therefore $BD \cdot CG = AC \cdot DE,$
$= CA \cdot AF.$

Now $BD \cdot DG = BH \cdot AE;$

therefore $BH \cdot AE + CA \cdot AF = BD \cdot DG + BD \cdot GC$
$= BD \cdot DC.$

In triangles AHD, ADE

since $\angle ADH = \angle AED$, and $\angle DAH$ is common;

therefore (both figures) $HA \cdot AE = AD^2;$

therefore $BH \cdot AE + CA \cdot AF + HA \cdot AE = BD \cdot DC + AD^2;$

therefore $BA \cdot AE + CA \cdot AF = BD \cdot DC + AD^2.$

FIGURE 33.

$$BA \cdot AE + HA \cdot AE = BH \cdot AE + BD \cdot GC,$$
$$= BD \cdot DG + BD \cdot GC,$$
$$= BD \cdot DC + BD \cdot GC;$$

therefore $BA \cdot AE + AD^2 = BD \cdot DC + CA \cdot AF;$

therefore $BA \cdot AE - CA \cdot AF = BD \cdot DC - AD^2.$

It may also be pointed out that the lemma which Simson employed before he had discovered Lemma 10 of the *Loci Plani*, namely,

If AB be a straight line, C and D two points in it, C lying between A *and* B, *then*

$$AD^2 \cdot BC + BD^2 \cdot AC = AC^2 \cdot BC + BC^2 . AC + CD^2 \cdot AB$$

contains a theorem given by Euler in the *Novi Commentarii Academiae . . . Petropolitanae*, vol. i., p. 49 (1747).

For $(AD^2 - AC^2)BC + (BD^2 - BC^2)AC = CD^2 \cdot AB.$

But $AD - AC = BD + BC = CD;$

therefore $(AD + AC)BC + (BD - BC)AC = CD \cdot AB;$

therefore $AD\ BC + BD \cdot AC = CD \cdot AB;$

which is Euler's theorem.

Note on a property of a quadrilateral.

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The property is that if any quadrilateral, ABCD, skew or otherwise, have its sides AB, DC divided in E, F so that

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$$AE : EB = DF : FC = AD : BC,$$

then the direction of the line EF shall bisect the angle between the directions of BC, AD.

This extension of Euclid's VI. 3 follows immediately from the proposition that if AE : EB = DF : FC, then all such lines as EF are parallel to one plane, namely, the plane parallel to BC, AD; and that they each cut similar lines drawn with reference to BC, AD.

Dr Mackay has kindly supplied to me the following references bearing on the subject :—Legendre's *Geometry*, Book V., Prop. 16, (Brewster's Edition, 1824, p. 119 ; Hutton's *Course of Mathematics*, 12th edition, 1843, Vol. II., p. 224 ; *The Mathematician*, Vol. III., Supplementary Number, pp. 36–38.

Note on a possible definition of a plane.

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