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SOME TRIGONOMETRIC EXTREMAL PROBLEMS AND DUALITY

SZILÁRD GY. RÉVÉSZ

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Abstract

In this paper we present a minimax theorem of infinite dimension. The result contains several earlier duality results for various trigonometrical extremal problems including a problem of Fejér. Also the present duality theorem plays a crucial role in the determination of the exact number of zeros of certain Beurling zeta functions, and hence leads to a considerable generalization of the classical Beurling's Prime Number Theorem. The proof used functional analysis.

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1. Introduction

In [6] Ruzsa proved a duality property between certain extremal quantities. For a different kind of extremal problem a similar duality phenomenon was described in [3]. Later it turned out that the two types of extremal problems are in fact special cases of a class of extremal problems [5]. This class can be parametrized by a continuous variable r, where $0 \le r \le 1$, and the extremal problems in [3] and [6] belong to the special cases $r = 0$ and $r = 1$, respectively. So it was natural to look for a more general formulation of duality to cover the general class of extremal problems as well.

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In this paper we present a minimax theorem of several dimensions, which is general enough to cover both the above-mentioned extremal problems and the extremal problem of [2]. Actually, we need to introduce the problem in a several dimensional setting for the sake of [2], where in a certain analytic number theoretical problem this general duality plays a crucial role.

Most of the work presented here is contained in the author's thesis [4]. The author would like to express his gratitude to Professor I. Z. Ruzsa for calling his attention to the paper [6] and for giving useful comments in the course of his work.

2. The theorem

Let $d \in \mathbb{N}$ and denote $T^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. We define

(1) $\mathbb{Z}_+^d := \{\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{Z}^d : \exists j < d,\ z_1 = \cdots = z_j = 0,\ z_{j+1} > 0\}.$

For any $M, L \subset \mathbb{Z}_+^d$ we introduce

(2) $$\mathscr{F}(M, L) := \left\{ f \in \mathscr{T} : f \geq 0,\ f(x) = 1 + \sum_{\mathbf{k} \in \mathbf{Z}_+^d} a(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{x}), \right.$$
$$\left. a(\mathbf{k}) \leq 0\ (\mathbf{k} \notin M),\ a(\mathbf{k}) \geq 0\ (\mathbf{k} \notin L) \right\},$$

where $\mathscr{T}$ denotes the set of trigonometric polynomials of d variables, and

$$\mathscr{M}(M, L) := \left\{ \tau \in BM(T^d):\ d\tau(\mathbf{x}) \sim 2 \sum_{\mathbf{k} \in \mathbf{Z}_+^d} t(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{x}), \right.$$
$$\left. t(\mathbf{k}) \leq 0\ (\mathbf{k} \notin M),\ t(\mathbf{k}) \geq 0\ (\mathbf{k} \notin L) \right\},$$

where $BM(T^d)$ stands for the regular Borel measures of T^d.

We consider a fixed $\rho \in BM(T^d)$ with Fourier-Lebesgue series

(4) $$d\rho(\mathbf{x}) \sim 1 + 2 \sum_{\mathbf{k} \in \mathbf{Z}_+^d} r(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{x}).$$

Our goal is to find the extremal quantity

(5) $$\alpha_\rho(M, L) := \inf\{\langle f, \rho \rangle : f \in \mathscr{F}(M, L)\}.$$

where the scalar product of f and ρ is

(6) $$\langle f, \rho\rangle = \frac{1}{(2\pi)^d}\int_{T^d} f(\mathbf{x})\,d\rho(\mathbf{x}) = 1 + \sum_{\mathbf{k}\in\mathbb{Z}_+^d} a(\mathbf{k})r(\mathbf{k}).$$

Observe that, with the notation

(7) $$\overline{M} := \mathbb{Z}_+^d\backslash M, \quad \overline{L} := \mathbb{Z}_+^d\backslash L,$$

we get

(8) $$\langle f, \tau\rangle \le 0 \quad \text{for all } f\in\mathscr{F}(M, L) \text{ and } \tau\in\mathscr{M}(\overline{M}, \overline{L}).$$

Hence if $\sigma\in BM(T^d)$ satisfies

(9) $$\sigma - \rho =: \tau \in \mathscr{M}(\overline{M}, \overline{L}),$$

then

$$\langle f, \sigma\rangle \le \langle f, \rho\rangle, \qquad f\in\mathscr{F}(M, L),$$

and hence taking infinum over $f\in\mathscr{F}(M, L)$ we obtain

(10) $$\alpha_\sigma(M, L) \le \alpha_\rho(M, L).$$

On the other hand suppose that σ satisfies, for some $t\in\mathbb{R}$, the inequality

(11) $$\sigma \ge t\lambda \quad (d\lambda(\mathbf{x}) = dx_1\,dx_2\cdots dx_d).$$

Introducing the extremal quantity

(12) $$\omega_\rho(H, K) := \sup\{t : \exists\tau\in\mathscr{M}(H, K),\ \sigma = \rho + \tau \ge t\lambda\},$$

we see from (9), (10), (11) and $\langle f, \sigma\rangle \ge \langle f, t\lambda\rangle = t$, that

(13) $$\alpha_\rho(M, L) \ge \omega_\rho(\overline{M}, \overline{L}).$$

Given this observation our aim is to prove the sharpness of (13).

THEOREM. *Let M, $L\subset\mathbb{Z}_+^d$ and $\rho\in BM(T^d)$ be arbitrary. We have*

$$\alpha_\rho(M, L) = \omega_\rho(\overline{M}, \overline{L}).$$

3. Proof of the duality theorem

Since ρ, M, L and $\overline{M}$, $\overline{L}$ are fixed once and for all, we use α, $\mathscr{F}$ and ω, $\mathscr{M}$ without writing out ρ, M, L and ρ, $\overline{M}$, $\overline{L}$ respectively. In view of (13) it is enough to prove $\omega\ge\alpha$.

We can suppose $\alpha > -\infty$ and as $f\equiv 1\in\mathscr{F}$, we have $\alpha\le 1$. For any $\mathbf{m}\in M$ and $\mathbf{l}\in L$, $1+\cos(\mathbf{m}\cdot\mathbf{x})\in\mathscr{F}$ and $1-\cos(\mathbf{l}\cdot\mathbf{x})\in\mathscr{F}$. Now if $\alpha = 1$

then $1 = \alpha \le \langle 1 + \cos(\mathbf{m}\cdot\mathbf{x}), \rho\rangle = 1 + r(\mathbf{m})$ and $1 = \alpha \le \langle 1 - \cos(\mathbf{l}\cdot\mathbf{x}), \rho\rangle = 1 - r(\mathbf{l})$, hence $r(\mathbf{m}) \ge 0$ $(\mathbf{m} \in M)$, $r(\mathbf{l}) \le 0$ $(\mathbf{l} \in L)$ and so $\tau := \lambda - \rho \in \mathscr{M}$, and so $\sigma = \rho + \tau = \lambda$ can occur in (12) and $\omega = 1$.

Therefore, we can suppose $-\infty < \alpha < 1$. Moreover, we can suppose $\alpha = 0$, since for any $\alpha \in (-\infty, 1)$ we can consider

$$\rho^* = \frac{1}{1-\alpha}(-\alpha\cdot\lambda + \rho), \quad \alpha^* = \alpha_{\rho^*}(M, L), \quad \omega^* = \omega_{\rho^*}(\overline{M}, \overline{L})$$

and trivially

$$\alpha^* = \frac{-\alpha}{1-\alpha} + \frac{1}{1-\alpha}\alpha = 0, \quad \omega^* = \frac{-\alpha}{1-\alpha} + \frac{1}{1-\alpha}\omega$$

proves $\alpha = \omega$ if $\omega^* = 0$. Hence we take now $\alpha = 0$ and prove $\omega \ge 0$.

Denote

$$\mathscr{P} := \{h \in C(T^d) : h > 0\}, \tag{14}$$

and with conditions on the coefficients identical to $\mathscr{F}$, introduce

$$\mathscr{G} := \left\{ g \in \mathscr{T} : \langle g, \rho\rangle = 0,\ g(\mathbf{x}) = 1 + \sum_{\mathbf{n}\in\mathbf{Z}_+^d} a(\mathbf{n})\cos(\mathbf{n}\cdot\mathbf{x}), \right.$$
$$\left. a(\mathbf{n}) \le 0\ (\mathbf{n} \notin M),\ a(\mathbf{n}) \ge 0\ (\mathbf{n} \notin L) \right\}. \tag{15}$$

In the Banach space $C(T^d)$, where the norm is the supremum norm as usual, $\mathscr{P}$ forms an open, nonvoid convex cone, and $\mathscr{G}$ is another convex set. Then $\mathscr{G}$ is nonvoid since for any $f_0 \in \mathscr{F}$ with $0 \le \langle f_0, \rho\rangle < 1$ (such f_0 must exist since $\alpha = 0 < 1$) we have

$$g_0(\mathbf{x}) := \frac{1}{1 - \langle f, \rho\rangle}(f - \langle f, \rho\rangle) \in \mathscr{G}.$$

Moreover, $\mathscr{P} \cap \mathscr{G} = \varnothing$. Indeed, for any $g \in \mathscr{P} \cap \mathscr{G}$, $0 < \min g \le 1$ and so with $\delta = \frac{1}{2}\min g$ we have

$$F := \frac{g - \delta}{1 - \delta} \in \mathscr{F},$$

and hence $0 = \alpha \le \langle F, \rho\rangle = -\delta/(1-\delta) < 0$, a contradiction. Therefore we can apply the separation theorem of convex sets (cf. [1, Corollary 2.2.2]) to $\mathscr{P}$ and $\mathscr{G}$, which furnishes a nontrivial continuous linear functional I satisfying

$$I\mathscr{P} \ge 0 \ge I\mathscr{G}, \qquad I1 = 1. \tag{16}$$

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Here $I1 = 1$ is a matter of normalization since $I\mathscr{P} \geq 0$ and $I1 = 0$ would imply $I\mathscr{P} = 0$ and hence $I = 0$, and therefore $I1 > 0$ is guaranteed. The separation constant can be chosen to be 0 since $\mathscr{P}$ is a cone and $I\mathscr{P} \neq 0$ implies $I\mathscr{P} = (0, \infty)$ or $[0, \infty)$ once $I\mathscr{P}$ is bounded from below. Also, we can suppose that I is even in the sense that

$$I(f(\mathbf{x})) = \frac{1}{2^d} \sum_{\varepsilon_1, \dots, \varepsilon_d = \pm 1} I(f(\varepsilon_1 x_1 + \cdots + \varepsilon_d x_d)) \tag{17}$$

since we can define a new functional by the right hand side of (17) if it does not hold for I itself. Applying the representation theorem of F. Riesz (cf. [1, Theorem 4.10.1]) we obtain a $\mu \in BM(T^d)$ which satisfies, according to (16) and (17),

$$I = \frac{1}{(2\pi)^d} \int_{T^d} \cdot \, d\mu, \quad d\mu(\mathbf{x}) \sim 1 + \sum_{\mathbf{k} \in \mathbf{Z}_+^d} b(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{x}). \tag{18}$$

We define the index sets

$$N_{\substack{+\\0\\-}} = \{\mathbf{k} \in \mathbb{Z}_+^d : r(\mathbf{k}) \gtreqless 0\}, \quad M_{\substack{+\\0\\-}} = N_{\substack{+\\0\\-}} \cap M, \quad L_{\substack{+\\0\\-}} = L \cap N_{\substack{+\\0\\-}}, \tag{19}$$

where the three alternatives $(+, 0, -)$ are to be understood separately. Accordingly, we denote the elements of M_+ by $\mathbf{m}_+$, elements of M_0 by $\mathbf{m}_0$, etc. For any $g_0 \in \mathscr{G}$ and $a > 0$

$$g_0 + a\cos(\mathbf{m}_0 \cdot \mathbf{x}) \in \mathscr{G}, \quad g_0 - a\cos(\mathbf{l}_0 \cdot \mathbf{x}) \in \mathscr{G},$$

and hence (16) and (18) give after $a \to +\infty$ the inequalities

$$b(\mathbf{m}_0) \leq 0, \quad b(\mathbf{l}_0) \geq 0. \tag{20}$$

Suppose now that $\mathbf{n} \in M_- \cup L_+$. Clearly,

$$g_{\mathbf{n}}(\mathbf{x}) := 1 - \frac{1}{r(\mathbf{n})} \cos(\mathbf{n} \cdot \mathbf{x}) \in \mathscr{G},$$

and so (16) and (18) now yield $1 \leq b(\mathbf{n})/r(\mathbf{n})$ and

$$s := \inf\left\{\frac{b(\mathbf{n})}{r(\mathbf{n})} : \mathbf{n} \in M_- \cup L_+\right\} \geq 1. \tag{21}$$

Finally we define for $\mathbf{k}$, $\mathbf{n} \in N_+ \cup N_-$ the function

$$f_{\mathbf{k},\mathbf{n}}(\mathbf{x}) := \frac{1}{r(\mathbf{k})} \cos(\mathbf{k} \cdot \mathbf{x}) - \frac{1}{r(\mathbf{n})} \cos(\mathbf{n} \cdot \mathbf{x}).$$

Then $\langle f_{\mathbf{k},\mathbf{n}}, \rho \rangle = 0$ and for a fixed $g_0 \in \mathscr{G}$ and $a > 0$ we have

$$\begin{aligned} g_0 - af_{\mathbf{k},\mathbf{n}} &\in \mathscr{G} \quad (\mathbf{k} \in L_+, \mathbf{n} \in L_-), \\ g_0 + af_{\mathbf{k},\mathbf{n}} &\in \mathscr{G} \quad (\mathbf{k} \in M_+, \mathbf{n} \in M_-), \\ g_0 + af_{\mathbf{k},\mathbf{n}} &\in \mathscr{G} \quad (\mathbf{k} \in M_+, \mathbf{n} \in L_+), \\ g_0 - af_{\mathbf{k},\mathbf{n}} &\in \mathscr{G} \quad (\mathbf{k} \in M_-, \mathbf{n} \in L_-). \end{aligned} \tag{22}$$

Again, we refer to (16), $I\mathscr{G} \le 0$ and (18) to obtain after $a \to +\infty$ the inequalities

$$\text{(23)} \qquad \begin{aligned} \frac{b(\mathbf{l}_+)}{r(\mathbf{l}_+)} &\ge \frac{b(\mathbf{l}_-)}{r(\mathbf{l}_-)}, & \frac{b(\mathbf{m}_-)}{r(\mathbf{m}_-)} &\ge \frac{b(\mathbf{m}_+)}{r(\mathbf{m}_+)}, \\ \frac{b(\mathbf{l}_+)}{r(\mathbf{l}_+)} &\ge \frac{b(\mathbf{m}_+)}{r(\mathbf{m}_+)}, & \frac{b(\mathbf{m}_-)}{r(\mathbf{m}_-)} &\ge \frac{b(\mathbf{l}_-)}{r(\mathbf{l}_-)}. \end{aligned}$$

Comparing (23) and (21) we obtain that for a certain real $s \ge 1$

$$\text{(24)} \qquad \frac{b(\mathbf{n})}{r(\mathbf{n})} \ge s \ge \frac{b(\mathbf{k})}{r(\mathbf{k})} \qquad (\mathbf{n} \in M_- \cup L_+, \ \mathbf{k} \in M_+ \cup L_-).$$

Now let us define

$$\text{(25)} \qquad \tau := \left(1 - \frac{1}{s}\right)\lambda + \frac{1}{s}\mu - \rho, \quad d\tau(\mathbf{x}) \sim 2 \sum_{\mathbf{k} \in \mathbf{Z}_+^d} t(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{x}).$$

Then it is easy to check that the constant term in the Fourier-Lebesgue series of τ is zero, and (24) along with (20) can be expressed as

$$\text{(26)} \qquad t(\mathbf{m}) \le 0 \quad (\mathbf{m} \in M), \qquad t(\mathbf{l}) \ge 0 \quad (\mathbf{l} \in L)$$

whence

$$\text{(27)} \qquad \tau \in \mathscr{M}.$$

Now with $t = 1 - 1/s \ge 0$ we infer from $\mu \ge 0$ and (25)–(27) that $\sigma := \tau + \rho \ge t\lambda$ and so $\omega \ge t \ge 0$, which completes the proof of our theorem.

References

[1] R. E. Edwards, *Functional analysis*, (Holt-Rinehart-Winston, New York, Toronto, London, 1965).

[2] Sz. Gy. Révész, 'On Beurling's prime number theorem', preprint, Mathematical Institute of the Hungarian Academy of Sciences, No. 5, 1990.

[3] Sz. Gy. Révész, 'Extremal problems and a duality phenomenon', Proceedings of the IMACS conference held in Dalian, China, 1989, edited by Alan G. Law, to appear.

[4] Sz. Gy. Révész, *Polynomial extremal problems* (in Hungarian), (thesis for the "candidate degree", 1988, Budapest).

[5] Sz. Gy. Révész, 'On a class of extremal problems', *Journal of Approx. Th. and Appl.*, to appear.

https://doi.org/10.1017/S1446788700032985 Published online by Cambridge University Press

[6] I. Z. Ruzsa, 'Connections between the uniform distribution of a sequence and its differences', *Topics in classical number theory*, Coll. Math. Soc. J. Bolyai 34, edited by P. Erdös and G. Halász, pp. 1419–1443 (North-Holland, Amsterdam, New York, Budapest, 1981).

Mathematical Institute
Hungarian Academy of Sciences
Budapest, POB 127, 1364
Hungary