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Growth Rates of 3-dimensional Hyperbolic Coxeter Groups are Perron Numbers

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Abstract. In this paper we consider the growth rates of 3-dimensional hyperbolic Coxeter polyhedra with at least one dihedral angle of the form $\frac{\pi}{k}$ for an integer $k \geq 7$. Combining a classical result by Parry with a previous result of ours, we prove that the growth rates of 3-dimensional hyperbolic Coxeter groups are Perron numbers.

1 Introduction

Let $\mathbb{H}^d$ denote the upper half-space model of hyperbolic d-space and $\overline{\mathbb{H}}^d$ its closure in $\mathbb{R}^d \cup \{\infty\}$. A convex polyhedron $P \subset \overline{\mathbb{H}}^d$ of finite volume is called a *Coxeter polyhedron* if all of its dihedral angles are of the form $\frac{\pi}{k}$ for an integer $k \geq 2$ or $k = \infty$, *i.e.*, the intersection of the respective facets is a point on the boundary $\partial\mathbb{H}^d$. The set S of reflections with respect to the facets of P generates a discrete group Γ, called a *hyperbolic Coxeter group*, and the pair (Γ, S) is called the *Coxeter system* associated with P. Then P becomes a fundamental domain for Γ. If P is compact (resp. non-compact), the hyperbolic Coxeter group Γ is called *cocompact* (resp. *cofinite*). The *growth series* $f_S(t)$ of a Coxeter system (Γ, S) is the formal power series $\sum_{l=0}^{\infty} a_l t^l$ where a_l is the number of elements of Γ whose word length with respect to S is equal to l. Then $\tau_\Gamma := \limsup_{l\to\infty} \sqrt[l]{a_l}$ is called the *growth rate* of the Coxeter system (Γ, S). By means of the Cauchy–Hadamard theorem, τ_Γ is equal to the reciprocal of the radius of convergence R of $f_S(t)$. The growth series and the growth rate of a hyperbolic Coxeter polyhedron P is defined to be the growth series and the growth rate of the Coxeter system (Γ, S) associated with P, respectively. It is known that the growth rate of a hyperbolic Coxeter polyhedron is a real algebraic integer bigger than 1 [3]. Recall that a real algebraic number $\tau > 1$ is a Perron number if all other algebraic conjugates are less than τ in absolute value. By results of Parry [13] in the compact case (resp. by results of [9–12] in the case of certain families of non-compact hyperbolic Coxeter polyhedra), the growth rate τ_Γ is a Salem number (resp. a Perron number). In [18], we proved that the growth rate of any non-compact Coxeter polyhedron in $\mathbb{H}^3$ with all dihedral angles of type $\frac{\pi}{k}$ for $2 \leq k \leq 6$ is a Perron number. The main result of this work is the extension of our result [18] to non-compact Coxeter polyhedra P in $\mathbb{H}^3$ having at least one dihedral angle of the form $\frac{\pi}{k}$ for some integer $k \geq 7$. More precisely, we shall prove the following Theorem A, which, together with the results of Parry [13,18], can be summarized in Theorem B.

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Theorem A *The growth rates of non-compact* 3*-dimensional hyperbolic Coxeter polyhedra having at least one dihedral angle of the form* $\frac{\pi}{k}$ *for some integer* $k \geq 7$ *are Perron numbers.*

Theorem B *The growth rates of* 3*-dimensional hyperbolic Coxeter groups are Perron numbers.*

Theorem B settles the 3-dimensional case of a conjecture by Kellerhals and Perren [7] and states that the set of growth rates of 3-dimensional hyperbolic Coxeter polyhedra consists of Perron numbers only. As for their small representatives, notice that the minimal growth rate among all compact Coxeter polyhedra was found by Kellerhals–Kolpakov [6], while the minimal growth rate among all non-compact ones was discovered by Kellerhals [5].

In Section 2, we provide the necessary background and review a useful formula from [18], which allow us to calculate the growth function of a hyperbolic Coxeter polyhedron. In Section 3, we establish the growth function of a non-compact hyperbolic Coxeter polyhedron with at least one dihedral angle of the form $\frac{\pi}{k}$ for some $k \geq 7$.

2 Preliminaries

In this section, we introduce the relevant notation and review some useful identities in [18] in order to calculate the growth functions of hyperbolic Coxeter polyhedra.

Definition 2.1 (Coxeter system, Coxeter graph, growth rate)

(i) A *Coxeter system* (Γ, S) consists of a group Γ and a finite set of generators $S \subset \Gamma$, $S = \{s_i\}_{i=1}^{N}$, with relations $(s_i s_j)^{m_{ij}}$ for each i, j, where $m_{ii} = 1$ and $m_{ij} \geq 2$ or $m_{ij} = \infty$ for $i \neq j$. We call Γ a *Coxeter group*. For any subset $I \subset S$, we define Γ_I to be the subgroup of Γ generated by $\{s_i\}_{i \in I}$. Then Γ_I is called the *Coxeter subgroup* of Γ generated by I.

(ii) The Coxeter graph of (Γ, S) is constructed as follows: Its vertex set is S. If $m_{ij} \geq 3$ $(s_i \neq s_j \in S)$, we join the pair of vertices by an edge and label it with m_{ij}. If $m_{ij} = \infty$ $(s_i \neq s_j \in S)$, we join the pair of vertices by a bold edge.

(iii) The *growth series* $f_S(t)$ of a Coxeter system (Γ, S) is the formal power series $\sum_{l=0}^{\infty} a_l t^l$ where a_l is the number of elements of Γ whose word length with respect to S is equal to l. Then $\tau = \limsup_{l \to \infty} \sqrt[l]{a_l}$ is called the *growth rate* of (Γ, S).

A Coxeter group Γ is *irreducible* if the Coxeter graph of (Γ, S) is connected. In this paper, we are interested in Coxeter groups that act discontinuously on the hyperbolic space $\mathbb{H}^d$.

Definition 2.2 (hyperbolic polyhedron) A subset $P \subset \overline{\mathbb{H}}^d$ is called a *hyperbolic polyhedron* if P can be written as the intersection of finitely many closed half spaces: $P = \bigcap H_i^-$, where H_i^- is the closed domain of $\mathbb{H}^d$ bounded by a hyperplane H_i.

Suppose that $H_i \cap H_j \neq \varnothing$ in $\mathbb{H}^d$. Then we define the *dihedral angle* between H_i and H_j as follows: let us choose a point $x \in H_i \cap H_j$ and consider the outer normal vectors u_i and u_j. Then the dihedral angle between H_i and H_j is defined as the real number $\theta \in [0, \pi)$ satisfying $\cos\theta = -(u_i, u_j)$, where $(\cdot, \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^d$ at x.

If $H_i \cap H_j = \varnothing$ in $\mathbb{H}^d$, then $\overline{H_i} \cap \overline{H_j} \in \overline{\mathbb{H}}^d$ is a point at the ideal boundary $\partial\mathbb{H}^d$ of $\mathbb{H}^d$, and we define the dihedral angle between H_i and H_j to be equal to zero.

Definition 2.3 (hyperbolic Coxeter polyhedron) A hyperbolic polyhedron $P \subset \mathbb{H}^d$ of finite volume is called a *hyperbolic Coxeter polyhedron* if all of its dihedral angles have the form $\frac{\pi}{k}$ for an integer $k \geq 2$ or $k = \infty$ if the intersection of the respective bounding hyperplanes is a point on the boundary $\partial\mathbb{H}^d$.

Notice that a hyperbolic polyhedron in $\mathbb{H}^d$ is of finite volume if and only if it is the convex hull of finitely many points in $\overline{\mathbb{H}}^d$. If $P \subset \mathbb{H}^d$ is a hyperbolic Coxeter polyhedron, the set S of reflections with respect to facet hyperplanes of P generates the discrete group Γ. We call Γ the *d-dimensional hyperbolic Coxeter group* associated with P. Moreover, if P is compact (resp. non-compact), Γ is called *cocompact* (resp. *cofinite*).

We recall Solomon's formula and Steinberg's formula, which are very useful for calculating growth series.

Theorem 2.4 (Solomon's formula [15]) *The growth series $f_S(t)$ of an irreducible finite Coxeter system (Γ, S) can be written as $f_S(t) = [m_1 + 1; m_2 + 1; \ldots; m_p + 1]$, where $[n] = 1 + t + \cdots + t^{n-1}$, $[m; n] = [m][n]$,etc., and where $\{m_1, m_2, \ldots, m_p\}$ is the set of exponents of (Γ, S).*

The exponents of irreducible finite Coxeter groups are shown in Table 1 (see [4] for details).

Coxeter group	Exponents	Growth series
A_n	$1, 2, \ldots, n$	$[2; 3; \ldots; n+1]$
B_n	$1, 3, \ldots, 2n-1$	$[2; 4; \ldots; 2n]$
D_n	$1, 3, \ldots, 2n-3, n-1$	$[2; 4; \ldots; 2n-2; n]$
E_6	1,4,5,7,8,11	$[2; 5; 6; 8; 9; 12]$
E_7	1,5,7,9,11,13,17	$[2; 6; 8; 10; 12; 14; 18]$
E_8	1,7,11,13,17,19,23,29	$[2; 8; 12; 14; 18; 20; 24; 30]$
F_4	1,5,7,11	$[2; 6; 8; 12]$
H_3	1,5,9	$[2; 6; 10]$
H_4	1,11,19,29	$[2; 12; 20; 30]$
$I_2(m)$	$1, m-1$	$[2; m]$

Table 1: Exponents

Theorem 2.5 (Steinberg's formula [16]) *Let* (Γ, S) *be a Coxeter system. Denote by* Γ_T *the Coxeter subgroup of* Γ *generated by the subset* $T \subseteq S$*, and denote by* $f_T(t)$ *the growth series of the Coxeter system* (Γ_T, T)*. Set* $\mathcal{F} = \{T \subseteq S : \Gamma_T \text{ is finite}\}$*. Then*

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

By Theorem 2.4 and Theorem 2.5, the growth series of (Γ, S) is represented by a rational function $\frac{p(t)}{q(t)} (p, q \in \mathbb{Z}[t])$. The rational function $\frac{p(t)}{q(t)}$ is called the *growth function* of (Γ, S). The radius of convergence R of the growth series $f_S(t)$ is equal to the positive real root of $q(t)$ that has the smallest absolute value among all the roots of $q(t)$.

From now on, we restrict our attention to the 3-dimensional case. Suppose that P is a Coxeter polyhedron in $\mathbb{H}^3$, and let v be a vertex of P. Let $F_1, \dots, F_n$ be adjacent facets of P incident to v and let $\frac{\pi}{k_i}$ be the dihedral angle between F_i and F_{i+1}. By Andreev's theorem [1], the number of facets of P incident to v is at most 4 and $k_1, \dots, k_n$ satisfy the following conditions:

$$k_1 = k_2 = k_3 = k_4 = 2 \qquad \text{if } n = 4, \tag{2.1}$$

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \geq 1 \qquad \text{if } n = 3. \tag{2.2}$$

Note that a vertex v of P belongs to $\partial\mathbb{H}^3$ if and only if $k_1 = k_2 = k_3 = k_4 = 2$ or $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$, and we call such a vertex a *cusp*, for short. We shall use the following notation and terminology for the rest of the paper:

- If a vertex v of P satisfies the identity (2.1), we call v a *cusp of type* $(2, 2, 2, 2)$.
- If a vertex v of P satisfies the inequality (2.2), we call v a *vertex of type* (k_1, k_2, k_3).
- $v_{2,2,2,2}$ denotes the number of cusps of type $(2, 2, 2, 2)$.
- v_{k_1,k_2,k_3} denotes the number of vertices of type (k_1, k_2, k_3).
- V, E, F denotes the number of vertices, edges and facets of P.
- If an edge e of P has dihedral angle $\frac{\pi}{k}$, we call it a $\frac{\pi}{k}$*-edge*.
- e_k denotes the number of $\frac{\pi}{k}$-edges.
- The growth function $f_S(t)$ of the Coxeter system (Γ, S) associated with P is called the *growth function of* P.
- The growth rate of the Coxeter system (Γ, S) associated with P is called the *growth rate of* P.

It is easy to see (*cf.* [18], for example) that the following identities and inequality hold for P:

$$V - E + F = 2, \tag{2.3}$$

$$V = v_{2,2,2,2} + \sum_{k \geq 2} v_{2,2,k} + v_{2,3,3} + v_{2,3,4} + v_{2,3,5} + v_{2,3,6} + v_{2,4,4} + v_{3,3,3}, \tag{2.4}$$

$$E = \sum_{k \geq 2} e_k, \tag{2.5}$$

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(2.6) $2e_2 = 4v_{2,2,2,2} + 3v_{2,2,2} + 2\sum_{k=3}^{\infty} v_{2,2,k} + v_{2,3,3} + v_{2,3,4} + v_{2,3,5} + v_{2,3,6} + v_{2,4,4}$,

(2.7) $2e_3 = 3v_{3,3,3} + 2v_{2,3,3} + v_{2,2,3} + v_{2,3,4} + v_{2,3,5} + v_{2,3,6}$,

(2.8) $2e_4 = 2v_{2,4,4} + v_{2,2,4} + v_{2,3,4}$,

(2.9) $2e_5 = v_{2,2,5} + v_{2,3,5}$,

(2.10) $2e_6 = v_{2,2,6} + v_{2,3,6}$,

(2.11) $2e_k = v_{2,2,k} \qquad k \geq 7$,

(2.12) $v_{2,2,2,2} + v_{2,3,6} + v_{2,4,4} + v_{3,3,3} \geq 1$.

We use these identities and the last inequality to express growth functions of the 3-dimensional hyperbolic Coxeter polyhedra under consideration. The following proposition due to Komori–Umemoto [9] will be of fundamental importance when showing that their growth rates are Perron numbers.

Proposition 2.6 ([9, Lemma 1]) *Let $g(t)$ be a polynomial of degree $n \geq 2$ having the form*

$$g(t) = \sum_{k=1}^{n} n_k t^k - 1,$$

where n_k are non-negative integers. We assume that the greatest common divisor of $\{k \in \mathbb{N} \mid n_k \neq 0\}$ is 1. Then there exists a real number r_0, $0 < r_0 < 1$ that is the unique zero of $g(t)$ having the smallest absolute value among all zeros of $g(t)$.

Our aim is to express the growth functions of non-compact hyperbolic Coxeter polyhedra with at least one dihedral angle of type $\frac{\pi}{k}$ for $k \geq 7$ as rational functions whose denominator polynomials satisfy the conditions of Proposition 2.6. This will be done by using Steinberg's formula (see Theorem 2.5) and the relations (2.3)–(2.12). This strategy was already successfully applied in [18].

3 Non-compact Coxeter Polyhedra Some of Whose Dihedral Angles are $\frac{\pi}{k}$ for $k \geq 7$

In this section, we calculate the growth function $f_S(t)$ of a non-compact hyperbolic Coxeter polyhedron P some of whose dihedral angles are $\frac{\pi}{k}$ for $k \geq 7$ and prove the growth rate of P is a Perron number.

Theorem 3.1 *Let σ be the sum of the $\frac{\pi}{k}$-edges for $k \geq 7$ of a non-compact hyperbolic polyhedron P, that is, $\sigma = \sum_{k \geq 7} e_k$. Then we obtain the inequality $\sigma \leq F - 3$. Moreover, if the equality $\sigma = F - 3$ holds, then P has a unique cusp of type $(2,2,2,2)$, and all other vertices of P are of type $(2,2,k)$ for $k \geq 7$.*

In order to prove Theorem 3.1, we use the following deformation argument for Coxeter polyhedra studied by Kolpakov in [8]. We present it in a modified form that is more suitable for further account.

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Theorem 3.2 ([8, Propositions 1 and 2])

(i) *Suppose that a non-compact hyperbolic Coxeter polyhedron $P \subset \mathbb{H}^3$ has some $\frac{\pi}{k}$-edges for $k \geq 7$. Then all of the $\frac{\pi}{k}$-edges can be contracted to cusps of type $(2,2,2,2)$. The hyperbolic Coxeter polyhedron $\widehat{P}$ that is obtained from P by contracting all $\frac{\pi}{k}$-edges for $k \geq 7$ of P is called the pinched Coxeter polyhedron of P.*

(ii) *If a hyperbolic Coxeter polyhedron P has some cusps of type $(2,2,2,2)$, then there exists a unique Coxeter polyhedron that is obtained from P by opening one cusp of type $(2,2,2,2)$. (See Fig. 1.)*

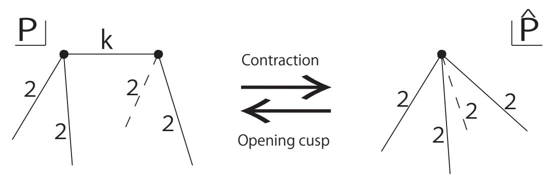

Figure 1

In the sequel, $\widehat{P}$ denotes the pinched Coxeter polyhedron obtained from P, and $\widehat{V}, \widehat{E}, \widehat{F}, \widehat{v}_{2,2,2,2}, \widehat{v}_{k_1,k_2,k_3}$, and $\widehat{e}_k$ denote respectively the number of vertices, edges, facets, cusps of type $(2,2,2,2)$, vertices of type (k_1,k_2,k_3), and $\frac{\pi}{k}$-edges of $\widehat{P}$.

Proof of Theorem 3.1 Suppose that P is a non-compact hyperbolic Coxeter polyhedron and the sum of the numbers of the $\frac{\pi}{k}$-edges for $k \geq 7$ of P is σ. By substituting identities (2.4)–(2.10) for identity (2.3), we can see the following identity for $\widehat{P}$:

$$\widehat{F} - 2 = \widehat{v}_{2,2,2,2} + \frac{1}{2}(\text{the number of vertices of } \widehat{P} \text{ with valency } 3). \tag{3.1}$$

Even if we contract the all $\frac{\pi}{k}$-edges for $k \geq 7$ of P, the number of facets of $\widehat{P}$ is equal to the number of faces of P, so that we obtain the following relations for $\widehat{P}$:

$$F = \widehat{F}, \tag{3.2}$$

$$\widehat{v}_{2,2,2,2} = v_{2,2,2,2} + \sigma. \tag{3.3}$$

Then, by substituting identities (3.2) and (3.3) for (3.1), we see that

$$F - 2 = v_{2,2,2,2} + \sigma + \frac{1}{2}(\text{the number of vertices of } \widehat{P} \text{ with valency } 3). \tag{3.4}$$

Identity (3.4) implies that $\sigma \leq F-2$. Moreover, if P satisfies the identity $\sigma = F-2$, then all of the vertices of $\widehat{P}$ are cusps of type $(2,2,2,2)$ obtained from P by contracting all $\frac{\pi}{k}$-edges for $k \geq 7$ of P. This observation means that all of the vertices of P are of type $(2,2,k)$ for $k \geq 7$. Therefore, P has no cusps. This fact contradicts to the assumption that P is non-compact. Thus, we obtain the inequality. $\sigma \leq F-3$.

Suppose that $\sigma = F-3$. Then, identity (3.4) is rewritten as

$$F - 2 = v_{2,2,2,2} + F - 3 + \frac{1}{2}(\text{the number of vertices of } \widehat{P} \text{ with valency } 3). \tag{3.5}$$

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Since any $\frac{\pi}{k}$-edge for $k \geq 3$ is adjacent to two vertices with valency 3, if P has at least one cusp of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$, then P has at least three vertices with valency 3.

Therefore, by identity (3.5), we obtain the inequality

$$F - 2 \geq v_{2,2,2,2} + F - 3 + \frac{3}{2} = v_{2,2,2,2} + F - \frac{3}{2}.$$

Hence, if P has at least one cusp of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$, we arrive at a contradiction. This implies that if $\sigma = F - 3$, P has a unique cusp of type $(2,2,2,2)$, and all other vertices of P are of type $(2,2,k)$ for $k \geq 7$. ∎

3.1 The Growth Rates in the Case of $\sigma = F - 3$

By Theorem 3.1, P has a unique cusp which is furthermore of type $(2,2,2,2)$. Apply Theorem 3.2(ii) and consider the unique hyperbolic polyhedron $\widetilde{P}$ obtained by opening this cusp in P. Then $\widetilde{P}$ is a compact Coxeter polyhedron whose growth rate is a Salem number. By a result of Kolpakov [8, Theorem 5], the growth rate of P is then a Pisot number and therefore also a Perron number.

3.2 The Growth Rates in the Case of $\sigma \leq F - 4$

In this subsection, we prove the following theorem.

Theorem 3.3 *Suppose that $\sigma \leq F - 4$ and P satisfies the following inequality*

$$v_{2,2,2,2} + e_3 + e_4 + e_5 + e_6 + F - 8 \geq 0. \tag{3.6}$$

Then the growth rate of P is a Perron number.

In order to prove Theorem 3.3, we shall use the following notation and terminology introduced in [14].

Definition 3.4 (abstract polyhedron) An *abstract polyhedron C* is a simple graph on the 2-dimensional sphere S^2 all of its vertices are 3-valent or 4-valent. If each edge of an abstract polyhedron C is labeled with $\frac{\pi}{k}$ for an integer $k \geq 2$, C is called an *abstract Coxeter polyhedron*.

For any hyperbolic Coxeter polyhedron P, the boundary ∂P is homeomorphic to S^2. This implies that the 1-skeleton of P provides an abstract Coxeter polyhedron C. We call C the *abstract Coxeter polyhedron associated with P*. Suppose that C is an abstract Coxeter polyhedron and that v is a vertex with valency i for $i = 3$ or $i = 4$. Let $c_1, \dots, c_i$ be the edges of C incident to v and denote by $\frac{\pi}{k_i}$ the label of the edge c_i.

- If a vertex v of C with valency 3 satisfies the inequality $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} > 1$, we call v a *spherical vertex of type (k_1, k_2, k_3)*.
- If a vertex v of C with valency 3 satisfies the equality $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$, we call v a *Euclidean vertex of type (k_1, k_2, k_3)*.
- If a vertex v of C with valency 3 satisfies the inequality $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1$, we call v a *hyperbolic vertex of type (k_1, k_2, k_3)*.

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- If a vertex v of C with valency 4 satisfies the equality $k_1 = k_2 = k_3 = k_4 = 2$, we call v a *Euclidean vertex of type* $(2,2,2,2)$.
- A vertex v of C with valency 4 different from a Euclidean vertex is called a *hyperbolic vertex of valency 4*.
- $\mathcal{V}_{k_1,k_2,k_3}$ denotes the number of spherical vertices of type (k_1,k_2,k_3) of C.
- $\mathcal{E}_k$ denotes the number of edges labeled by $\frac{\pi}{k}$ of C.
- $\mathcal{F}$ denotes the number of faces of C.

A spherical, Euclidean or hyperbolic vertex v of type (k_1,k_2,k_3) of C corresponds to a spherical, Euclidean or hyperbolic Coxeter triangle Δ_{k_1,k_2,k_3} whose interior angles are $\frac{\pi}{k_1}$, $\frac{\pi}{k_2}$ and $\frac{\pi}{k_3}$, respectively. We denote by $f_{k_1,k_2,k_3}(t)$ the growth function of Δ_{k_1,k_2,k_3}. Then the *abstract growth function* $f_C(t)$ of C is defined by the identity

$$\frac{1}{f_C(t^{-1})} := 1 - \frac{\mathcal{F}}{[2]} + \sum_{k\geq 2}\frac{\mathcal{E}_k}{[2;k]} - \sum_{\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}>1}\frac{\mathcal{V}_{k_1,k_2,k_3}}{f_{k_1,k_2,k_3}(t)}.$$

In the sequel, let P be a non-compact finite volume hyperbolic Coxeter polyhedron and C be the abstract Coxeter polyhedron associated with P. Then we can see that the abstract growth function $f_C(t)$ of C is equal to the growth function $f_S(t)$ of P.

Suppose that P has some dihedral angles $\frac{\pi}{k}$ for $k \geq 7$ and C is the abstract Coxeter polyhedron associated with P. Let C' be the abstract Coxeter polyhedron obtained from C by changing one of the labels of C from $\frac{\pi}{k}$ to $\frac{\pi}{6}$ (see Fig. 2).

Theorem 3.5 ([1], Andreev's theorem) *Let C be an abstract polyhedron other than a tetrahedron or a triangular prism, and suppose that non-obtuse labels are given corresponding to each edge of C. There is a hyperbolic polyhedron P of finite volume in $\mathbb{H}^3$ whose 1-skeleton provides C if and only if the following conditions are satisfied:*

(i) *if three distinct edges of C meet at a vertex, then the sum of the labels is greater than or equal to π;*
(ii) *if four distinct edges of C meet at a vertex, then all the labels equal $\frac{\pi}{2}$;*
(iii) *if three faces of C are pairwise adjacent but do not meet at a vertex, then the sum of the labels on the edges formed by adjacent faces is less than π ;*
(iv) *if four faces of C are cyclically adjacent but do not meet at a vertex, then the sum of the labels on the edges formed by adjacent faces is less than 2π;*
(v) *if a face F_i is adjacent to faces F_j and F_k, while F_j and F_k are not adjacent but have a common vertex which F_i does not share, then at least one of the labels on the edges formed by F_i with F_j or with F_k is different from $\frac{\pi}{2}$.*

By Andreev's theorem [1], the endpoints of a $\frac{\pi}{k}$-edge of P are vertices of type $(2,2,k)$ for $k \geq 7$ so that the abstract polyhedron C' has at least one Euclidean vertex and no hyperbolic vertices of valency 4. Then the growth function $f_S(t)$ of P differs from the abstract growth function $f_{C'}(t)$ of C' in the terms related to changing the label. This implies the following identity by using the relation $1/([k](t^{-1})) = t^{k-1}/[k]$:

(3.7)
$$\frac{1}{f_S(t)} = \frac{1}{f_{C'}(t)} + \Big\{ \Big(-\frac{t^6}{[2;6]} + \frac{2t^7}{[2;2;6]} \Big) + \Big(\frac{t^k}{[2;k]} - \frac{2t^{k+1}}{[2;2;k]} \Big) \Big\}$$
$$= \frac{1}{f_{C'}(t)} + \frac{(t-1)}{[2;2;6;k]} \sum_{n=6}^{k-1} t^n.$$

Proof of Theorem 3.3 Let $P \subset \mathbb{H}^3$ be a non-compact finite volume Coxeter polyhedron with $F \geq 4$ faces. Observe that the theorem holds for $F = 4$ without any further restriction, since the growth rate of a finite volume Coxeter tetrahedron P has been shown to be a Perron number by [9]. Therefore, assume that $F \geq 5$. The proof of the theorem proceeds by induction on the number σ of $\frac{\pi}{k}$-edges with $k \geq 7$ of P. More specifically, denote by P_σ such a polyhedron with dihedral angles $\frac{\pi}{k_1}, \dots, \frac{\pi}{k_\sigma}$ where $k_1, \dots, k_\sigma \geq 7$. In order to prove that the growth rate of P_σ is a Perron number, we show that the growth function $f_{S_\sigma}(t)$ of P_σ satisfies the identity

$$\frac{1}{f_{S_\sigma}(t)} = \frac{(t-1)Q_\sigma(t)}{[2;2;6;k_1;\dots;k_\sigma](1+2t^2+2t^4+2t^6+2t^8+t^{10})},$$

where $Q_\sigma(t)$ is the integer polynomial of degree $k_1 + \cdots + k_\sigma + 16 - \sigma$ whose constant term is equal to -1 and the coefficients of $Q_\sigma(t)$ except its constant term are non-negative.

Step 1: In the case where $\sigma = 1$, consider the abstract Coxeter polyhedron C_1' whose labels lie in the set $\{\frac{\pi}{k} \mid k = 2,3,4,5,6\}$ by construction. By the calculation of [18, subsection 3.5], $\frac{1}{f_{C_1'}(t)}$ is written as

$$\frac{1}{f_{C_1'}(t)} = \frac{(t-1)}{[2;4;6;10]} H_{2,3,4,5,6}(t),$$

where $H_{2,3,4,5,6}(t)$ is the integer polynomial of degree 17. Then, by using mathematica, we see that the polynomial $H_{2,3,4,5,6}$ is divisible by the polynomial $[2] = t+1$:

$$\frac{1}{f_{C_1'}(t)} = \frac{(t-1)}{[4;6;10]} G_{2,3,4,5,6}(t),$$

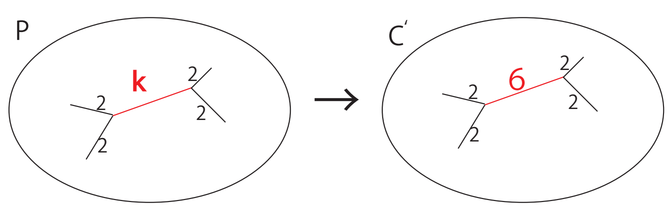

Figure 2

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where $G_{2,3,4,5,6}(t) := \frac{H_{2,3,4,5,6}(t)}{[2]}$ is the integer polynomial of degree 16. By using mathematica, $G_{2,3,4,5,6}(t)$ can be rewritten as follows:

$$\begin{aligned}
G_{2,3,4,5,6}(t) = {} & (v'_{2,3,6} + v'_{2,4,4} + v'_{3,3,3} + v'_{2,2,2,2} - 1)t^{16} \\
& + (v'_{2,3,6} + v'_{2,4,4} + v'_{3,3,3} + F' - 4)t^{15} \\
& + (\tfrac{1}{2}v'_{2,2,3} + \tfrac{1}{2}v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + \tfrac{1}{2}v'_{2,2,6} + v'_{2,3,3} \\
& \quad + v'_{2,3,4} + v'_{2,3,5} + 3v'_{2,3,6} + 3v'_{2,4,4} + \tfrac{5}{2}v'_{3,3,3} + 3v'_{2,2,2,2} - 4)t^{14} \\
& + (\tfrac{1}{2}v'_{2,2,2} + \tfrac{1}{2}v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + \tfrac{1}{2}v'_{2,2,6} + \tfrac{1}{2}v'_{2,3,3} + v'_{2,3,4} \\
& \quad + v'_{2,3,5} + 3v'_{2,3,6} + \tfrac{5}{2}v'_{2,4,4} + 3v'_{3,3,3} + v'_{2,2,2,2} + 2F' - 10)t^{13} \\
& + (\tfrac{3}{2}v'_{2,2,3} + v'_{2,2,4} + \tfrac{3}{2}v'_{2,2,5} + \tfrac{3}{2}v'_{2,2,6} + 2v'_{2,3,3} \\
& \quad + \tfrac{5}{2}v'_{2,3,4} + 3v'_{2,3,5} + 5v'_{2,3,6} + 5v'_{2,4,4} + \tfrac{9}{2}v'_{3,3,3} + 5v'_{2,2,2,2} - 8)t^{12} \\
& + (v'_{2,2,2} + v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + v'_{2,2,6} + v'_{2,3,3} + 2v'_{2,3,4} \\
& \quad + \tfrac{5}{2}v'_{2,3,5} + 4v'_{2,3,6} + 4v'_{2,4,4} + 4v'_{3,3,3} + 2v'_{2,2,2,2} + 3F' - 16)t^{11} \\
& + (2v'_{2,2,3} + \tfrac{3}{2}v'_{2,2,4} + \tfrac{5}{2}v'_{2,2,5} + 2v'_{2,2,6} + 3v'_{2,3,3} \\
& \quad + \tfrac{7}{2}v'_{2,3,4} + \tfrac{9}{2}v'_{2,3,5} + 6v'_{2,3,6} + 6v'_{2,4,4} + 6v'_{3,3,3} + 6v'_{2,2,2,2} - 11)t^{10} \\
& + (v'_{2,2,2} + v'_{2,2,4} + v'_{2,2,6} + v'_{2,3,3} + 2v'_{2,3,4} \\
& \quad + 3v'_{2,3,5} + 4v'_{2,3,6} + 4v'_{2,4,4} + 4v'_{3,3,3} + 2v'_{2,2,2,2} + 4F' - 20)t^{9} \\
& + (2v'_{2,2,3} + \tfrac{3}{2}v'_{2,2,4} + 3v'_{2,2,5} + 2v'_{2,2,6} + 3v'_{2,3,3} \\
& \quad + \tfrac{7}{2}v'_{2,3,4} + 5v'_{2,3,5} + 6v'_{2,3,6} + 6v'_{2,4,4} + 6v'_{3,3,3} + 6v'_{2,2,2,2} - 12)t^{8} \\
& + (v'_{2,2,2} + v'_{2,2,4} + v'_{2,2,6} + v'_{2,3,3} + 2v'_{2,3,4} \\
& \quad + 3v'_{2,3,5} + 4v'_{2,3,6} + 4v'_{2,4,4} + 4v'_{3,3,3} + 2v'_{2,2,2,2} + 4F' - 20)t^{7} \\
& + (2v'_{2,2,3} + \tfrac{3}{2}v'_{2,2,4} + \tfrac{5}{2}v'_{2,2,5} + 2v'_{2,2,6} + 3v'_{2,3,3} \\
& \quad + \tfrac{7}{2}v'_{2,3,4} + \tfrac{9}{2}v'_{2,3,5} + 5v'_{2,3,6} + 5v'_{2,4,4} + 5v'_{3,3,3} + 5v'_{2,2,2,2} - 11)t^{6} \\
& + (v'_{2,2,2} + v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + v'_{2,2,6} + v'_{2,3,3} + 2v'_{2,3,4} \\
& \quad + \tfrac{5}{2}v'_{2,3,5} + 3v'_{2,3,6} + 3v'_{2,4,4} + 3v'_{3,3,3} + 2v'_{2,2,2,2} + 3F' - 16)t^{5} \\
& + (\tfrac{3}{2}v'_{2,2,3} + v'_{2,2,4} + \tfrac{3}{2}v'_{2,2,5} + \tfrac{3}{2}v'_{2,2,6} + 2v'_{2,3,3} \\
& \quad + \tfrac{5}{2}v'_{2,3,4} + 3v'_{2,3,5} + 3v'_{2,3,6} + 3v'_{2,4,4} + \tfrac{7}{2}v'_{3,3,3} + 3v'_{2,2,2,2} - 8)t^{4} \\
& + (\tfrac{1}{2}v'_{2,2,2} + \tfrac{1}{2}v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + \tfrac{1}{2}v'_{2,2,6} + \tfrac{1}{2}v'_{2,3,3} \\
& \quad + v'_{2,3,4} + v'_{2,3,5} + v'_{2,3,6} + \tfrac{3}{2}v'_{2,4,4} + v'_{3,3,3} + v'_{2,2,2,2} + 2F' - 10)t^{3} \\
& + (\tfrac{1}{2}v'_{2,2,3} + \tfrac{1}{2}v'_{2,2,4} + \tfrac{1}{2}v'_{2,2,5} + \tfrac{1}{2}v'_{2,2,6} + v'_{2,3,3} \\
& \quad + v'_{2,3,4} + v'_{2,3,5} + v'_{2,3,6} + v'_{2,4,4} + \tfrac{3}{2}v'_{3,3,3} + v'_{2,2,2,2} - 4)t^{2} \\
& + (F' - 4)t - 1,
\end{aligned}$$

where F', $v'_{2,2,2,2}$ and v'_{k_1,k_2,k_3} denote respectively the number of faces, Euclidean vertices of type $(2,2,2,2)$ and spherical vertices of type (k_1,k_2,k_3) of C'_1. We denote n_i by the i-th coefficient of the polynomial $G_{2,3,4,5,6}(t)$. By using identities (2.3)–(2.10) and inequality (2.12), we can see that the following inequalities:

(3.8) $\qquad n_i \geq 0 \qquad (i = 1, 3, 5, 7, 9, 11, 13, 15),$

(3.9) $\qquad n_i + n_{i+1} \geq 0 \qquad (i = 1, \dots, 15),$

(3.10) $\qquad n_i + n_{i+1} + n_{i+2} \geq 0 \qquad (i = 1, \dots, 14).$

Using identity (3.7), we can see that

$$\begin{aligned}
\frac{1}{f_{S_1}(t)} &= \frac{1}{f_{C_1'}(t)} + \frac{(t-1)}{[2;2;6;k_1]} \sum_{i=6}^{k_1-1} t^i \\
&= \frac{(t-1)}{[4;6;10]} G_{2,3,4,5,6}(t) + \frac{(t-1)}{[2;2;6;k_1]} \sum_{i=6}^{k_1-1} t^i \\
&= \frac{(t-1)}{[2;2;5;6](1+t^2)(1-t+t^2-t^3+t^4)} G_{2,3,4,5,6}(t) + \frac{(t-1)}{[2;2;6;k_1]} \sum_{i=6}^{k_1-1} t^i \\
&= \frac{(t-1)\left\{ [k_1] G_{2,3,4,5,6}(t) + (1+2t^2+2t^4+2t^6+2t^8+t^{10}) \sum_{i=6}^{k_1-1} t^i \right\}}{[2;2;6;k_1](1+2t^2+2t^4+2t^6+2t^8+t^{10})}.
\end{aligned}$$

Let $Q_1(t) := [k_1] G_{2,3,4,5,6}(t) + (1+2t^2+2t^4+2t^6+2t^8+t^{10}) \sum_{i=6}^{k_1-1} t^i$,

$$\begin{aligned}
[k_1] G_{2,3,4,5,6}(t) &= \Big(\sum_{j=0}^{k_1-1} t^j \Big) \Big(\sum_{i=1}^{16} n_i t^i - 1 \Big) \\
&= \sum_{j=0}^{k=1-1} \sum_{i=1}^{16} n_i t^{i+j} - \sum_{j=0}^{k_1-1} t^j \\
&= \sum_{i=1}^{k_1+15} \Big\{ \chi_{[1,k_1]}(i) n_1 + \cdots + \chi_{[16,k_1+15]}(i) n_{16} \Big\} t^i - \sum_{j=0}^{k_1-1} t^j,
\end{aligned}$$

and

$$(1+2t^2+2t^4+2t^6+2t^8+t^{10}) \sum_{i=6}^{k_1-1} t^i =$$
$$\sum_{i=8}^{k_1+9} \Big\{ 2\big(\chi_{[8,k_1+1]} + \chi_{[10,k_1+3]} + \chi_{[12,k_1+5]} + \chi_{[14,k_1+7]} \big)(i) + \chi_{[16,k_1+9]}(i) \Big\} t^i + \sum_{j=6}^{k_1-1} t^j,$$

where $\chi_{[p,q]}$ is defined to be the simple function on the closed interval $[p, q]$. Then the degree of $Q_1(t)$ is $k_1 + 15$, so that we can represent $Q_1(t)$ as $\sum_{i=1}^{k_1+15} n_i^{(1)} t^i - 1$ and $n_i^{(1)}$ is written as follows:

$$\begin{aligned}
n_i^{(1)} = \sum_{j=1}^{16} \chi_{[j,k_1+j-1]}(i) n_j + 2\big(\chi_{[8,k_1+1]} + \chi_{[10,k_1+3]} + \chi_{[12,k_1+5]} + \chi_{[14,k_1+7]} \big)(i) \\
+ \chi_{[16,k_1+9]}(i) - \chi_{[1,5]}(i).
\end{aligned}$$

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Therefore, by combining inequalities (3.8), (3.9), and (3.10), we can obtain the following inequalities and identities:

$$\begin{aligned}
&n_i^{(1)} \geq 0 \quad 6 \leq i \leq k_1 + 15),\\
&n_5^{(1)} = n_5 + n_4 + n_3 + n_2 + n_1 - 1,\\
&n_4^{(1)} = n_4 + n_3 + n_2 + n_1 - 1,\\
&n_3^{(1)} = n_3 + n_2 + n_1 - 1,\\
&n_2^{(1)} = n_2 + n_1 - 1 = v'_{2,2,2,2} + e'_3 + e'_4 + e'_5 + e'_6 + F' - 9,\\
&n_1^{(1)} = n_1 - 1 = F' - 5.
\end{aligned}$$

Since C'_1 is obtained from P_1 by changing one dihedral angle from $\frac{\pi}{k_1}$ to $\frac{\pi}{6}$, $n_2^{(1)}$ can be rewritten as

$$n_2^{(1)} = v_{2,2,2,2} + e_3 + e_4 + e_5 + e_6 + F - 8. \tag{3.11}$$

Equality (3.11) together with $F' = F \geq 5$ mean that the coefficients of $Q_1(t)$ except its constant term are non-negative under the assumption of Theorem 3.3. Therefore, by Proposition 2.6, the growth rate of P_1 is a Perron number.

Step 2: We assume that the following identity holds for the growth function $f_{S_{\sigma-1}}(t)$ of $P_{\sigma-1}$ for $\sigma \geq 2$ as inductive hypothesis:

$$\frac{1}{f_{S_{\sigma-1}}(t)} = \frac{(t-1)Q_{\sigma-1}(t)}{[2;2;5;6;k_1;\ldots;k_{\sigma-1}](1+2t^2+2t^4+2t^6+2t^8+t^{10})},$$

where $Q_{\sigma-1}(t)$ is a polynomial of degree $k_1+\cdots+k_{\sigma-1}+16-(\sigma-1)$ and the coefficients of $Q_{\sigma-1}(t)$ except its constant term are non-negative. By identity (3.7) we deduce that the following identities hold:

$$\begin{aligned}
&\frac{1}{f_{S_\sigma}(t)}\\
&= \frac{(t-1)}{[2;2;6]}\left\{\frac{Q_{\sigma-1}(t)}{[k_1;\ldots;k_{\sigma-1}](1+2t^2+2t^4+2t^6+2t^8+t^{10})} + \frac{\sum_{n=6}^{k_\sigma-1} t^n}{[k_\sigma]}\right\}\\
&= \frac{(t-1)\{[k_\sigma]Q_{\sigma-1}(t) + [k_1;\ldots;k_{\sigma-1}](1+2t^2+2t^4+2t^6+2t^8+t^{10})\sum_{n=6}^{k_\sigma-1} t^n\}}{[2;2;6;k_1;\ldots;k_\sigma](1+2t^2+2t^4+2t^6+2t^8+t^{10})}.
\end{aligned}$$

Let $Q_\sigma(t) := [k_\sigma]Q_{\sigma-1}(t)+[k_1;\ldots;k_{\sigma-1}](1+2t^2+2t^4+2t^6+2t^8+t^{10})\sum_{n=6}^{k_\sigma-1} t^n$ and $R(t) := [k_1;\ldots;k_{\sigma-1}](1+2t^2+2t^4+2t^6+2t^8+t^{10})\sum_{n=6}^{k_\sigma-1} t^n$. Note that the coefficients of $R(t)$ is non-negative. Moreover, the coefficients of i-th terms are positive for $6 \leq i \leq k_\sigma - 1$:

$$\begin{aligned}
\deg [k_\sigma]Q_\sigma(t) &= (k_\sigma - 1) + \deg Q_{\sigma-1}\\
&= k_1 + \cdots + k_\sigma + 16 - \sigma,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\deg R(t) &= (k_1 - 1) + \cdots + (k_{\sigma-1} - 1) + 10 + (k_\sigma - 1)\\
&= k_1 + \cdots + k_\sigma + 10 - \sigma.
\end{aligned} \tag{3.13}$$

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Equalities (3.12) and (3.13) imply that the degree of $Q_\sigma(t)$ is equal to $k_1+\cdots+k_\sigma+16-\sigma$. We denote by $n_i^{(\sigma-1)}$ the i-th coefficient of the polynomial $Q_{\sigma-1}(t)$, so that $Q_{\sigma-1}(t)$ can be rewritten as $\sum_{i\geq 1} n_i^{(\sigma-1)} t^i - 1$:

$$\begin{aligned}
Q_\sigma(t) &= [k_\sigma]\Big(\sum_{i\geq 1} n_i^{(\sigma-1)} t^i\Big) - [k_\sigma] + R(t)\\
&= \Big(\sum_{i=0}^{6} t^i + \sum_{i=7}^{k_\sigma-1} t^i\Big)\Big(\sum_{i\geq 1} n_i^{(\sigma-1)} t^i\Big) - \Big(1+\sum_{i=1}^{5} t^i + \sum_{i=6}^{k_\sigma-1} t^i\Big) + R(t)\\
&= \Big(\sum_{i\geq 1} n_i^{(\sigma-1)} t^i\Big) + \Big(\sum_{i\geq 2} n_{i-1}^{(\sigma-1)} t^i\Big) + \Big(\sum_{i\geq 3} n_{i-2}^{(\sigma-1)} t^i\Big) + \Big(\sum_{i\geq 4} n_{i-3}^{(\sigma-1)} t^i\Big)\\
&\quad + \Big(\sum_{i\geq 5} n_{i-4}^{(\sigma-1)} t^i\Big) + \Big(\sum_{i\geq 6} n_{i-5}^{(\sigma-1)} t^i\Big) + \Big(\sum_{i\geq 7} n_{i-6}^{(\sigma-1)} t^i\Big)\\
&\quad + \sum_{j=7}^{k_\sigma-1}\sum_{i\geq 1} n_i^{(\sigma-1)} t^{i+j} + \Big\{R(t) - \sum_{i=6}^{k_\sigma-1} t^i\Big\} - \sum_{i=1}^{5} t^i - 1,
\end{aligned}$$

and hence we obtain the following inequality and identities once we represent $Q_k(t)$ as $\sum n_i^{(\sigma)} t^i - 1$:

$$\begin{aligned}
n_i^{(\sigma)} &\geq 0 \ (i \geq 6),\\
n_5^{(\sigma)} &= n_5^{(\sigma-1)} + n_4^{(\sigma-1)} + n_3^{(\sigma-1)} + n_2^{(\sigma-1)} + n_1^{(\sigma-1)} - 1,\\
n_4^{(\sigma)} &= n_4^{(\sigma-1)} + n_3^{(\sigma-1)} + n_2^{(\sigma-1)} + n_1^{(\sigma-1)} - 1,\\
n_3^{(\sigma)} &= n_3^{(\sigma-1)} + n_2^{(\sigma-1)} + n_1^{(\sigma-1)} - 1,\\
n_2^{(\sigma)} &= n_2^{(\sigma-1)} + n_1^{(\sigma-1)} - 1,\\
n_1^{(\sigma)} &= n_1^{(\sigma-1)} - 1 = n_1^{(1)} - (\sigma-1).
\end{aligned}$$

By the result of *Step 1*,

$$n_1^{(\sigma)} = n_1^{(1)} - (\sigma-1) = F - 4 - \sigma.$$

Therefore, the coefficients of $Q_\sigma(t)$ except its constant term are non-negative and the constant term of $Q_\sigma(t)$ is equal to -1 if P satisfies the inequality $F-4 \geq k$. Therefore, by Proposition 2.6, the growth rate of P_σ is a Perron number. ∎

3.3 The Proof of Theorem A

By Theorem 3.3, condition (3.6) is sufficient in order to deduce that the growth rate of P is a Perron number when $F-4 \geq \sigma$. First, suppose that P is a non-compact hyperbolic Coxeter polyhedron with $F \geq 7$. Since P has at least 1 cusp, we get the inequality

$$v_{2,2,2,2} + e_3 + e_4 + e_5 + e_6 + F - 8 \geq 1 + 7 - 8 = 0$$

which allows us to conclude. Therefore, it remains to consider non-compact Coxeter polyhedra with $F = 5$ or $F = 6$ faces and that do not satisfy inequality (3.6) of Theorem 3.3. Figure 3 shows all possible combinatorial structures of acute-angled convex polyhedra with $4, 5$ or 6 faces [2].

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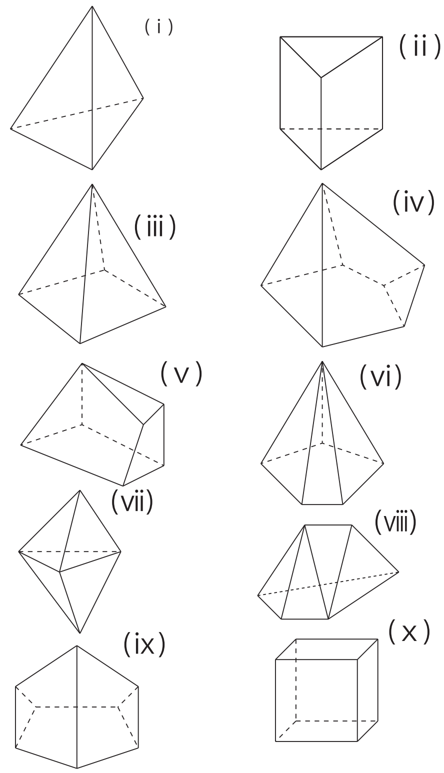

Figure 3

We use Andreev's Theorem (see Section 3.2) in order to describe a non-compact hyperbolic Coxeter polyhedron with 5 or 6 faces that does not satisfy inequality (3.6).

By Theorem 3.2 and Andreev's Theorem, it is not difficult to see that a non-compact finite volume hyperbolic Coxeter polyhedron P with 5 or 6 faces and with at least one $\frac{\pi}{k}$-edge for $k \geq 7$ has to be of combinatorial type (ii), (iv), (v), (viii), (ix), (x). If the combinatorial structure of P is (viii), P has 2 cusps of type $(2,2,2,2)$, and if the combinatorial structure is (ix) or (x), P has at least one of cusps of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$. Hence, inequality (3.6) holds for polyhedra P of type (viii), (iv), or (x), and by Theorem 3.3, their growth rates are Perron numbers.

Consider finally Coxeter polyhedra P of type (ii), (iv), or (v). First and by means of Theorem 3.2, we determine which edges of P subject to (ii), (iv), or (v) can be of the form $\frac{\pi}{k}$ for $k \geq 7$. In this way, we can deduce that each such polyhedron P results from opening cusps of type $(2,2,2,2)$ as shown in Figure 4.

In Figure 4, labels on edges mean the dihedral angles and $k, k_1, k_2 \geq 7$. If the inequality (3.6) does not hold for the case of (iv) or (v), all of the dihedral angles other than $\frac{\pi}{k_1}, \frac{\pi}{k_2}$ are $\frac{\pi}{2}$, since $v_{2,2,2,2} = 1$.

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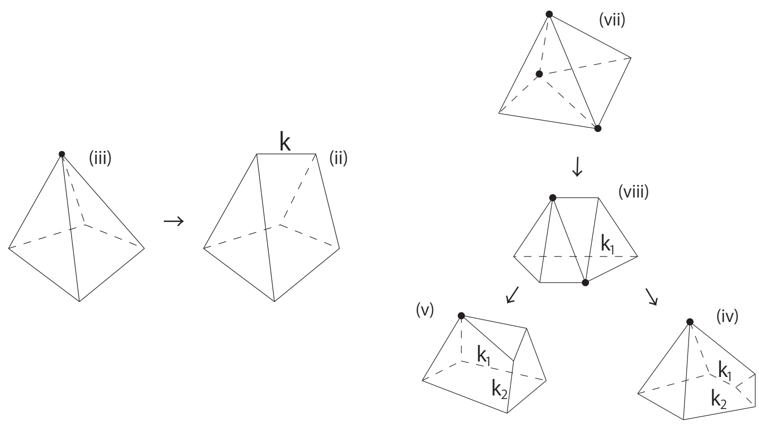

Figure 4

Proposition 3.6 *Suppose that the combinatorial structure of P is* (iv) *or* (v)*. Then the growth rate of P is a Perron number.*

Proof By means of Steinberg's formula (see Theorem 2.5), we can calculate the growth function $f_S(t)$ of P as follows:

$$\begin{aligned}\frac{1}{f_S(t)} &= 1 - \frac{6t}{[2]} + \frac{9t^2}{[2;2]} + \frac{t^{k_1}}{[2;m_1]} + \frac{t^{k_2}}{[2;k_2]} - \frac{2t^3}{[2;2;2]} - \frac{2t^{k_1+1}}{[2;2;k_1]} - \frac{2t^{k_2+1}}{[2;2;k_2]}\\ &= \frac{(t-1)\big\{(2t+1)[k_1;k_2] - (t+1)([k_1]+[k_2])\big\}}{[2;2;2;k_1;k_2]}.\end{aligned}$$

Let $Q(t) := (2t+1)[k_1;k_2] - (t+1)([k_1]+[k_2])$. We can assume that $k_1 \geq k_2$ without loss in generality.

If $k_1 = k_2$, $Q(t)$ can be rewritten as,

$$\begin{aligned}Q(t) &= [k_1]\big\{(2t+1)[k_1] - (2t+2)\big\}\\ &= [k_1]\Big(2\sum_{i=0}^{k_1-1} t^{i+1} + \sum_{i=0}^{k_1-1} t^i - 2t - 2\Big)\\ &= [k_1](2t^{k_1} + 3t^{k_1-1} + 3t^{k_1-2} + \cdots + 3t^2 + t - 1).\end{aligned}$$

If $k_1 > k_2$, $Q(t)$ can be rewritten as

$$\begin{aligned}Q(t) &= (2t+1)\Big\{(t^{k_1-1} + \cdots + t^{k_2})[k_2] + [k_2]^2\Big\}\\ &\quad - (t+1)\Big\{(t^{k_1-1} + \cdots + t^{k_2}) + 2[k_2]\Big\}\\ &= (2t+1)(t^{k_1-1} + \cdots + t^{k_2})[k_2] - (t+1)(t^{k_1-1} + \cdots + t^{k_2})\\ &\quad + [k_2]\big\{(2t+1)[k_2] - (2t+2)\big\}\\ &= [k_1](2t^{k_2} + 3t^{k_2-1} + \cdots + 3t^2 + t) + t(t^{k_1-1} + \cdots + t^{k_2}) - [k_2].\end{aligned}$$

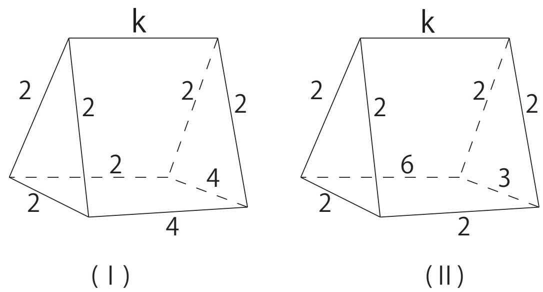

Figure 5

By the above calculation, the coefficients of $Q(t)$ except its constant term are non-negative.

Therefore, we can apply Proposition 2.6 to conclude that the growth rate is a Perron number. ∎

It remains to study the growth rates of non-compact Coxeter triangular prisms P (see Figure 4). Since P has at least one vertex at infinity, P has precisely one $\frac{\pi}{k}$-edge for $k \geq 7$. By contraction of this edge to a vertex of type $(2,2,2,2)$ (see Theorem 3.2), P deforms into exactly one among the hyperbolic Coxeter pyramid $\widehat{P}$ which have been entirely classified by Tumarkin [17]. In this way, we can deduce a precise configuration for P (see Figure 5) and prove the following result.

Proposition 3.7 *Suppose that P is a Coxeter triangular prism and P does not satisfy the inequality* (3.6). *Then P has the dihedral angles as in Figure* 5*, and the growth rate of P is a Perron number.*

Proof

Case (I): By means of Steinberg's formula, we can calculate the growth function $f_S(t)$ of P, and hence the growth function is written as

$$\frac{1}{f_S(t)} = \frac{(t-1)(2t^{k+2}+3t^{k+1}+4t^k+\cdots+4t^4+3t^3+t^2-1)}{[2;2;4;k]}.$$

Case (II): The growth function is calculated in the same manner:

$$\frac{1}{f_S(t)} = \frac{R(t)}{[2;2;2;3;6;k]},$$

where

$$R(t) = 2t^{k+8}+5t^{k+7}+7t^{k+6}+7t^{k+5}+6t^{k+4}+5t^{k+3}+3t^{k+2}+t^{k+1} - t^9-4t^8-7t^7-8t^6-7t^5-6t^4-4t^3-t^2+t+1.$$

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Therefore $f_S(t)$ can be rewritten as

$$\frac{1}{f_S(t)} = \frac{(t-1)(2t^{k+4}+3t^{k+3}+4t^{k+2}+5t^{k+1}+6t^k+\cdots+6t^6+5t^5+3t^4+2t^3+t^2-1)}{[2;2;6;k]}$$

Hence, we can apply Proposition 2.6 to conclude that the growth rate is a Perron number. ■

Proof of Theorem A Let P be a non-compact hyperbolic Coxeter polyhedron having at least one dihedral angle of the form $\frac{\pi}{k}$ for some integer $k \geq 7$ and let σ be the number of $\frac{\pi}{k}$-edges of P with $k \geq 7$. By Theorem 3.1, P satisfies the inequality $\sigma \leq F-3$. If the equality $\sigma = F-3$ holds for P, by combining with the observation in Section 3.1, the growth rate of P is a Perron number. If the inequality $\sigma \leq F-4$ holds for P, there are two cases that can be considered. First, the case where P satisfies inequality (3.6). In this case, by Theorem 3.3, the growth rate of P is a Perron number. Second, the case where P does not satisfy the inequality (3.6). In this case, P has to be of combinatorial type (ii), (iv), or (v) (see Figure 3). By Proposition 3.6 (resp. Proposition 3.7), if the combinatorial structure of P is (iv) or (v) (resp. (ii)), the growth rate of P is a Perron number. ■

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