J. Igusa
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ON A CERTAIN POISSON FORMULA[1]

JUN-ICHI IGUSA

Introduction. Let G denote a locally compact commutative group with a lattice Γ, G^* its dual, and $\langle g, g^*\rangle = g^*(g)$ for every (g, g^*) in $G \times G^*$; let Γ_* denote the annihilator of Γ in G^* and dg the Haar measure on G such that G/Γ is of measure 1. Finally, let F denote an L^1-function on G and F^* its Fourier transform defined by

$$F^*(g^*) = \int_G F(g)\langle g, g^*\rangle dg \ .$$

Then, under suitable conditions on F, we have

$$\sum_{\gamma \in \Gamma} F(\gamma) = \sum_{\gamma^* \in \Gamma_*} F^*(\gamma^*) \ ,$$

in which both sides are absolutely convergent. This is a classical Poisson formula.

Let X denote a locally compact commutative group, dx a Haar measure on X, and f a continuous mapping of X to the above group G; for every Φ in the Schwartz-Bruhat space $S(X)$ of X, define a function $F^* = F^*_\Phi$ on G^* as

$$F^*_\Phi(g^*) = \int_X \Phi(x)\langle f(x), g^*\rangle dx \ .$$

Then, under suitable conditions on f, the Haar measure dx decomposes into a family of tempered measures $d\mu_g$, where $\mathrm{Supp}\,(d\mu_g)$ is contained in $f^{-1}(g)$ for every g in G, such that the above Poisson formula holds for $F = F_\Phi$ defined by

$$F_\Phi(g) = \int_X \Phi(x) d\mu_g(x) \ .$$

This variant is due to Weil [9]; it is an "abstract form" of the Siegel formula.

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Let k denote a global field, i.e., a number field or a function field of one variable with a finite constant field; let the subscripts A and k denote the adelization relative to k and the taking of k-rational points, respectively; let ψ denote a non-trivial character of k_A/k and identify k_A with its dual by $(i, i^*) \to \psi(ii^*)$. We shall fix a universal domain $\boldsymbol{K}$ containing k and identify $\boldsymbol{K}$ with an affine line over the prime field. We shall change our notation slightly: let X denote an affine space and f a morphism of X to $\boldsymbol{K}$ defined over k; let $|dx|_A$ denote the Haar measure on X_A such that X_A/X_k is of measure 1. Then we can take k_A as $G = G^*, k$ as $\Gamma = \Gamma_*, X_A$ as $X, |dx|_A$ as dx, and f_A as f. During the past several years, we became interested in proving a Poisson formula (of Weil's type) in the above setup. In this paper, we shall consider the special case where f is homogeneous of degree at least 2 and "strongly non-degenerate" in the sense that it is submersive everywhere except at the origin 0 of X. We shall show, in that case, that the Poisson formula holds if char (k) does not divide deg (f) and

$$\dim(X) > 2 \deg(f) .$$

In this formula, everything is explicitly defined; for a complete statement, we refer to § 5, Th. 5. It appears that the simplicity of the above condition is quite remarkable. We have included an additional section on some numerical coefficients of certain asymptotic expansions.

1. A review of some results. We shall keep the notation in the introduction. Let v denote a valuation on the global field k and k_v the corresponding local field; let ψ_v denote the product of the canonical injection $k_v \to k_A$ and the non-trivial character ψ of k_A/k. We recall that X is an affine space defined over k; we introduce coordinates in X with respect to a k-base of X_k. Let $(x_1, \cdots, x_n), (y_1, \cdots, y_n)$ denote coordinates of x, y in X; then $[x, y] = x_1y_1 + \cdots + x_ny_n$ defines a non-degenerate symmetric bilinear form on $X \times X$. Let X_v denote the vector space over k_v of k_v-rational points of X; let $|dx|_v$ denote the autodual (or "self-dual") measure on X_v relative to the bicharacter $(x, y) \to \psi_v([x, y])$ of $X_v \times X_v$. Then the restricted product measure of all $|dx|_v$ becomes the autodual measure on X_A relative to the bicharacter $(x, y) \to \psi([x, y])$ of $X_A \times X_A$; this measure coincides with the Haar measure $|dx|_A$ on X_A such that X_A/X_k is of measure 1.

If we take 1 as a k-base of k, what we have said can be applied to the universal domain $\boldsymbol{K}$ instead of X: we shall denote by $|di|_v$ the autodual measure on k_v relative to the bicharacter $(i, i^*) \to \psi_v(ii^*)$ of $k_v \times k_v$ and by $|di|_A$ the restricted product measure of all $|di|_v$, etc. We shall denote by $|\ \ |_v$ the absolute value on k_v defined by $|d(i_0 i)|_v = |i_0|_v |di|_v$ for every $i_0 \neq 0$ in k_v.

We recall that $f: X \to \boldsymbol{K}$ is a morphism defined over k; it gives rise to a continuous mapping, in fact a k_v-analytic mapping, $f_v: X_v \to k_v$ for every v. If there is no ambiguity, we shall denote f_v also by f. Let X' denote the set of points of X where f is submersive, i.e., where the cotangent vector df does not vanish; then X' is a Zariski open subset of X defined over k. We observe that f is strongly non-degenerate if and only if $X - X' \subset \{0\}$; for a moment, we shall only assume that $X - X' \subset f^{-1}(0)$. We put $U(i) = f^{-1}(i) \cap X'$ for every i in $\boldsymbol{K}$; we have $U(i) = f^{-1}(i)$ if $i \neq 0$. For every i in k_v, let $U(i)_v$ denote the set of k_v-rational points of $U(i)$; then $U(i)_v$ becomes a k_v-analytic manifold, and the union of all $U(i)_v$ coincides with the similarly defined open subset X'_v of X_v. Moreover, there exists a Borel measure $|\theta_i|_v$ on each $U(i)_v$ such that

$$\int_{X_v} \phi(x) |dx|_v = \int_{k_v} \left(\int_{U(i)_v} \phi(x) |\theta_i(x)|_v \right) |di|_v$$

for every continuous function ϕ on X_v with compact support contained in X'_v; the measure $|\theta_i|_v$ admits an explicit analytic expression; cf. [9], pp. 12–13.

We define a function F_Φ on $k_v^\times = k_v - \{0\}$ for every Φ in the Schwartz-Bruhat space $\mathscr{S}(X_v)$ of X_v as

$$F_\Phi(i) = \int_{U(i)_v} \Phi(x) |\theta_i(x)|_v \ .$$

We also define a function F_Φ^* on k_v as

$$F_\Phi^*(i^*) = \int_{X_v} \Phi(x) \psi_v(f_v(x) i^*) |dx|_v \ .$$

Finally, for every quasicharacter ω of $k_v^\times$ which is bounded around 0, we put

$$Z(\omega, \Phi) = \int_{X_v} \omega(f_v(x)) \Phi(x) |dx|_v \ .$$

In AE we developed a coherent theory of the above three types of functions; in the following, we shall recall some of our results:

Suppose first that k_v is an $\boldsymbol{R}$-field, i.e., v is archimedean; then, for every quasicharacter ω of $k_v^\times$ and t in $k_v^\times$, we have

$$\omega(t) = |t|_v^s(|t|^{-1}t)^p ,$$

in which s is in $\boldsymbol{C}$ and p in $\boldsymbol{Z}$; we have $p = 0, 1$ if $k_v = \boldsymbol{R}$. Conversely, for every s in $\boldsymbol{C}$ and p in $\boldsymbol{Z}$, the above prescription defines a quasicharacter of $k_v^\times$. The complex power (or the "local zeta function") $Z(\omega, \Phi)$ has a meromorphic continuation to the whole complex Lie group of quasicharacters of $k_v^\times$ with poles only on the negative real axis of the s-plane. If $-\lambda$ is a pole of $Z(\omega, \Phi)$ and

$$\sum_{i=1}^{m_\lambda} b_{\lambda,i}(p)(s+\lambda)^{-i}$$

the principal part of its Laurent expansion around $-\lambda$, we have the following asymptotic expansion:

$$F_\Phi^*(\gamma^{-1}t) \approx \sum_\lambda \sum_{i=1}^{m_\lambda} a_{\lambda,i}^*(|t|^{-1}t)|t|_v^{-\lambda}(\log|t|_v)^{i-1}$$

as $|t|_v \to \infty$. The constant γ on the left hand side is an element of $k_v^\times$ defined by $\psi_v(t) = \boldsymbol{e}(\gamma t)$, $\boldsymbol{e}(2\,\mathrm{Re}\,(\gamma t))$ for every t in $k_v = \boldsymbol{R}, \boldsymbol{C}$, respectively. The coefficients $a_{\lambda,i}^*(u)$ on the right hand side are determined by $b_{\lambda,i}(p)$ as follows: put $m_v = 2, 2\pi, d = \frac{1}{2}, 1$ for $k_v = \boldsymbol{R}, \boldsymbol{C}$, respectively, and

$$b_p(s) = i^{|p|}(2d\pi)^{d(1-2s)}\Gamma(ds + \tfrac{1}{2}|p|)/\Gamma(d(1-s) + \tfrac{1}{2}|p|) ;$$

then

$$a_{\lambda,i}^*(u) = (1/m_v)\sum_p \Big(\sum_{j=1}^{m_\lambda}((-1)^{i+j}/(i-1)!\,(j-i)!)b_{\lambda,j}(p) \cdot (d^{j-i}b_p(s)/ds^{j-i})_{s=\lambda}\Big)u^p .$$

We refer to AE-II, Th. 2 for the proof (in the case where $\gamma = 1$).

Suppose next that $K = k_v$ is a $\mathfrak{p}$-field, i.e., v is non-archimedean; let R denote the maximal compact subring of K, P its maximal ideal, and $R/P = \boldsymbol{F}_q$; let π denote an element of $P - P^2$ and write an arbitrary element t of $K^\times$ as $\pi^e u$ with e in $\boldsymbol{Z}$ and u in $K_1^\times = R - P$; then, for every quasicharacter ω of $K^\times$, we have

$$\omega(t) = z^e \chi(u)\ ,$$

in which z is in $C^\times$ and χ is a character of $K_1^\times$. Conversely, for every z in $C^\times$ and χ in the dual of $K_1^\times$, the above prescription defines a quasi-character of $K^\times$; and the complex power $Z(\omega, \Phi)$ becomes a rational function of z provided that char $(k) = 0$. If we write

$$Z(\omega, \Phi) \equiv \sum_{\alpha} \sum_{i=1}^{m_\alpha} b_{\alpha,i}(\chi)(1-\alpha^{-1}z)^{-i} \qquad \mathrm{mod}\ C[z, z^{-1}]\ ,$$

we have $|\alpha| > 1$ and

$$F_\Phi^*(\pi^e u) = \Big[(1-q^{-1})^{-1} \sum_{\alpha,\chi} \sum_{i=1}^{m_\alpha} \binom{i-d-e-e_\chi-1}{i-1} b_{\alpha,i}(\chi) g_{\chi^{-1}} \alpha^{e_\chi} \chi(u)$$
$$+ \sum_{\alpha} \sum_{i=1}^{m_\alpha} \Big(\sum_{j=0}^{\infty} \binom{i-d-e+j-1}{i-1} \alpha^{-j}\Big) b_{\alpha,i}(1)\Big] \alpha^{d+e}$$

for all small e, in which d is the largest integer such that $\psi_v = 1$ on P^{-d} and e_χ is the smallest positive integer such that $\chi = 1$ on $1 + P^{e_\chi}$; g_χ for $\chi \neq 1$ is a complex number of absolute value $q^{-\frac{1}{2}e_\chi}$ and $g_1 = -q^{-1}$. Again we refer to AE-II, Th. 2 for the proof (of an equivalent statement).

About the function F_Φ, we have only to know the following: if k_v is an $\boldsymbol{R}$-field, F_Φ is an infinitely differentiable function on $k_v^\times$ such that $F_\Phi(i)$ tends to 0 as $|i|_v \to \infty$ more rapidly than any negative power of $|i|_v$; if k_v is a p-field, F_Φ is a locally constant function on $k_v^\times$ with bounded support in k_v. Moreover, the limit $F_\Phi(0)$ of $F_\Phi(i)$ as $|i|_v \to 0$ exists for every Φ in $\mathscr{S}(X_v)$ if and only if F_Φ^* is an L^1-function on k_v (for every Φ); cf. AE-II, Th. 2.

Finally, we recall that the information about $Z(\omega, \Phi)$ comes from the existence of a "Hironaka resolution" of the singularities of f. If $X^\#$ is the projective space obtained from X by "adding" a hyperplane at infinity, say E, and $\mathscr{I}$ the sheaf of ideals associated with the divisor of zeros of the extension $f^\#$ of f to a function on $X^\#$, the Hironaka resolution of $(E, \mathscr{I})$ is the one defined by his "Main Theorem II (N)" in [3], p. 176. In the case where f is strongly non-degenerate, without any assumption on the characteristic of k, the Hironaka resolution exists and is unique; as a morphism, it is simply the monoidal transformation of $X^\#$ with the origin 0 of X as its center; and it is "tame" if char (k) does not divide deg (f). In particular, the above "provision" can be replaced by this condition. In the following sections, we shall tacitly assume that f is strongly non-degenerate; we put $\dim(X) = n$ and $\deg(f) = m \geqq 2$.

2. Asymptotic formulas. We shall first consider the case where k_v is an $\boldsymbol{R}$-field and (leaving the ambiguity of a numerical constant) determine the "first term" of the asymptotic expansion of $F^*_\Phi(i^*)$ as $|i^*|_v \to \infty$; we recall that Φ is an arbitrary Schwartz function on X_v.

THEOREM 1. *We have*

$$F^*_\Phi(i^*) \approx c_v \Phi(0) |i^*|_v^{-n/m} + \cdots$$

as $|i^*|_v \to \infty$, *in which* $c_v = c_0 + c_1 \operatorname{sgn}(i^*)$ *if* $k_v = \boldsymbol{R}$; c_0, c_1, *and* c_v *for* $k_v = \boldsymbol{C}$ *are independent of* Φ *and* i^*.

Proof. Consider the complex power $Z(\omega, \Phi)$; then it becomes a finite sum of the following four types of integrals:

$$\int_{k_v} \omega(t^m) |t|_v^{n-1} \phi_1(t) |dt|_v , \qquad \int_{k_v^2} \omega(y_1^m y_2) |y_1|_v^{n-1} \phi_2(y) |dy|_v ,$$
$$\int_{k_v} \omega(t) \phi_3(t) |dt|_v , \qquad \int_{X_v} \omega(f_v(x)) \phi_4(x) |dx|_v ,$$

in which all ϕ's are Schwartz functions; of these ϕ_1, ϕ_2, ϕ_3 have compact supports; Supp (ϕ_4) does not contain 0; and $\phi_1(0) = \text{const.}\ \Phi(0), \phi_2(0, t) = \Phi(0)\theta(t)$ with the "const." and the θ both independent of Φ. Therefore, if $\omega(t) = |t|_v^s (|t|^{-1} t)^p$ for every t in $k_v^\times$, then the poles of the meromorphic continuation of $Z(\omega, \Phi)$ are among the following sequences in the s-plane:

$$-n/m - (1/2dm)\text{-times } 0, 1, 2, \cdots,$$
$$-1 - (1/2d)\text{-times } 0, 1, 2, \cdots,$$

in which $d = \frac{1}{2}$ or 1 according as $k_v = \boldsymbol{R}$ or $\boldsymbol{C}$. Moreover, the order of a pole $-\lambda$ is at most 2. For our purpose, we have only to examine the principal parts at those poles which are not smaller than $-n/m$.

In the case where $k_v = \boldsymbol{R}$, the principal parts in question are as follows:

(1) $A_p \Phi(0)(s + n/m)^{-1}$;

(2) $A'_{\lambda,p}(s + \lambda)^{-1}$, where λ is a positive integer at most equal to n/m and $A'_{\lambda,p} \neq 0$ only if $\lambda \not\equiv p \bmod 2$;

(3) $A''_p \Phi(0)(s + n/m)^{-2}$ if n/m is an integer, where $A''_p \neq 0$ implies $n/m \not\equiv p \bmod 2$.

Moreover A_p in (1) and A''_p in (3) are independent of Φ. Each one of these principal parts contributes to the asymptotic expansion of $F^*_\Phi(\gamma^{-1}t)$ as $|t|_v \to \infty$, in which $\psi_v(t) = \boldsymbol{e}(\gamma t)$. The contribution can be determined by the formula in the previous section: from (1) we get

$$(\tfrac{1}{2})\Big(\sum_p A_p b_p(n/m)\,(\operatorname{sgn} t)^p\Big)\Phi(0)|t|_v^{-n/m};$$

from (2) we get no contribution because $A'_{\lambda,p} \neq 0$ implies $\Gamma(\frac{1}{2}(1-\lambda+p)) = \infty$, hence

$$b_p(\lambda) = i^p \pi^{(1/2)(1-2\lambda)}\Gamma(\tfrac{1}{2}(\lambda+p))/\Gamma(\tfrac{1}{2}(1-\lambda+p)) = 0;$$

and, for a similar reason, from (3) we only get

$$-(\tfrac{1}{2})\Big(\sum_p A''_p (db_p(s)/ds)_{s=n/m}\,(\operatorname{sgn} t)^p\Big)\Phi(0)|t|_v^{-n/m}.$$

This implies our theorem for $k_v = \boldsymbol{R}$.

In the case where $k_v = \boldsymbol{C}$, the principal parts in question are as follows:

(1) $A_p\Phi(0)(s+n/m)^{-1}$, where $A_p \neq 0$ implies $p = 0$;

(2) $A'_{\lambda,p}(s+\lambda)^{-1}$, where $\lambda = 1 + \frac{1}{2}|p| + i$ for some non-negative integer i such that λ is at most equal to n/m;

(3) $A''_p\Phi(0)(s+n/m)^{-2}$ if n/m is an integer, and $A''_p \neq 0$ implies $p = 0$.

Moreover A_p in (1) and A''_p in (3) are independent of Φ. Each one of these principal parts contributes to the asymptotic expansion of $F^*_\Phi(\gamma^{-1}t)$ as $|t|_v \to \infty$, in which $\psi_v(t) = \boldsymbol{e}(2\operatorname{Re}(\gamma t))$: from (1) we get

$$(1/2\pi)A_0 b_0(n/m)\Phi(0)|t|_v^{-n/m};$$

from (2) we get no contribution (as in the previous case); and from (3) we only get

$$-(1/2\pi)A''_0(db_0(s)/ds)_{s=n/m}\Phi(0)|t|_v^{-n/m}.$$

This implies our theorem for $k_v = \boldsymbol{C}$. q.e.d.

Remark. The constants c_0, c_1 for $k_v = \boldsymbol{R}$ and c_v for $k_v = \boldsymbol{C}$ have the following properties: c_0, c_v are *real* and c_1 is *pure imaginary*; and $c_1 = 0$ if m is odd. These properties can be proved in two ways. One way is to make the above proof more precise: we observe that the "const."

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in $\phi_1(0) = \text{const.}\,\Phi(0)$ and $\theta(t)$ in $\phi_2(0, t) = \Phi(0)\theta(t)$ are real. This implies that A_p and A''_p are real. Moreover, in the case where $k_v = \boldsymbol{R}$, we get $mp \equiv 0 \bmod 2$ from $A_p \neq 0$ and also from $A''_p \neq 0$. The properties of c_0, c_1, and c_v follow immediately from these. Another way is simply to manipulate the asymptotic formula in the theorem: we observe that the complex conjugation applied to $F^*_\Phi(i^*)$ has the effect of replacing (Φ, i^*) by $(\bar{\Phi}, -i^*)$. If $k_v = \boldsymbol{C}$, the asymptotic formula as $|i^*|_v \to \infty$ of the complex conjugate of $F^*_\Phi(i^*)$ can be obtained in two different ways; and we get $\bar{c}_v = c_v$. If $k_v = \boldsymbol{R}$, we similarly get $\bar{c}_0 + \bar{c}_1 = c_0 - c_1$; this implies that c_0 is real and c_1 pure imaginary. In the integral defining $F^*_\Phi(i^*)$, we replace x by $-x$; then $F^*_\Phi(i^*)$ and $\Phi(0)$ remain unchanged. If m is odd, however, this has the effect of changing the sign of i^*. Therefore, by passing to the asymptotic formula, we get $c_1 = 0$.

We shall consider the case where $K = k_v$ is a p-field and determine $F^*_\Phi(i^*)$ for all large $|i^*|_v$; we recall that Φ is an arbitrary locally constant function with compact support on X_v.

THEOREM 2. *We have*

$$F^*_\Phi(\pi^e u) = \Big[\sum_{\alpha^m = q^n} \Big(\sum_{\chi^m = 1} c_{\alpha,\chi}\chi(u)\Big)\alpha^e\Big]\Phi(0)$$

for all small e in $\boldsymbol{Z}$ and for every u in $K_1^\times$, in which $c_{\alpha,\chi}$ are certain constants independent of Φ and $i^* = \pi^e u$.

Proof. We proceed as in the proof of Th. 1: in the non-archimedean case, we only have the first three types of integrals where ϕ's are locally constant functions with compact support; and $\phi_1(0) = \text{const.}\,\Phi(0)$, $\phi_2(0, t) = \Phi(0)\theta(t)$. Therefore, if $\omega(\pi^e u) = z^e\chi(u)$ for every e in $\boldsymbol{Z}$ and u in $K_1^\times$, we get

$$Z(\omega, \Phi) \equiv \sum_{\alpha^m = q^n} A_{\alpha,\chi}\Phi(0)(1 - \alpha^{-1}z)^{-1} + A'(1 - q^{-1}z)^{-1} + A''\delta_{mn}\Phi(0)(1 - q^{-1}z)^{-2} \quad \bmod \boldsymbol{C}[z, z^{-1}]\,,$$

in which $A_{\alpha,\chi}, A', A''$ are constants and δ_{mn} is the Kronecker delta; $A_{\alpha,\chi}, A''$ are independent of Φ; $A_{\alpha,\chi} = 0$ unless $\chi^m = 1$ and $A' = A'' = 0$ unless $\chi = 1$. By applying the formula we recalled in the previous section, we get

$$F_\Phi^*(\pi^e u) = \Big[\sum_{\alpha^m=q^n} ((1-q^{-1})^{-1} \sum_{\chi^m=1} A_{\alpha,\chi} g_{\chi^{-1}} \alpha^{e_\chi} \chi(u) + (1-\alpha^{-1})^{-1} A_{\alpha,1}) \alpha^{d+e} + (1-q^{-1})^{-2} A'' \delta_{mn} q^{d+e}\Big] \Phi(0)$$

for all small e. This can be rewritten as stated in the theorem. q.e.d.

Th. 1 and Th. 2 have the following implication:

COROLLARY. *We have*

$$|F_\Phi^*(i^*)| \leqq \text{const.} \max(1, |i^*|_v)^{-n/m}$$

for every i^* in k_v, in which the "const." depends on Φ but not on i^*.

We might mention that such an estimate was difficult to obtain in the archimedean case. In fact, even for a special f, known estimates are considerably less precise than ours.

3. Asymptotic formulas ("good case"). We have introduced coordinates in X with respect to a k-base of X_k. In terms of the coordinates $(x_1, \cdots, x_n)$ of $x, f(x)$ becomes a homogeneous polynomial of degree m with coefficients in k. Since f is strongly non-degenerate, there exist a positive integer ρ and n^2 polynomials $A_{ij}(x)$ in $k[x_1, \cdots, x_n]$ satisfying

$$x_i^\rho = \sum_{j=1}^n A_{ij}(x) f_j(x)$$

for $1 \leqq i \leqq n$, in which $f_j(x) = \partial f / \partial x_j$. For a non-archimedean valuation v on k, we have started using the notation $K = k_v, R, P$, etc.; we shall use the notation $R^{(n)}$, instead of R^n, to denote $R \times \cdots \times R$; we similarly define $P^{(n)}$ etc. Also we shall denote by X_v° the $R^{(n)}$ considered as a (compact open) subset of X_v. We choose v so that the coefficients of $f(x)$ and $A_{ij}(x)$ for all i, j are in R and $d = 0$, i.e., $\psi_v = 1$ on R but not on P^{-1}. We have excluded only a finite number of valuations on k and achieved the following: (1) $f(x)$ is in $R[x_1, \cdots, x_n]$; (2) if $\bar{f}(x)$ is the element of $\boldsymbol{F}_q[x_1, \cdots, x_n]$ obtained from $f(x)$ by reducing its coefficients modulo P and $\bar{a}_1, \cdots, \bar{a}_n$ elements of any extension field of $\boldsymbol{F}_q$ not all 0, at least one $\bar{f}_i(x) = \partial \bar{f}(x)/\partial x_i$ does not vanish at $\bar{a} = (\bar{a}_1, \cdots, \bar{a}_n)$; (3) X_v° is of measure 1.

We shall assume that v is such a "good valuation" on k; and, for the sake of completeness, we shall prove the following two elementary lemmas:

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LEMMA 1. *Let t denote an element of R of order r and $N_e(t)$ the number of $a \bmod P^e$, where a is in $R^{(n)}$, satisfying $f(a) \equiv t \bmod P^e$; then, if $e > r$, we have*

$$q^{-(n-1)e}N_e(t) = \Big(\sum_{0 \leqq mi < r} q^{-(n-m)i}\Big) q^{-(n-1)}(N_1(0) - 1) \\ + \delta'_{r0} q^{-(n/m-1)r-(n-1)} N_1(\pi^{-r}t) ,$$

in which $\delta'_{r0} = 1$ or 0 according as $r \equiv 0$ or $r \not\equiv 0 \bmod m$; if $r = \infty$, i.e., *if $t = 0$, we have*

$$q^{-(n-1)e}N_e(0) = q^{e+n[-e/m]} + \Big(\sum_{0 \leqq mi < e} q^{-(n-m)i}\Big) q^{-(n-1)}(N_1(0) - 1) ,$$

in which [ ] is the Gauss symbol.

Proof. Suppose first that $t \neq 0$. If $f(a) \equiv t \bmod P^e$, $a \equiv 0 \bmod P^i$, and $e > r$, then we get $mi \leqq r$; hence

$$N_e(t) = \sum_{0 \leqq mi \leqq r} q^{(m-1)ni} \cdot \text{card} ,$$

where the "card" is the number of $a \bmod P^{e-mi}$ satisfying

$$f(a) \equiv \pi^{-mi}t \bmod P^{e-mi} , \qquad a \not\equiv 0 \bmod P .$$

By the usual lifting process of a solution $\bmod P$ to solutions modulo higher powers of P, we get

$$\text{"card"} = \begin{cases} (N_1(0) - 1) q^{(n-1)(e-mi-1)} \\ N_1(\pi^{-r}t) q^{(n-1)(e-mi-1)} \end{cases}$$

according as $mi < r$ or $mi = r$; the rest is clear.

Suppose next that $t = 0$. If $a \equiv 0 \bmod P^i$ and $mi \geqq e$, we obviously have $f(a) \equiv 0 \bmod P^e$; hence

$$N_e(0) = 1 + \sum_{e \leqq mi < me} (1 - q^{-n}) q^{n(e-i)} + \sum_{0 \leqq mi < e} q^{(m-1)ni} \cdot \text{card} ,$$

where the "card" is the number of $a \bmod P^{e-mi}$ satisfying $f(a) \equiv 0 \bmod P^{e-mi}$, $a \not\equiv 0 \bmod P$. Hence, by the usual lifting process, we get

$$\text{"card"} = (N_1(0) - 1) q^{(n-1)(e-mi-1)} ;$$

the rest is straightforward. q.e.d.

LEMMA 2. *Let Φ denote the characteristic function of X_v°; then for every i^* in $K^\times$ of order $-e \leqq 0$, we have*

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$$F_\Phi^*(i^*) = \begin{cases} q^{n[-e/m]} \\ q^{-n((e-1)/m+1)} \sum_{t \bmod P} \psi_v(\pi^{e-1} i^* t) N_1(t) \end{cases}$$

according as $e \not\equiv 1$ *or* $e \equiv 1 \bmod m$.

Proof. By Lemma 1 and by the orthogonality of characters (of a finite commutative group), we get

$$\begin{aligned} F_\Phi^*(i^*) &= \int_{X_v^\circ} \psi_v(f(x)i^*)\,|dx|_v \\ &= \sum_{t \bmod P^e} \psi_v(ti^*) q^{-ne} N_e(t) \\ &= q^{-e}(q^{e+n[-e/m]} \\ &\quad + \Big(\sum_{e-1 \leqq mi < e} q^{-(n-m)i}\Big) q^{-(n-1)}(N_1(0) - 1) \\ &\quad + \delta'_{e-1,0} q^{-(n/m-1)(e-1)-n+1} \sum{}' \psi_v(ti^*) N_1(\pi^{-(e-1)} t)) , \end{aligned}$$

where the summation $\sum'$ is relative to $t \bmod P^e$ satisfying $\operatorname{ord}(t) = e-1$. The rest is straightforward. q.e.d.

In the notation of Lemma 2, we have

$$(*) \qquad |F_\Phi^*(i^*)| \leqq |i^*|_v^{-n/m}$$

for every i^* in $K - P$ provided that $e = -\operatorname{ord}(i^*) \not\equiv 1 \bmod m$; we shall examine the case where $e \equiv 1 \bmod m$: we change our notation slightly and denote by t an element of $\boldsymbol{F}_q$, by ψ any non-trivial character of $\boldsymbol{F}_q$, and by $N(t)$ the number of solutions of $\bar{f}(x) - t = 0$ in $\boldsymbol{F}_q^n$. Then by Lemma 2 we can rewrite (*) as

$$(**) \qquad \Big| q^{-n} \sum_t \psi(t) N(t) \Big| \leqq q^{-n/m} ,$$

in which the summation is taken over $\boldsymbol{F}_q$; it is equal to the sum of $\psi(\bar{f}(a))$ for a running over $\boldsymbol{F}_q^n$. In the special case where $m = 2$, it is well known (and easy to show) that (**) holds with the equality sign. On the other hand, if $n \geqq 3$, then $\bar{f}(x) - tx_0^m$ is absolutely irreducible for every t in $\boldsymbol{F}_q$ and $\bar{f}(x) - tx_0^m = 0$ defines a non-singular projective hypersurface over $\boldsymbol{F}_q$. Therefore we can apply the "Riemann-Weil hypothesis" proved recently (after the works of Grothendieck and others) by Deligne [1]; see also Dwork [2]:

In the above notation, "there exist complex numbers $\alpha_1, \alpha_2, \cdots$ of absolute value $q^{(1/2)n-1}$ such that

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$$N(0) = q^{n-1} + (-1)^n (q-1) \sum_i \alpha_i \ ;$$

also there exist complex numbers $\alpha_1(t), \alpha_2(t), \cdots$ of absolute value $q^{(1/2)(n-1)}$ depending on $t \neq 0$ such that

$$N(t) = q^{n-1} - (-1)^n \left(\sum_i \alpha_i + \sum_j \alpha_j(t) \right);$$

the number of α_i's and the number of $\alpha_j(t)$'s depend only on m and n."

If we use this result, we get

$$q^{-n} \sum_t \psi(t) N(t) = (-1)^n q^{-n} \left(q \sum_i \alpha_i - \sum_{t \neq 0} \psi(t) \left(\sum_j \alpha_j(t) \right) \right),$$

hence

$$\left| q^{-n} \sum_t \psi(t) N(t) \right| \leqq c \cdot q^{-(1/2)(n-1)}$$

for some constant c depending only on m and n. Since we have $\frac{1}{2}(n-1) - n/m \geqq 1/6$ for $n > m \geqq 3$, the inequality:

$$c \cdot q^{-(1/2)(n-1)} > q^{-n/m}$$

implies $c^6 > q$; the number of such v's is finite. We have thus obtained the following theorem:

THEOREM 3. *Suppose that* $n > m$, *i.e.*, $\dim(X) > \deg(f)$; *then there exists a finite set* S *of valuations on* k *containing the set* S_∞ *of archimedean valuations such that if* v *is not in* S *and* Φ *is the characteristic function of* X_v°, *we have*

$$|F_\Phi^*(i^*)| \leqq \max(1, |i^*|_v)^{-n/m}$$

for every i^* *in* k_v.

We recall that we have the standing hypothesis that f is strongly non-degenerate and $m \geqq 2$.

4. A tempered measure on X_A. We shall start from a local consideration; the following lemma follows from what we have reviewed in §1, from the corollary in §2 (of Th. 1 and Th. 2), and from AE-II, Th. 3:

LEMMA 3. *Suppose that* $n > m$; *then the image measure of* $|\theta_0|_v$ *under the inclusion* $U(0)_v \to X_v$ *is tempered and*

$$F_{\Phi_v}(i) = \int_{X_v} \Phi_v(x)|\theta_i(x)|_v$$

defines a continuous L^1*-function* F_{Φ_v} *on* k_v *for every* Φ_v *in* $\mathscr{S}(X_v)$. *Moreover its Fourier transform* $F^*_{\Phi_v}$ *is also a continuous* L^1*-function on* k_v; *and we have*

$$F_{\Phi_v}(i) = \int_{k_v} F^*_{\Phi_v}(i^*)\psi_v(-ii^*)|di^*|_v$$

for every i *in* k_v.

For every i in k_v, we have recalled the definition of $U(i)_v$ in § 1. Suppose that v is a good valuation. If i is v-integral and different from 0, we define $U(i)_v^\circ$ as the (compact open) subset of $U(i)_v$ consisting of v-integral points; if $i = 0$, we define $U(0)_v^\circ$ as the subset of $U(0)_v$ consisting of v-primitive points, i.e., v-integral points with v-units among their coordinates. Then, in the notation of the previous section, we have

$$\begin{cases} \displaystyle\int_{U(i)_v^\circ} |\theta_i|_v = q^{-(n-1)}N(\bar{i}) \\ \displaystyle\int_{U(0)_v^\circ} |\theta_0|_v = q^{-(n-1)}(N(0) - 1) \end{cases}$$

provided that $\bar{i}$, the residue class of $i \bmod P$, is different from 0. We can apply Deligne's result to the right hand sides; and we get

$$\left|\int_{U(i)_v^\circ} |\theta_i|_v - 1\right| \leqq c\cdot q^{-(1/2)(n-1)},\ c\cdot q^{-(1/2)n+1}$$

according to the cases, in which c is a constant independent of v; the upper bound $c\cdot q^{-(1/2)n+1}$ can obviously be used in both cases.

We shall change our notation slightly and denote by i an element of k; then the above estimates hold for almost all v. Consider the following infinite product:

$$\prod_v{}' \int_{U(i)_v^\circ} |\theta_i|_v$$

extended over the set of good valuations or over its subset defined by $|i|_v = 1$ according as $i = 0$ or $i \neq 0$. Then, if $\frac{1}{2}n - 1 > 1$, i.e., if $n > 4$, it is convergent (in the usual sense if we exclude a finite number of factors which may be 0). Therefore the restricted product measure

$|\theta_i|_A$ of all $|\theta_i|_v$ is defined on the adelization $U(i)_A$ of $U(i)$ provided that $n > 4$; this provision can be replaced by $n > 3$ for $i \neq 0$. Moreover the image measure of $|\theta_i|_A$ under $U(i)_A \to X_A$ is tempered for every $i \neq 0$; by Lemma 3 it is also tempered for $i = 0$ if $n > m$. Therefore, if $n > \max(m, 4)$, then

$$F_\Phi(i) = \int_{U(i)_A} \Phi(x)|\theta_i(x)|_A$$

is defined for every Φ in $\mathscr{S}(X_A)$ and i in k.

THEOREM 4. *Suppose that* $n > \max(m+1, 4)$ *and let* C *denote a compact subset of* $\mathscr{S}(X_A)$; *then the series*

$$\sum_{i \in k} F_\Phi(i)$$

for Φ *varying in* C *has a "dominant series."*

Proof. There exists a finite set S of valuations on k containing S_∞ and an element $\phi_v \geqq 0$ of $\mathscr{S}(X_v)$ for each v, which is equal to the characteristic function of X_v° for every v not in S, such that

$$|\Phi(x)| \leqq \phi(x) = \left(\prod_v \phi_v\right)(x)$$

for every Φ in C and x in X_A. For the proof, we refer to [8], Lemma 5; see also [5], Lemma 7. We may assume that every v not in S is a good valuation. We have only to show that the series of $F_\phi(i)$ for i running over $k^\times$ is convergent.

First of all, we have

$$F_\phi(i) = \prod_v F_{\phi_v}(i)$$

for every i. We observe that if v is not in S_∞, the image of $\operatorname{Supp}(\phi_v)$ under f_v is compact. Therefore the set of all i for which $F_{\phi_v}(i) \neq 0$ is bounded in k_v. In particular, if v is not in S, then $F_{\phi_v}(i) \neq 0$ implies $|i|_v \leqq 1$. Since $F_{\phi_v}(i) = 0$ implies $F_\phi(i) = 0$, we may restrict i in $k^\times$ by the condition that $|i|_v \leqq 1$ for every v not in S and $|i|_v \leqq$ const. for every v in $S - S_\infty$. Then, in the case where k is a function field, we just get a finite set; hence there is no problem. Therefore we may assume that k is a number field. In that case, i is contained in a fractional ideal of k; and its image in

$$k_\infty = k \underset{Q}{\otimes} \boldsymbol{R} = \prod_{v \in S_\infty} k_v$$

is a lattice in this vector space.

If v is not in S and $0 < |i|_v \leqq 1$, by Lemma 1 we have

$$F_{\phi_v}(i) = q^{-(n-1)e} N_e(i)$$

for any $e > \mathrm{ord}_v(i)$. Therefore $F_{\phi_v}(i)$ is equal to $q^{-(n-1)} N_1(i)$ if $\mathrm{ord}_v(i) = 0$; it has

$$(1 - q^{-(n-m)})^{-1} q^{-(n-1)} N_1(0) + q^{-(n-m)-(n-1)} \max_{|t|_v = 1} N_1(t)$$

as an upper bound if $\mathrm{ord}_v(i) > 0$ provided that $n > m$. We evaluate these further by Deligne's result and we get

$$F_{\phi_v}(i) \leqq (1 - 2q^{-2})^{-1}(1 + cq^{-(1/2)n+1})$$

for some constant c independent of v and i provided that $n > m + 1$. Therefore, if $n > \max(m + 1, 4)$, we get

$$\prod_{v \notin S} F_{\phi_v}(i) \leqq c_1$$

for some constant c_1 independent of i.

On the other hand, if $n > m$, then F_{ϕ_v} is a continuous function on k_v for every v; it has a compact support for v not in S_∞; and $F_{\phi_v}(i_v)$ tends to 0 as $|i_v|_v \to \infty$ more rapidly than any negative power of $|i_v|_v$ for v in S_∞. This follows from Lemma 3 and from what we have reviewed in § 1. Therefore we get

$$\prod_{v \in S} F_{\phi_v}(i) \leqq c_2 \prod_{v \in S_\infty} \max(1, |i|_v)^{-2}$$

for some constant c_2 independent of i. If we denote the summation over the above-mentioned lattice in k_∞ by $\sum'$, by putting these together, we get

$$\sum_{i \in k^\times} F_\phi(i) \leqq c_1 c_2 \sum_i{}' \prod_{v \in S_\infty} \max(1, |i|_v)^{-2};$$

and the right hand side is convergent, say, by [5], Lemma 12. q.e.d.

We recall that each $F_\phi(i)$ is a tempered measure on X_A. Therefore Th. 4 shows that the series of $F_\phi(i)$ for i running over k also defines a tempered measure on X_A. We shall add the following remark:

Remark. Suppose that k is a number field and denote by k^o its principal order, by Q an integral ideal of k, and by $|Q|$ its absolute norm; assume that $f(x)$ is in $k^o[x_1, \cdots, x_n]$ and, for every $i \neq 0$ in k^o, let $N_Q(i)$ denote the number of $a \bmod Q$, where a is in $(k^o)^{(n)}$, satisfying $f(a) \equiv i \bmod Q$. Then the limit (if it exists) of $|Q|^{-(n-1)}N_Q(i)$ as Q becomes divisible by any given integral ideal of k is called the "singular series" associated with $f(x)$ and i; cf. [7]. This is related to $F_\Phi(i)$ as follows:

We decompose X_A into $X_0 \times X_\infty$, where X_0 is the restricted product of X_v for all v not in S_∞ and X_∞ the product of X_v for v in S_∞; similarly, we decompose $U(i)_A$ into $U(i)_0 \times U(i)_\infty$ and $|\theta_i|_A$ into $|\theta_i|_0 \otimes |\theta_i|_\infty$. Let Φ_0 denote the characteristic function of the product of X_v° for all v not in S_∞, Φ_∞ an arbitrary Schwartz function on X_∞, and $\Phi = \Phi_0 \otimes \Phi_\infty$. Then we have

$$F_\Phi(i) = \left(\lim_Q |Q|^{-(n-1)}N_Q(i)\right) \cdot \int_{X_\infty} \Phi_\infty(x)|\theta_i(x)|_\infty$$

provided that $n \geqq 4$.

5. Poisson formula. We shall turn our attention to another type of functions; we shall first prove the following general lemma:

LEMMA 4. *Let r denote a non-negative integer, ε a positive real number, and σ_v for each valuation v on k a real number; suppose that $\sigma_v > r$ for all v and $\sigma_v \geqq r + 1 + \varepsilon$ for almost all v. Then the series*

$$\sum_i \prod_v \max(1, |i_1|_v, \cdots, |i_r|_v)^{-\sigma_v},$$

in which $i = (i_1, \cdots, i_r)$ runs over k^r, is convergent.

Proof. In the case where k is a number field, this lemma was proved in [5] as Prop. 1. Suppose, therefore, that k is a function field of genus g; let F_{q_0} denote the algebraic closure in k of the prime field. Since k contains a prime divisor of arbitrarily large degree, we can choose one, say p_∞, satisfying $\deg(p_\infty) \geqq 2g$; let $|\ \ |_\infty$ denote the usual absolute value at p_∞. For any non-negative integer e, let $L(ep_\infty)$ denote the vector space over F_{q_0} of elements of k with poles only at p_∞ and with orders at most e; then $L(p_\infty)$ contains an element t not in F_{q_0}.

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Let k° denote the integral closure of $F_{q_0}[t]$ in k and λ, α real numbers satisfying $\lambda \geqq 1, \alpha > 1$; then we have

$$\sum_{i \in k^\circ} \max(\lambda, |i|_\infty)^{-\alpha} \leqq c \cdot \lambda^{1-\alpha}$$

for some constant c independent of λ. This is a counterpart of [5], Lemma 12; and it can be proved as follows:

We observe that k° is the union of $L(ep_\infty)$ for $e = 0, 1, 2, \cdots$. Moreover, if q denotes the power of q_0 with $\deg(p_\infty)$ as its exponent, we have $|t|_\infty = q$. Therefore the left hand side is equal to

$$q_0 \lambda^{-\alpha} + \sum_{e=1}^{\infty} \operatorname{card}(L(ep_\infty) - L((e-1)p_\infty)) \max(\lambda, q^e)^{-\alpha},$$

in which $L(0) = F_{q_0}$. And by the Riemann-Roch theorem we get

$$\operatorname{card}(L(ep_\infty)) = (q_0)^{1-g} q^e$$

for $e = 1, 2, \cdots$. If we split the above summation into two parts by $q^e \leqq \lambda$ and $q^e > \lambda$, we can convince ourselves that both parts are of order $\lambda^{1-\alpha}$.

Once we have that, the rest can be proved in the same way as "Prop. 1." In fact, since the group of units of k° is $(F_{q_0})^\times$, the proof is simpler. q.e.d.

Let Φ denote an arbitrary element of $\mathscr{S}(X_A)$ and C a compact subset of k_A; then, if $n > m$, there exists a finite set S of valuations on k with the following properties: Th. 3 holds for S and every v not in S is a good valuation; Φ decomposes into the product of the characteristic function of X_v° for all v not in S and a Schwartz-Bruhat function Φ_S of the product X_S of X_v for v in S; for every v not in S, the image of C under the canonical projection $k_A \to k_v$ is contained in the unit disc R.

Let $|dx|_S$ denote the product measure of $|dx|_v$ for all v in S; then, by using the corollary in §1 (of Th. 1 and Th. 2) and Th. 3, we get

$$\begin{aligned} |F_\Phi^*(i^* + i)| &\leqq \prod_{v \notin S} \max(1, |i|_v)^{-n/m} \\ &\quad \cdot \left| \int_{X_S} \Phi_S(x) \prod_{v \in S} \psi_v(f(x)(i_v^* + i)) |dx|_S \right| \\ &\leqq c \cdot \prod_v \max(1, |i|_v)^{-n/m} \end{aligned}$$

for every $i^* = (i_v^*)_v$ in C and i in k, in which the constant c is independent of i^* and i. Actually we can show that such a constant exists even if we let Φ vary in a compact subset of $\mathscr{S}(X_A)$. At any rate, by taking $r = 1$ and $\sigma_v = n/m$ in Lemma 4, we see that the following series:

$$\Theta(i^*) = \sum_{i \in k} F_\Phi^*(i^* + i)$$

for i^* varying in C has a dominant series if $n > 2m$. Since k_A/k is compact, we may assume that $k_A = k + C$. In this way, we see that the above series defines a continuous function on k_A/k and that F_Φ^* is a continuous L^1-function on k_A. Let

$$\Theta(i^*) = \sum_{i \in k} a(i)\psi(ii^*)$$

denote the Fourier expansion of $\Theta(i^*)$; then we get

$$a(i) = \int_{k_A} F_\Phi^*(i^*)\psi(-ii^*)\,|di^*|_A\ .$$

By applying Lemma 3, we see that the right hand side coincides with $F_\Phi(i)$ for every i; hence

$$\sum_{i \in k} F_\Phi(i)\psi(ii^*) = \sum_{i \in k} F_\Phi^*(i^* + i)$$

for every i^* in k_A provided that the left hand side is absolutely convergent. Since $n > 2m$ implies $n > \max(m+1, 4)$, the absolute convergence is guaranteed by Th. 4. And, by putting $i^* = 0$, we get the following theorem:

THEOREM 5. *Suppose that $n > 2m$, i.e., $\dim(X) > 2\deg(f)$; then for every Φ in $\mathscr{S}(X_A)$ we have*

$$\sum_{i \in k} \int_{U(i)_A} \Phi(x)\,|\theta_i(x)|_A = \sum_{i^* \in k} \int_{X_A} \Phi(x)\psi(f(x)i^*)\,|dx|_A\ ,$$

in which both sides are absolutely convergent.

We observe that the integrand $\Phi_{i^*}(x) = \Phi(x)\psi(f(x)i^*)$ on the right hand side is in $\mathscr{S}(X_A)$ and that the above tempered measure on X_A takes the same value at $\Phi(x)$ and $\Phi_{i^*}(x)$. The correspondence $\Phi \to \Phi_{i^*}$ uniquely extends to a unitary operator on $L^2(X_A)$. This invariance property is preserved even if we introduce a tempered measure $E(\Phi)$ as

$$E(\Phi) = \sum_{i^* \in k} \int_{X_A} \Phi(x)\psi(f(x)i^*)|dx|_A + \Phi(0) .$$

In the special case where $m = 2$, if we put $f(x, y) = f(x + y) - f(x) - f(y)$, the new measure $E(\Phi)$ is also invariant under the following twisted Fourier transformation:

$$\hat{\Phi}(x) = \int_{X_A} \Phi(y)\psi(f(x, y))|dy|_A ;$$

cf. [9], p. 64. In this way, we see that $E(\Phi)$ is invariant under a group of unitary operators which is isomorphic to $SL_2(k)$. It is an interesting problem to examine whether this classical result has a generalization of some kind to the case where $m \geqq 3$.

6. The constants $i_v(\psi_v \circ f)$**.** We shall add some remarks on the numerical constants which appeared in the asymptotic formulas for $F_\Phi^*(i^*)$ as $|i^*|_v \to \infty$: consider $\psi_v(f(x))$ as a tempered distribution $T(x) = T_v(x)$ on X_v as

$$T[\Phi] = \int_{X_v} \Phi(x)\psi_v(f(x))|dx|_v$$

for every Φ in $\mathscr{S}(X_v)$; let T^* denote its Fourier transform defined by $T^*[\Phi] = T[\Phi^*]$, where

$$\Phi^*(x) = \int_{X_v} \Phi(y)\psi_v([x, y])|dy|_v .$$

Then T^* is an analytic function on X_v for $k_v = \boldsymbol{R}, \boldsymbol{C}$. We have learned this fact from L. Ehrenpreis; it can be proved as follows:

Suppose first that $k_v = \boldsymbol{R}$; for our purpose, we may assume that $\psi_v(t) = \boldsymbol{e}(t)$ for every t in $\boldsymbol{R}$. If $P(\xi)$ is a polynomial in n letters $\xi_1, \cdots, \xi_n$ with complex coefficients, we shall denote by $P(\partial/\partial x)$ the differential operator obtained from $P(\xi)$ by replacing each ξ_i by $\partial/\partial x_i$; as before, we put $f_i(x) = \partial f/\partial x_i$ for $1 \leqq i \leqq n$. We observe that the distribution T^* on $X_v = \boldsymbol{R}^n$ satisfies

$$f_i(\partial/\partial x)T^* + (2\pi(-1)^{1/2})^{m-1}x_i T^* = 0$$

for $1 \leqq i \leqq n$, in which $m = \deg(f) \geqq 2$. Therefore the analyticity of T^* follows from the following theorem:

"Let $P_1(\xi), P_2(\xi), \cdots$ denote a finite number of homogeneous polynomials of the same positive degree in $\xi_1, \cdots, \xi_n$ with real coefficients such that they do not vanish simultaneously at any point of $\boldsymbol{R}^n - \{0\}$; let $D_1, D_2, \cdots$ denote linear partial differential operators of the form

$$D_i = P_i(\partial/\partial x) + \text{lower terms},$$

in which the coefficients of the lower terms are analytic on some open subset U of $\boldsymbol{R}^n$. Then any distribution S on U satisfying $D_i S = 0$ for every i is analytic on U." (This theorem itself follows from the standard theorem in, e.g., [4], p. 178 by observing that

$$D = \sum_i P_i(\partial/\partial x) D_i$$

is "elliptic" and $DS = 0$.)

Suppose next that $k_v = \boldsymbol{C}$; let $(z_1, \cdots, z_n)$, instead of $(x_1, \cdots, x_n)$, denote coordinates on X_v; put $x_i = \operatorname{Re}(z_i)$, $y_i = \operatorname{Im}(z_i)$, and $F(x, y) = 2 \operatorname{Re}(f_v(z))$; then $F(x, y)$ has the same property as the $f_v(x)$ for $k_v = \boldsymbol{R}$: it is a homogeneous polynomial of degree m in $x_1, \cdots, x_n, y_1, \cdots, y_n$ with real coefficients such that its $2n$ partial derivatives do not vanish simultaneously at any point of $\boldsymbol{R}^{2n} - \{0\}$. Therefore T^* is analytic on (the underlying real vector space of) $X_v = \boldsymbol{C}^n$.

THEOREM 6. *In the notation of Th.* 1, *we have* $T_v^*(0) = c_0 + c_1$ *or* c_v *according as* $k_v = \boldsymbol{R}$ *or* $\boldsymbol{C}$.

Proof. We have

$$c_v \Phi(0) = \lim_{t \to \infty} |t|_v^n F_\Phi^*(t^m)$$

for every Φ in $\mathscr{S}(X_v)$. We choose an element of $\mathscr{S}(X_v)$ with compact support; call it ϕ and take its Fourier transform ϕ^* as Φ; then we get

$$\begin{aligned} c_v \phi^*(0) &= \lim_{t \to \infty} \int_{X_v} \phi(x) T^*(t^{-1}x) |dx|_v \\ &= T^*(0) \int_{X_v} \phi(x) |dx|_v . \end{aligned}$$

We have used the fact that $T^*(t^{-1}x)$ tends uniformly to $T^*(0)$ on the compact subset $\operatorname{Supp}(\phi)$ of X_v. Since ϕ is an arbitrary element of $\mathscr{S}(X_v)$ with compact support, we get $c_v = T^*(0)$. q.e.d.

We shall consider the case where $K = k_v$ is a p-field and prove the following counterpart of Th. 6:

THEOREM 7. *The constant $T_v^*(0)$ is meaningful; and, in the notation of Th. 2, we have*

$$T_v^*(0) = \sum_{\alpha^m = q^n} \Big(\sum_{\chi^m = 1} c_{\alpha,\chi} \chi(1) \Big) .$$

Proof. As a tempered distribution on X_v, T^* is a finitely additive function on the family of all compact open subsets of X_v. Let e denote a non-negative integer and ϕ_e the characteristic function of $(P^e)^{(n)}$; then the integral of ϕ_e over X_v is $q^{-((1/2)d+e)n}$ and, if $\Phi = \phi_0$, we have

$$\lim_{e\to\infty} q^{((1/2)d+e)n} T^*[\phi_e] = \lim_{e\to\infty} q^{(d+e)n} F_\Phi^*(\pi^{-(d+e)m}) .$$

According to Th. 2, the expression on the right hand side (under the limit sign) is equal to

$$\sum_{\alpha^m = q^n} \Big(\sum_{\chi^m = 1} c_{\alpha,\chi} \chi(1) \Big)$$

for all large e. We observe that this finite sum does not depend on the choice of the coordinates by which the sequence $\phi_0, \phi_1, \cdots$ is defined. Therefore we may call the above limit the "derivative" of T^* at 0 and denote it by $T^*(0)$. q.e.d.

We shall change our notation and denote $T_v^*(0)$ by $i_v(\psi_v \circ f)$ for every v. Suppose that v is a good valuation and $a_v \neq 0$ an element of $K = k_v$ of order e_0; then we get

$$i_v(\psi_v \circ a_v f) = \begin{cases} q^{n[e_0/m]} \\ q^{n((e_0+1)/m-1)} \displaystyle\sum_{t \bmod P} \psi_v(\pi^{-e_0-1} a t) N_1(t) \end{cases}$$

according as $e_0 \not\equiv -1$ or $e_0 \equiv -1 \bmod m$. This can be derived from Lemma 2. In particular, by taking $a_v = 1$, we get $i_v(\psi_v \circ f) = 1$. Therefore we can define $i(\psi \circ f)$ as

$$i(\psi \circ f) = \prod_v i_v(\psi_v \circ f) ;$$

the product on the right hand side is really a finite product. In the case where $m = 2$, $i_v(\psi_v \circ f)$ is equal to Weil's $\gamma_v(\psi_v \circ f)$ except for the square root of the absolute value in k_v of the discriminant of f; cf. [8], p. 161. We recall that the product of $\gamma_v(\psi_v \circ f)$ is 1; cf. op. cit., p. 179. By an elementary product formula, the product of the absolute value in k_v of the discriminant of f is 1; hence we get $i(\psi\circ f)=1$. We shall give an example indicating that a theorem concerning $i(\psi\circ f)$ (even if it exists) can not have such a simple form for $m\geqq 3$.

EXAMPLE. We take $k=\boldsymbol{Q}$, $\psi_\infty(t)=\boldsymbol{e}(-t)$ for every t in $\boldsymbol{Q}_\infty=\boldsymbol{R}$, $\psi_p(t)=\boldsymbol{e}(\langle t\rangle)$ for every t in $\boldsymbol{Q}_p$, where $\langle t\rangle$ denotes the "fractional part" of t. Then, for any a_∞ in $\boldsymbol{R}^\times$, we get

$$i_\infty(\psi_\infty(a_\infty x^m))=(2/m)\Gamma(1/m)(2\pi|a_\infty|)^{-1/m}-\text{times}$$
$$\boldsymbol{e}(-\operatorname{sgn}(a_\infty)/4m)\ \text{or}\ \cos(\pi/2m)$$

according as m is even or odd. Let χ denote a character of the group of units of $\boldsymbol{Z}_p$ and put

$$g_\chi=p^{-e_\chi}\Big(\sum_{u \bmod p^{e_\chi}}\chi(u)\boldsymbol{e}(p^{-e_\chi}u)\Big),$$

in which $u\not\equiv 0 \bmod p$ and e_χ is (as in § 1) the smallest positive integer such that $\chi=1$ on $1+p^{e_\chi}\boldsymbol{Z}_p$. Then, for any $a_p=p^{e_0}u_0$ in $\boldsymbol{Q}_p^\times$ of order e_0 satisfying $0\leqq e_0<m$, we get

$$i_p(\psi_p(a_px^m))=1+(1/m)\sum_{\alpha^m=p}\Big(\sum_{\chi^m=1}g_{\chi^{-1}}\alpha^{e_0+e_\chi}\chi(u_0)\Big).$$

If we take $m=2$, as a special case of what we have recalled, we get $i(\psi(ax^2))=1$ for every a in $\boldsymbol{Q}^\times$. In particular, we have

$$i(\psi(x^2))=(1+i)\cdot\tfrac{1}{2}(1-i)=1.$$

However, if we take $m=4$ and $a=1$, we get

$$i(\psi(x^4))=(1+\zeta)\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{4})(2\pi)^{-(1/4)}\zeta^{-1},$$

in which $\zeta=\boldsymbol{e}(1/16)$; this is not even a real number. Finally, if we take $m=3$ and $a=9$, we get $i_3(\psi_3(9x^3))=0$, hence $i(\psi(9x^3))=0$. (We can show that the situation does not improve even if we extend $\boldsymbol{Q}$ by adjoining m-th roots of unity.)

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The Johns Hopkins University