similarly, $S_2 \in \mathscr{C}$, and it follows that $\mathscr{C}$ is a normal convexity.

The resulting non-compact convexity does not satisfy part (ii) of our Hahn-Banach theorem. In the picture below, the fat line indicates a convex closed set C of X. All points outside of C and outside of the dotted line constitute a convex open set O of X:



O is not biconvex, since its complement includes the vertical fiber over $1/3$. No extension of O, which is disjoint with C, is convex, since such an extension must contain a point of the dotted line and hence it contains a full horizontal fiber of X.

15. *Remarks.* (1) The equivalence of (i) and (ii) in Theorem 2 (quoted from [**14**]) has been obtained independently and earlier by R. E. Jamison in his 1974 dissertation (University of Washington, Seattle).

(2) In a forthcoming paper of the author, entitled "Pseudo-boundaries and Pseudo-interiors for Topological Convexities," we have extended convexity theory to non-compact spaces. In comparison with the present results, a considerable sharpening has been obtained—even in the case of compact spaces—of Theorem 10 on biconvex enlargements.

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