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A LEBESGUE DECOMPOSITION THEOREM FOR C^* ALGEBRAS

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1. **Introduction**. This paper, by generalizing von Neumann's proof of the Radon–Nikodym and Lebesgue decomposition theorems [3], obtains analogous results for positive linear functionals on a C^* algebra. The concept of "absolute continuity" used and the Radon–Nikodym portion of the resulting theorem are due to Dye [2].

2. **Definitions and notation**. (i) Let $\mathbf{B}$ be a fixed C^* algebra. (For basic facts regarding C^* algebras one may consult Dixmier's book [1].) Let $\mathbf{P}$ be the positive cone of $\mathbf{B}^*$. To each $p \in \mathbf{P}$ there corresponds (a) a Hilbert space $L^2(p)$ which is the completion of $\mathbf{B}$ in the norm $\|T\|_p = p(T^*T)^{1/2}$, (b) a representation π_p of $\mathbf{B}$ on $L^2(p)$ which is the extension of left multiplication from $\mathbf{B}$ to $L^2(p)$, and (c) a distinguished element l_p of $L^2(p)$, cyclic with respect to the representation π_p, such that for $T \in \mathbf{B}$, $p(T) = \langle \pi_p(T)l_p, l_p\rangle_p$. This triple is uniquely determined by p in the sense that if $\mathbf{H}$ is any Hilbert space, π a representation of $\mathbf{B}$ on $\mathbf{H}$, and $x \in \mathbf{H}$ a cyclic vector for π such that for $T \in \mathbf{B}$, $\langle \pi(T)x, x\rangle_{\mathbf{H}} = p(T)$; then there is a unique unitary operator $U: \mathbf{H} \to L^2(p)$ such that $Ux = l_p$, and for $T \in \mathbf{B}$, $U\pi(T)U^* = \pi_p(T)$. For $T \in \mathbf{B}$ we use the notation T^p for the element $\pi_p(T)l_p \in L^2(p)$.

(ii) Let $f, p \in \mathbf{P}$. f is *dominated* by p if there exists a constant $K > 0$ such that $Kp - f \in \mathbf{P}$. For such f the bilinear form $f(S^*T)$, at first defined only for $S, T \in \mathbf{B}$, may be transferred to the subspace $\mathbf{B}^p = \{T^p \mid T \in \mathbf{B}\}$ of $L^2(p)$, and then extended to all $L^2(p)$. This extended form is written $[x, y]_f^p$, $x, y \in L^2(p)$, to distinguish it from the usual inner product on $L^2(p)$. It is uniquely determined by the equation

$$[T^p, S^p]_f^p = f(S^*T) \quad \text{for } T, S \in \mathbf{B}.$$

There is a unique bounded operator H on $L^2(p)$ such that $[x, y]_f^p = \langle x, Hy\rangle_p$. H is positive and commutes with all operators $\pi_p(\mathbf{B})$. Conversely any positive operator H on $L^2(p)$, commuting with $\pi_p(\mathbf{B})$, defines by $f(T) = \langle T^p, Hl_p\rangle_p$ an element f of $\mathbf{P}$ dominated by p. Consider the range subspace: $\mathbf{R}(H)$ = closed linear span in $L^2(p)$ of $H\mathbf{B}^p$ = closed linear span of $\pi_p(\mathbf{B})Hl_p$. It is well known that $\mathbf{R}(H) = \mathbf{R}(H^{1/2})$, therefore $H^{1/2}l_p$ is a cyclic vector for the representation $\pi_p \mid \mathbf{R}(H)$. Furthermore

$$\langle \pi_p(T)H^{1/2}l_p, H^{1/2}l_p\rangle_p = f(T), \quad T \in \mathbf{B}$$

therefore $(\mathbf{R}(H), \pi_p \mid \mathbf{R}(H), H^{1/2}l_p)$ may be identified with $(L^2(f), \pi_f, l_f)$.

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(iii) Now let H be a positive operator on $L^2(p)$, not necessarily bounded, but such that (a) H commutes, in the sense appropriate to unbounded operators, with $\pi_p(\mathbf{B})$; and (b) l_p is in the domain of $H^{1/2}$. Consider the functional

$$f(T) = \langle \pi_p(T)H^{1/2}l_p, H^{1/2}l_p\rangle_p, \quad T \in \mathbf{B}. \tag{1}$$

Clearly $f \in \mathbf{P}$. In view of (a) we may identify $(L^2(f), \pi_f, l_f)$ with $(\mathbf{R}(H), \pi_p \mid \mathbf{R}(H), H^{1/2}l_p)$ just as in (ii). In general for $f, p \in \mathbf{P}$ we call f *almost dominated* by p if for any sequence $\{T_n\} \subseteq \mathbf{B}$ such that T_n^p converges to zero (in $L^2(p)$) and $\{T_n^f\}$ is Cauchy (in $L^2(f)$), then $T_n^f \to 0$. This is abbreviated $f \ll p$. The functional f defined by (1) is almost dominated by p. For suppose that $\{T_n\} \subseteq \mathbf{B}$ with $T_n^p \to 0$ and $\{T_n^f\}$ Cauchy. Since T_n^f may be identified with $\pi_p(T_n)H^{1/2}l_p = H^{1/2}T_n^p$, $T_n^f \to 0$ follows from the fact that $H^{1/2}$ is a closed operator. It turns out that the functionals f of this form are all the functionals almost dominated by p. This is the Radon–Nikodym part of the theorem to follow. (See Dye [2].)

(iv) Let $f, p \in \mathbf{P}$, and let H be the unique bounded operator on $L^2(p+f)$ such that,

$$[x, y]_f^{p+f} = \langle x, Hy\rangle_{p+f}, \quad x, y \in L^2(p+f). \tag{2}$$

Then also $[x, y]_p^{p+f} = \langle x, (I-H)y\rangle_{p+f}$. We may identify $L^2(f)$ with $\mathbf{R}(H)$ and $L^2(p)$ with $\mathbf{R}(I-H)$. With this identification $L^2(p+f) = L^2(p) + L^2(f)$. Clearly $L^2(p+f) = L^2(p) \oplus L^2(f)$ iff $\mathbf{R}(H)$ is orthogonal to $\mathbf{R}(I-H)$. This occurs iff $\mathbf{K}(H)^\perp = \mathbf{R}(H) = \mathbf{R}(I-H)^\perp = \mathbf{K}(I-H)$, or iff H is the projection of $L^2(p+f)$ onto $L^2(f)$. In this case we say that f is *singular* to p, abbreviated $f \perp p$.

3. Existence of the Lebesgue decomposition.

THEOREM 1. *Let $p, f \in \mathbf{P}$. Then there exist $f_1, f_2 \in \mathbf{P}$ such that $f = f_1 + f_2$ and* (a) *$f_1 \ll p$ and* (b) *$f_2 \perp p$.*

Proof. The Set Up. Consider $L^2(p+f)$. Let H satisfy (2) and identify $L^2(f)$ with $\mathbf{R}(H)$ and $L^2(p)$ with $\mathbf{R}(I-H)$. Let P be the projection of $L^2(p+f)$ onto $L^2(p)$. Since $\mathbf{R}(I-H) = \mathbf{K}(I-H)^\perp =$ (the fixed points of $H)^\perp$, P is the support projection of $I-H$, in particular a spectral projection of H.

Define for $T \in \mathbf{B}$,

$$\begin{aligned} f_1(T) &= [T^{p+f}, Pl_{p+f}]_f^{p+f} = \langle T^{p+f}, PHl_{p+f}\rangle_{p+f} \\ &= \langle \pi_{p+f}(T)H^{1/2}Pl_{p+f}, H^{1/2}Pl_{p+f}\rangle_{p+f}, \end{aligned}$$

and

$$\begin{aligned} f_2(T) &= [T^{p+f}, (I-P)l_{p+f}]_f^{p+f} = \langle T^{p+f}, H(I-P)l_{p+f}\rangle_{p+f} \\ &= \langle \pi_{p+f}(T)(I-P)l_{p+f}, (I-P)l_{p+f}\rangle_{p+f}. \end{aligned}$$

Clearly $f_1, f_2 \in \mathbf{P}$, and $f = f_1 + f_2$.

Since P is a spectral projection of H, H leaves invariant $L^2(p) = PL^2(p+f)$. Let $\bar{H} = H \mid L^2(p)$. 1 is not in the point spectrum of $\bar{H}$, therefore $\bar{H}(I-\bar{H})^{-1}$ exists as a possibly unbounded, positive operator on $L^2(p)$. Clearly the domain of $\bar{H}^{1/2}(I-\bar{H})^{-1/2}$ contains $(I-H)^{1/2}l_{p+f}=l_p$. And as a function of $\bar{H}$, $\bar{H}(I-\bar{H})^{-1}$ commutes with all operators with which $\bar{H}$ commutes, including therefore $\pi_p(\mathbf{B})$. We have

$$(\bar{H}^{1/2}(I-\bar{H})^{-1/2})l_p = PH^{1/2}(I-H)^{-1/2}P(I-H)^{1/2}l_{p+f} = H^{1/2}Pl_{p+f}$$

therefore f_1 is of the form (1). By §2(iii), f_1 is almost dominated by p. This proves (a).

From the definition of f_2, $L^2(f_2)$ may be identified with $\mathbf{R}(I-P)$. Since this is orthogonal to $\mathbf{R}(P(I-H))=\mathbf{R}(I-H)=L^2(p)$, $L^2(p+f_2)$ may be identified with $\mathbf{R}(P(I-H)+(I-P))$. From this it is clear that $L^2(p+f)=L^2(p)\oplus L^2(f)$. This proves (b).

COROLLARY. (Dye [2]). *If f is almost dominated by p, then f is of the form (1).*

Proof. It suffices to prove, in the set up of Theorem 1, that $P=I$. Let $x\in L^2(p+f)$, and take a sequence $\{T_n\}\subseteq\mathbf{B}$ such that $T_n^{p+f}\to x$. Suppose that $Px=0$. Then $x\in\mathbf{K}(I-H)$ and

$$\begin{aligned}T_n^p = \pi_p(T_n)l_p = \pi_{p+f}(T_n)(I-H)^{1/2}l_{p+f} &= (I-H)^{1/2}T_n^{p+f}\\ \to (I-H)^{1/2}x &= 0.\end{aligned}$$

Since $f\ll p$, and $\{T_n^f\}$ is clearly Cauchy, it follows that $T_n^f\to 0$. Therefore,

$$\begin{aligned}\|x\|_{p+f}^2 &= \lim \|T_n^{p+f}\|_{p+f}^2 = \lim \|T_n^p\|_p^2+\lim \|T_n^f\|_f^2\\ &= 0.\end{aligned}$$

This proves that $P=I$, hence $f_2=0$.

4. Uniqueness of the decomposition.

THEOREM 2. *Let p, f be elements of* $\mathbf{P}$, *and let $f=f_1+f_2$ be the decomposition of Theorem 1. Let $f=g_1+g_2$ be a second decomposition of f such that $g_1\ll p$ and $g_2\perp p$. Then $f_1=g_1$ and $f_2=g_2$. To prove this we require the following:*

LEMMA. *Let $f, p, h\in\mathbf{P}$.* (a) *If $f\ll p$ and $p\perp h$, then $f\perp h$.* (b) *If $f\ll p$ and $f\perp p$, then $f=0$.*

Proof of the lemma. (a) Let P be the projection of $L^2(p+h)$ onto $L^2(p)$. Since $p\perp h$, $L^2(h)=\mathbf{R}(I-P)$. By the corollary to Theorem 1, there is a (possibly unbounded) operator H on $L^2(p)$ such that

$$f(T) = \langle\pi_p(T)H^{1/2}l_p, H^{1/2}l_p\rangle_p = \langle\pi_{p+h}(T)H^{1/2}Pl_{p+h}, H^{1/2}Pl_{p+h}\rangle_{p+h}.$$

Then in $L^2(p+h)$, $L^2(f+h)=\mathbf{R}(H^{1/2}P+(I-P))=\mathbf{R}(PHP+(I-P))$. From this realization of $L^2(f+h)$ it is clear that $f\perp h$.

(b) Let P be the projection on $L^2(p+f)$ such that $L^2(p)=\mathbf{R}(P)$ and $L^2(f)=\mathbf{R}(I-P)$. Let $x\in L^2(p+f)$, and suppose that $Px=0$. Exactly as in the proof of the corollary to Theorem 1 one shows that $x=0$. (In this case P plays the roles of P and H.) Thus $P=I$, and $f=0$.

Proof of Theorem 2. By our lemma (a), $f_1 \perp f_2$ and $g_1 \perp g_2$. Let P and Q be the projections of $L^2(f)$ onto $L^2(f_1)$ and $L^2(g_1)$ respectively. It suffices to prove that $P=Q$. Identifying $L^2(f)$ with $\mathbf{R}(H)\subseteq L^2(p+f)$, referring to the set up in the proof of Theorem 1, it is clear that the projection P just defined is the restriction of the projection P occurring in the proof of Theorem 1 from $L^2(p+f)$ to $L^2(f)$. Q, however, is only defined on $L^2(f)$. Let $x\in L^2(f)$ and let $\{T_n\}$ be a sequence in $\mathbf{B}$ such that $T_n^{p+f}\to x$. Suppose $Px=0$. As in the proof of the corollary to Theorem 1, this implies $x\in \mathbf{K}(I-H)$, and that $T_n^p\to 0$. Therefore $T_n^{g_1}\to 0$, since $g_1 \ll p$.

We also observe that

$$\lim T_n^f = \lim H^{1/2}T_n^{p+f} = H^{1/2}x = x,$$

so that

$$\|Qx\|_f^2 = \lim \|QT_n^f\|_f^2 = \lim \|T_n^{g_1}\|_{g_1}^2 = 0.$$

Thus $Q\le P$. What we have proved is that f_1 is in some sense the maximal part of f almost dominated by p.

Now consider the functional $h\in \mathbf{P}$ defined by

$$h(T) = \langle \pi_f(T)l_f, (P-Q)l_f\rangle_f.$$

h is clearly almost dominated by f_1 and g_2. Two applications of the lemma (a) show that also $h \perp f_1$. Therefore by the lemma (b) $h=0$. Thus $P=Q$, and the decomposition is unique.

5. **Remarks.** (a) Dye [2] has proven that if, in a sequence $\{f_n\}\subseteq \mathbf{P}$, all f_n are almost dominated by p, and $\sum \|f_n\|<\infty$, then $\sum f_n \ll p$. In particular $\{f \mid f\ll p\}$ is closed under addition.

(b) This same statement with "singular to" replacing "almost dominated by" is true for Abelian algebras, but far from true in general. A counter-example appears in the simplest of non-Abelian C^* algebras. Let $\mathbf{M}_2$ be the algebra of 2×2 complex matrices and consider the functionals

$$f_1\begin{pmatrix} a & b\\ c & d\end{pmatrix} = a,\quad f_2\begin{pmatrix} a & b\\ c & d\end{pmatrix} = d,\quad \text{and}\quad g\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \tfrac{1}{2}(a+b+c+d).$$

All three are pure states of $\mathbf{M}_2$; their corresponding representations are irreducible. Therefore any two of them must be either mutually singular or mutually almost dominating, since none can be decomposed into mutually singular parts.

The matrix

$$T = \begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix} 1 & -1\\ 0 & 0\end{pmatrix}^* \begin{pmatrix} 1 & -1\\ 0 & 0\end{pmatrix} \ge 0$$

satisfies $f_1(T)=f_2(T)=1$, $g(T)=0$. Therefore $f_1 \perp g$, and $f_2 \perp g$. But not only is f_1+f_2 not singular to g, but actually g is almost dominated by f_1+f_2, since the latter is a faithful trace on $\mathbf{M}_2$. It is worth mentioning in this context (as suggested by the referee) that $\{f \mid f \perp p\}$ is closed under increasing suprema.

(c) If **B** is a von Neumann algebra and f is a normal positive linear functional, then f_1 and f_2 of the Lebesgue decomposition of f are also normal, whether p is normal or not. This follows from the fact that π_f is normal when f is normal.

BIBLIOGRAPHY

1. J. Dixmier, *Les C*-Algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

2. H. Dye, "The Radon-Nikodym theorem for finite rings of operators", Trans. Amer. Math. Soc. **72** (1952), 243–280.

3. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar, New York (1955), 137–140.

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