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GROUPS OF INFINITE RANK WITH NORMALITY CONDITIONS ON SUBGROUPS WITH SMALL NORMAL CLOSURE

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Abstract

Groups of infinite rank in which every subgroup is either normal or contranormal are characterised in terms of their subgroups of infinite rank.

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1. Introduction

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property; if such an r does not exist, we will say that the group G has *infinite rank*. The investigation of the influence on a (generalised) soluble group of the behaviour of its subgroups of infinite rank has been developed in a series of recent papers (see, for instance, [2–5, 7, 8]). The aim of this paper is to provide some new contributions to this topic, by considering groups G in which every subgroup of infinite rank is either normal or contranormal. A subgroup H of G is said to be *contranormal* in G if it is not contained in a proper normal subgroup of G, that is, if $H^G = G$ (see, for instance, [13]). Groups satisfying this property will be called $\mathcal{AN}_\infty$-groups, in analogy with the symbol $\mathcal{AN}$ used to denote the class of groups in which every nonnormal subgroup is contranormal. The structure of $\mathcal{AN}$-groups has been studied in [14].

We will work within the universe of strongly locally graded groups, a class of generalised soluble groups that can be defined as follows. Recall that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\mathbf{\acute{P}}$, $\mathbf{\grave{P}}$, $\mathbf{R}$, $\mathbf{L}$ (we use the first chapter of the monograph [12] as a general reference for definitions and properties of closure operations on group classes). It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, any locally (soluble-by-finite) group is a $\bar{\mathfrak{D}}$-group and the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, Černikov proved that every $\bar{\mathfrak{D}}$-group of finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free nonabelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. For this reason, it is better in some cases to replace $\bar{\mathfrak{D}}$-groups by *strongly locally graded groups*, that is, groups in which every section belongs to $\bar{\mathfrak{D}}$. The class of strongly locally graded groups has been introduced in [5]. Most of our notation is standard and can be found in [11].

2. $\mathcal{AN}_\infty$-groups

As in many problems concerning groups of infinite rank, the existence of a proper normal subgroup of infinite rank plays a crucial role. Recall that a group G is said to be a *Dedekind group* if all its subgroups are normal.

LEMMA 2.1. *Let G be a strongly locally graded $\mathcal{AN}_\infty$-group and let N be a proper normal subgroup of infinite rank of G. Then every subgroup of N is normal in G.*

PROOF. Every subgroup of infinite rank of N is normal in G so, in particular, N is a Dedekind group (see [8, Theorem C]). Let L be a subgroup of finite rank of N. Since N is nilpotent, it contains a direct product $A_1 \times A_2$ such that both the subgroups A_1 and A_2 have infinite rank and $L \cap (A_1 \times A_2) = \{1\}$ (see [10]). Clearly the subgroups A_1 and A_2 are normal in G. Hence the subgroups of infinite rank LA_1 and LA_2 are normal in G, and $L = LA_1 \cap LA_2$ is normal in G. □

Our next lemma shows, in particular, that any strongly locally graded group of infinite rank whose proper normal subgroups have finite rank must admit a simple homomorphic image of infinite rank.

LEMMA 2.2. *Let G be a strongly locally graded group. Then every proper normal subgroup of G has finite rank if and only if the subgroup generated by all proper normal subgroups of G has finite rank.*

PROOF. Suppose that G has infinite rank but all its proper normal subgroups have finite rank. Clearly G is perfect and so it is not locally nilpotent, by [1, Lemma 2.3]. Hence G contains a proper normal subgroup N such that G/N is a simple group of infinite rank (see [5, Lemma 2.4]). Therefore N has finite rank. Let H be any proper normal subgroup of G. Since H has finite rank, HN also has finite rank and so it is a proper subgroup of G. Then $HN = N$ and it follows that $H \leq N$ so that N is the subgroup generated by all proper normal subgroups of G. □

The following result will be often used in our proofs.

LEMMA 2.3. *Let G be a group containing an abelian subgroup A of infinite rank and let H be a subgroup of G such that H^G has finite rank. Then there exists a subgroup B of A such that B has infinite rank and $H^G B$ is a proper subgroup of G.*

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Proof. Since H^G is a proper subgroup of G, we can take an element $x \in G \setminus H^G$. Then A contains a direct product $B \times C$ such that the subgroups B and C both have infinite rank and $BC \cap H^G\langle x\rangle = \{1\}$. Now

$$H^G B \cap H^G\langle x\rangle = H^G(B \cap H^G\langle x\rangle) = H^G,$$

so $x \notin H^G B$, and hence $H^G B$ is a proper subgroup of G. □

Proposition 2.4. *Let G be a strongly locally graded $\mathcal{AN}_\infty$-group. If G contains a proper normal subgroup of infinite rank, then G is an $\mathcal{AN}$-group.*

Proof. Let N be a proper normal subgroup of infinite rank of G. By Lemma 2.1, every subgroup of N is normal in G and so N is a Dedekind group. Let H be any subgroup of finite rank of G which is not contranormal, so that H^G is a proper normal subgroup of G. If H^G has infinite rank, then every subgroup of H^G is normal in G (by Lemma 2.1) and so H is normal in G. Suppose now that H^G has finite rank. Since N is a Dedekind group, it contains an abelian subgroup A of infinite rank. By Lemma 2.3, there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper normal subgroup of G. Therefore H is normal in G (by Lemma 2.1) and G is an $\mathcal{AN}$-group. □

It is now easy to prove the main result of this section.

Theorem 2.5. *Let G be a locally soluble $\mathcal{AN}_\infty$-group. Then G is an $\mathcal{AN}$-group.*

Proof. Since G is locally soluble, G contains a proper normal subgroup of infinite rank. Therefore G is an $\mathcal{AN}$-group, by Proposition 2.4. □

3. $\mathcal{SC}_\infty$-groups

In this section we will consider groups G in which every subgroup of infinite rank is either subnormal or contranormal. Groups satisfying this property will be called $\mathcal{SC}_\infty$-groups, in analogy with the symbol $\mathcal{SC}$ used to denote the class of groups in which every nonsubnormal subgroup is contranormal. This class is a natural extension of the class of $\mathcal{AN}$-groups, where the normality is replaced by subnormality. The structure of $\mathcal{SC}$-groups has been studied in [6]. We need the following elementary property.

Lemma 3.1. *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group and let K be a proper subnormal subgroup of infinite rank of G. Then every subgroup of infinite rank of K is subnormal in G.*

In particular, it follows that every proper subnormal subgroup of infinite rank of a $\mathcal{SC}_\infty$-group is soluble (see [9, Theorem 2]).

Theorem 3.2. *Let G be a torsion-free locally (soluble-by-finite) $\mathcal{SC}_\infty$-group. If G contains a proper normal subgroup of infinite rank, then G is an $\mathcal{SC}$-group.*

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Proof. Let N be a proper normal subgroup of G of infinite rank. Then N is soluble, by Lemma 3.1. Let H be any subgroup of G of finite rank such that H is not contranormal in G. Then H^G is a proper normal subgroup of G. Clearly, there exists a proper subnormal subgroup K of G, of infinite rank, which contains H. In fact, if H^G has infinite rank, we can put $K = H^G$; if H^G has finite rank, since N contains an abelian subgroup A of infinite rank, by Lemma 2.3 there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper subnormal subgroup of G and in this case we can chose $K = H^G B$. By Lemma 3.1, all subgroups of infinite rank of K are subnormal in G and hence K is nilpotent (by [9, Theorem 3]), so that H is subnormal in G. □

Recall that the *periodic radical* of a group G is the largest periodic normal subgroup of G. Moreover, G is a *Baer group* if all its cyclic subgroups are subnormal. The following lemma will be used to prove the last theorem of the paper.

Lemma 3.3. *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group containing a proper normal subgroup N of infinite rank. If the periodic radical of G has infinite rank, then every subgroup of N is subnormal in G.*

Proof. By Lemma 3.1, every subgroup of infinite rank of N is subnormal in G. So N is soluble and, in particular, a Baer group (see [9, Theorem 2]). Let H be any subgroup of finite rank of N. We can suppose that the largest periodic subgroup K of N has finite rank (otherwise H is subnormal in G, by [9, Theorem 5]). Denote by T the periodic radical of G and consider the subgroup NT. If NT is a proper normal subgroup of G, then all subgroups of infinite rank of NT are subnormal in G and, since T has infinite rank, H is subnormal in NT (by [9, Theorem 5]), and so it is subnormal in G.

Suppose that $G = NT$. Clearly, K is a periodic normal subgroup of G and hence it is contained in T. On the other hand, $T \cap N$ is contained in K, so $T \cap N = K$. Hence

$$\frac{N}{T \cap N} \simeq \frac{NT}{T} = \frac{G}{T}$$

is a torsion-free group and so T is the set of all elements of finite order of G.

Now G/T has infinite rank and all its subgroups of infinite rank are subnormal, so (by [9, Theorem 3]) it is nilpotent. Hence HT is a proper subnormal subgroup of G. By Lemma 3.1, every subgroup of infinite rank of HT is subnormal, but T has infinite rank and so H is subnormal in HT, by [9, Theorem 5]. Therefore H is subnormal in G. □

Theorem 3.4. *Let G be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group containing a proper normal subgroup of infinite rank. If the periodic radical of G has infinite rank, then G is an $\mathcal{SC}$-group.*

Proof. Let H be any subgroup of G of finite rank which is not contranormal in G. Then H^G is a proper normal subgroup of G. If H^G has infinite rank, then H is subnormal in G by Lemma 3.3. Suppose now that H^G has finite rank. If N is a proper normal

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subgroup of G of infinite rank, then N is soluble (by Lemma 3.1), and so it contains an abelian subgroup A of infinite rank. By Lemma 2.3, there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper subgroup of G. Therefore $H^G B$ is subnormal in G and, by Lemma 3.1, all its subgroups of infinite rank are subnormal in G, so that H is subnormal in G, by Lemma 3.3. This completes the proof of the theorem. □

The hypotheses of Theorems 3.2 and 3.4 cannot be weakened. Kurdachenko and Smith have proved the existence of a metabelian locally nilpotent group of infinite rank such that the largest periodic subgroup has finite rank and all subgroups of infinite rank are subnormal, but there exists a nonsubnormal subgroup of finite rank (see [9, Theorem 4]). Obviously, this subgroup cannot even be contranormal.

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