BULL. AUSTRAL. MATH. SOC. 05C99
VOL. 14 (1976), 289-292.

On the hamiltonian product of graphs

V. Krishnamoorthy

Let G_1 and G_2 be graphs and h_1, h_2 be hamiltonian paths (h-paths) in G_1 and G_2 respectively. The hamiltonian product $(G_1, h_1)*(G_2, h_2)$ was defined by Holton. If a hamiltonian cycle exists in G_2, it can give rise to $2n$ h-paths. Peckham conjectured that $(G_1, h_1)*(G_2, h_2) \cong (G_1, h_1)*(G_2, h_3)$ where h_2 and h_3 are any two of these $2n$ h-paths of G_2. He has proved the validity of this conjecture for those h_2, h_3 where h_3 is obtainable from h_2 by a rotation along the h-cycle of G_2. Here we disprove this conjecture for those h_2, h_3 where one is obtained from the other by a reflection of the h-cycle.

1. A counterexample

DEFINITION. Let h_1 be a hamiltonian path (h-path) in the graph G_1, given by $0, 1, 2, \ldots, m-1$ (in that order). Let h_2 be the h-path in G_2, given by $0, 1, 2, \ldots, n-1$ (in that order). The hamiltonian product (h-product) $G = (G_1, h_1)*(G_2, h_2)$ is defined as follows in [1].

$V(G) = V(G_1) \times V(G_2)$; (u, v) adj (w, x) in G iff

(i) $u = w$ and v adj x in G_2, or

Received 7 January 1976. The research was supported by CSIR India.

(ii) $v = x$ and u adj w in G_1 , or

(iii) $w = (u+1) \pmod m$ and $x = (v+1) \pmod n$, or

(iv) $w = (u-1) \pmod m$ and $x = (v-1) \pmod n$.

It can be easily seen that condition (iv) in the above definition is the same as condition (iii) and hence may be omitted.

THEOREM. *Let* G_2 *be a graph with an h-cycle* C $(0, 1, 2, \ldots, n-1)$ *such that no reflection of the regular* n*-gon* $0, 1, 2, \ldots, n-1$ *is an automorphism of* G_2 . *Let* h_2 *be the h-path* $0, 1, 2, \ldots, n-1$ *on* C *and* h_3 *the h-path* $0, n-1, n-2, \ldots, 2, 1$. *Then there exists a graph* G_1 *with a h-cycle* C_1 *such that* $(G_1, h_1)*(G_2, h_2) \not\cong (G_1, h_1)*(G_2, h_3)$ *where* h_1 *is an h-path in* C_1 .

Proof. Let G_1 and h_1 be as shown in Figure 1;

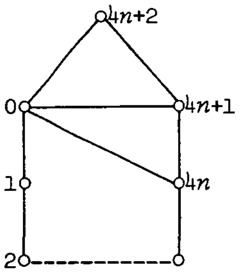

Figure 1

Suppose there exists an isomorphism

$$\alpha : G = (G_1, h_1)*(G_2, h_2) \to H = (G_1, h_1)*(G_2, h_3) .$$

Let us denote the vertices in G as $(r, s)_G$ and those in H as $(r, s)_H$. Clearly the vertices of maximum degree in G go to vertices of maximum degree in H and all these vertices have the first coordinate as 0 (since in G_1 , 0 has the maximum degree). It can be easily seen that the vertices of G , at a distance $2n$ from any vertex of maximum degree is $\{(2n, r)_G \mid 0 \le r \le n-1\}$. A similar observation holds for H also.

Hence

$$\alpha\left(\{(2n, r)_G \mid 0 \le r \le n-1\}\right) = \{(2n, r)_H \mid 0 \le r \le n-1\} .$$

The vertices $\{(2n-1, r)_G \mid 0 \le r \le n-1\}$ are those having the vertices of maximum degree $\Delta(G)$ at a distance $2n - 1$ and those of degree $\Delta(G) - 1$ at a distance greater than $2n - 1$. Hence, under α, they go to $\{(2n-1, r)_H \mid 0 \le r \le n-1\}$. Proceeding similarly we can show that

$$\alpha\left(\{(r, s)_G \mid 0 \le s \le n-1\}\right) = \{(r, s)_H \mid 0 \le s \le n-1\}$$

where $0 \le r \le 4n-1$. Now let $\alpha\left((0, 0)_G\right) = (0, r)_H$. Then $(1, 0)_G$ goes to $(1, r)_H$ or $(1, r-1)_H$. Suppose $(1, 0)_G$ goes to $(1, r)_H$. Then $(1, 1)_G$ goes to $(1, r-1)_H$ and proceeding similarly we end up with an automorphism of G_2 which is a reflection of the n-gon $(0, 1, \ldots, n-1)$, a contradiction. The case where $(1, 1)_G$ goes to $(1, r-1)_H$ is similar. This completes the proof.

Note 1. G_1 is chosen as above to simplify the proof. If G_1 need not have an h-cycle, then we can even use $K_3.P_{4n}$ as G_1 with the obvious h-path in it.

Note 2. Peckham [1] conjectured that if h_1 is an h-path in G_1 and if h_2 and h_3 are two h-paths obtained from an h-cycle C of G_2, then $G = \left(G_1, h_1\right)*\left(G_2, h_2\right) \cong \left(G_1, h_1\right)*\left(G_3, h_3\right) = H$. Our theorem gives counter examples to this conjecture. From the definition of h-product it follows that if h_3 can be got from h_2 by a rotation of the n-gon (given by C) then $G \cong H$ [1, Theorem 2]. In other words, it does not matter where h_2 starts on the n-gon, but only the orientation on the n-gon is important. It can be easily seen that if one reflection of the m-gon (imagined for h_1 as $0, 1, 2, \ldots, m-1$) or one reflection of the n-gon given by C is an automorphism of G_1 or G_2 respectively, then $G \cong H$. We conjecture that if no such reflection is an automorphism of G_1 or G_2 then $G \not\cong H$.

The above discussion shows that the graphs G_1 and G_2 in Figure 2

https://doi.org/10.1017/S0004972700025107 Published online by Cambridge University Press

give the smallest counter example to the conjecture of Peckham;

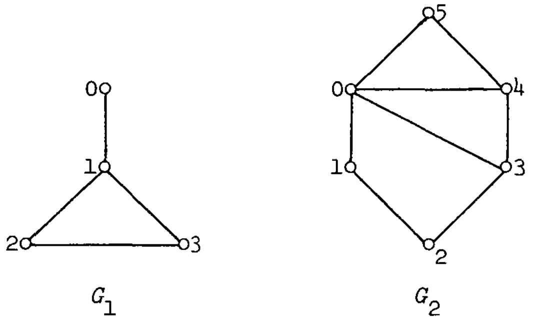

Figure 2

Reference

[1] I.A. Peckham, "The hamiltonian product of graphs", *Combinatorial mathematics*, 86-95 (Proc. Second Austral. Conf. Lecture Notes in Mathematics, 403. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

Department of Mathematics,
Indian Institute of Technology,
Madras,
Tamil Nadu,
India.