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A NOTE ON 1-FACTORS OF POINT DETERMINING GRAPHS

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A graph is called point determining if distinct points have distinct neighborhoods. In this note we provide a sufficient condition for a point determining graph to have a 1-factor. This is an extension of some results obtained by David P. Sumner.

All graphs considered in this note will be finite, undirected and without loops or multiple edges. Undefined terms can be found in [2]. If G is a graph, then the neighborhood of x in G is denoted by $N(x)$. G is called *point determining* if distinct points of G have distinct neighborhoods. If G is not point determining, we can obtain a point determining graph $\pi(G)$ from G by identifying points x and y of G with $N(x) = N(y)$. The set $\{a, b\}$ with $N(a) = N(b)$ is known as a *π-pair* of G. If G is a point determining graph, we let $G^0 = \{v \in G \mid G-v \text{ is point determining}\}$. In [4], Sumner gives some sufficient conditions on G^0 for a point determining graph G to have a 1-factor. His results are summarized in Theorem 1 below. In this note we shall extend his results by showing that if G^0 has no two disjoint π-pairs, and if $\pi(G^0)$ is a finite disjoint union of complete graphs of even order, then G has a 1-factor.

Throughout this note, $G + H$ will denote the join of G and H (see for example [2], p.21).

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THEOREM 1 ([4], Theorems 3, 4, 5, 7). *Let G be a point determining graph such that G^0 is any one of the following:*

(i) $K_{2n} + \overline{K}_m$, $1 \le m \le 2$, $n \ge 0$,

(ii) $K_2 \cup K_2$,

(iii) $K_{1,3}$,

then G has a 1-factor.

DEFINITION. Let G be a point determining graph and $p \in G - G^0$. Then by ([4], Lemma 5), any two π-pairs in $G - p$ are disjoint. Let the π-pairs of $G - p$ be $\pi_i = \{p_i, q_i\}$, $i = 1, 2, \ldots, n$. Suppose further that these π-pairs are labelled in such a way that if exactly one element of π_i is in G^0 , then q_i is that point. If both or neither of the elements of π_i are in G^0 , then let p_i be the element of π_i which is adjacent to p . We define $\pi(G; p) = G - \{p, p_1, \ldots, p_n\}$.

LEMMA 1 ([4], Theorem 1 (b)). *Let G be a point determining graph and $p \in G - G^0$. Then $(\pi(G; p))^0 \subseteq G^0$.*

LEMMA 2 ([4], Lemma 8). *If G is a point determining graph and G^0 has no two disjoint π-pairs, then for every $p \notin G^0$, $G - p$ has a unique π-pair.*

LEMMA 3 ([3], Theorem 3). *If G is point determining then no point of $G - G^0$ is adjacent to every point of G^0 .*

THEOREM 2. *Let G be a point determining graph with $G^0 = K_{2n+1} + \overline{K}_3$ $(n \ge 0)$. Then G has a 1-factor.*

Proof. By Theorem 1 *(iii)*, we may assume that $n > 0$. The result is true for graphs of low order. Suppose that G is a graph with $G^0 = K_{2n+1} + \overline{K}_3$ where $K_3 = \{x, y, z\}$ and that the result holds for all graphs of order less than $|G|$. Since G^0 is not point determining, $G \ne G^0$ and let $p \in G - G^0$. Now Lemma 2 asserts that $G - p$ contains a

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unique π-pair, say $\{a, b\}$ with $N(a) = N(b) - p$.

We first observe that $K_{2n+1} = G^0 - \{x, y, z\} \subseteq (\pi(G; p))^0$.

To see this, let $v \in K_{2n+1} = G^0 - \{x, y, z\}$. Then v is adjacent to every other elements of G^0 , so that if v is an element of the π-pair in $G - p$, then the other element of the π-pair is not in G^0 . Hence by the definition of $\pi(G; p)$, $v \in \pi(G; p)$. By Lemma 1, $(\pi(G; p))^0 \subseteq G^0$. Since no element of $\pi(G; p)$ is adjacent to all of $(\pi(G; p))^0$ by Lemma 3, it follows that $v \in (\pi(G; p))^0$.

We then have two cases to consider.

(I) $(\pi(G; p))^0 = G^0$.

In this case, the induction hypothesis asserts that $\pi(G; p)$ has a 1-factor. This 1-factor of $\pi(G; p)$ together with the edge bp forms a 1-factor of G .

(II) $(\pi(G; p))^0 \nsubseteq G^0$.

Recall that $(\pi(G; p))^0$ contains $K_{2n+1} = G^0 - \{x, y, z\}$. Now if $(\pi(G; p))^0 = G^0 - \{x, y, z\} = K_{2n+1}$, then one of x, y, z is a point of $\pi(G; p)$ adjacent to all of $(\pi(G; p))^0$, which is not possible by Lemma 3. Hence $(\pi(G; p))^0$ is either $K_{2n+1} + \overline{K}_1$ or $\overline{K}_{2n+1} + \overline{K}_2$. In any case, $\pi(G; p)$ has a 1-factor by Theorem 1.

Now if $b \notin G^0$, then $\pi(G; p) \simeq G - \{b, p\}$ and bp together with a 1-factor of $\pi(G; p)$ forms a 1-factor of G .

Suppose $b \in G^0$. If $a \in G^0$, then again $\pi(G; p) \simeq G - \{b, p\}$ and G has a 1-factor. If $a \notin G^0$, then $\pi(G; p) \simeq G - \{a, p\}$. Let $\{e_1, e_2, \ldots, e_m\}$ be a 1-factor of $\pi(G; p)$ with $e_1 = bw$ say. Then $\{aw, bp, e_2, \ldots, e_m\}$ is a 1-factor of G .

By induction, the theorem follows.

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THEOREM 3. *Let G be a point determining graph with G^0 containing no two disjoint π-pairs. Then $\pi(G^0)$ is a complete graph of even order if and only if $G^0 = K_{2n+1} + \overline{K}_m$ for some $n \geq 0$ and some $1 \leq m \leq 3$.*

Proof. If $G^0 = K_{2n+1} + \overline{K}_m$, then clearly the result is true. Conversely, suppose $\pi(G^0)$ is a complete graph of even order. If G^0 is point determining, then $\pi(G^0) = G^0 = K_{2n+1} + \overline{K}_1$. Hence assume that G^0 is not point determining. Let the π-pairs of G^0 be $\pi_1, \ldots, \pi_t$, $t \geq 1$. By the assumption on G^0, we have

(i) $\pi_i \cap \pi_j \neq \emptyset$,

(ii) $\pi_i \neq \pi_j$ for every $i \neq j$.

If $t = 1$, then $G^0 = K_{2n+1} + \overline{K}_2$. This is true because $\pi(G^0)$ is a complete graph of even order.

If $t \neq 1$, then $t \geq 3$. To see this, suppose $t = 2$. Then by (i) and (ii) we may let $\pi_1 = \{p, q\}$ and $\pi_2 = \{q, r\}$ where $p \neq r$. Since $N(p) = N(q) = N(r)$ holds in G^0, it follows that $\{p, r\}$ is also a π-pair of G^0 different from π_1 and π_2. This contradicts $t = 2$.

We further observe that the following is true:

(iii) $\pi_i \cap \pi_j \cap \pi_k = \emptyset$ for every $i < j < k$.

Suppose (iii) is false and let $\pi_i \cap \pi_j \cap \pi_k = \{a\}$ for some i, j and k. Then we have

$$\pi_i = \{a, p_i\}, \quad \pi_j = \{a, p_j\} \text{ and } \pi_k = \{a, p_k\}$$

say, where p_i, p_j and p_k are all distinct. Then $\{p_j, p_k\}$ is also a π-pair of G^0 with $\pi_i \cap \{p_j, p_k\} = \emptyset$ contradicting (i).

In view of (i), (ii) and (iii), we see that the π-pairs of G^0 can

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be written as $\{q_1, q_2\}, \{q_2, q_3\}, \ldots, \{q_t, q_1\}$, q_i all distinct and $t \geq 3$. Now if $t \geq 4$, then $\{q_1, q_3\}$ and $\{q_2, q_4\}$ are also π-pairs of G^0 which are disjoint. But this is not possible by (i). Hence $t = 3$. Since $\pi(G^0)$ is a complete graph of even order, it follows that $G^0 = K_{2n+1} + \overline{K}_3$.

REMARK 1. $\pi(G)$ is a complete graph of even (respectively odd) order if and only if $G = K_{2n+1} + \overline{K}_m$ (respectively $K_{2n} + \overline{K}_m$) for some $m \geq 1$ and some $n \geq 0$. However G contains no two disjoint π-pairs only when $1 \leq m \leq 3$.

Theorems 1, 2 and 3 lead to the following result.

THEOREM 4. *Let G be a point determining graph with G^0 containing no two disjoint π-pairs. If $\pi(G^0)$ is a complete graph of even order, then G has a 1-factor.*

Now if G^0 has no two disjoint π-pairs and that if $\pi(G^0)$ is a finite disjoint union of complete graphs of even order, then at most one component of G^0 is of the form $K_{2n+1} + \overline{K}_m$ with $n \geq 0$ and $2 \leq m \leq 3$.

COROLLARY. *Let G be a point determining graph with G^0 containing no two disjoint π-pairs. If $\pi(G^0)$ is a finite disjoint union of complete graphs of even order, then G has a 1-factor.*

REMARK 2. Let G be the path on seven points. Then $\pi(G^0) = 2K_2$ and G has no 1-factor. This example shows that the condition on G^0 having no two disjoint π-pairs cannot be omitted.

PROBLEM. Let G be a point determining graph with G^0 having no two disjoint π-pairs. If $\pi(G^0)$ has a 1-factor, does G admit a 1-factor?

References

[1] Dennis P. Geoffroy, "On 1-factors of point determining graphs", *Discrete Math.* **24** (1978), 167-175.

[2] Frank Harary, *Graph theory* (Addison-Wesley, Reading, Menlo Park, London, 1969).

[3] David P. Sumner, "Point determination in graphs", *Discrete Math.* **5** (1973), 179-187.

[4] David P. Sumner, "1-factors of point determining graphs", *J. Combin. Theory Ser. B* **16** (1974), 35-41.

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