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COMMUTATIVE ENDOMORPHISM RINGS

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Introduction. The problem of classifying the torsion-free abelian groups with commutative endomorphism rings appears as Fuchs' problems in [**4**, Problems 46 and 47]. They are far from solved, and the obstacles to a solution appear formidable (see [**4**; **5**]). It is, however, easy to see that the only dualizable abelian group with a commutative endomorphism ring is the infinite cyclic group. (An R-module M is called dualizable if $\mathrm{Hom}_R(M, R) \neq 0$.) Motivated by this, we study the class of prime rings R which possess a dualizable module M with a commutative endomorphism ring. A characterization of such rings is obtained in § 6, which as would be expected, places stringent restrictions on the ring and the module.

1. Throughout we will write homomorphisms of modules on the side opposite to the scalar action. Rings will not be assumed to contain identity elements unless otherwise indicated.

Given a left R-module M, we will let $M^* = \mathrm{Hom}_R(M, R)$ and $M^{**} = \mathrm{Hom}_R(M^*, R)$. There is a natural R-homomorphism $\delta_M \colon M \to M^{**}$ defined for $m \in M$, $f \in M^*$ via $[(m)\delta_M](f) = mf$. $K(M)$ will denote the kernel of δ_M; clearly $K(M) = \bigcap_{f \in M^*} \ker f$. If $K(M) = 0$, then M is called a *torsionless* left R-module. More generally, we will consider modules for which $K(M) \neq M$; or, equivalently, $M^* \neq 0$. It will be convenient, and descriptive, to call such a module *dualizable*. We note that $K(M)$ is a fully invariant submodule of M and that $M/K(M)$ is torsionless.

We now introduce a notation which will prove very convenient. While it is by no means new (it appeared in [**1**]), its use here stems from some recent lectures of S. A. Amitsur. For M a left R-module there is an R-R-bimodule homomorphism $(\ ,\ ) \colon M \otimes_E M^* \to R$, where $E = E(M) = \mathrm{Hom}_R(M, M)$, defined for $m \in M$, $f \in M^*$ by $(m, f) = (m)f$. There is also an E-E-bimodule homomorphism $[\ ,\ ] \colon M^* \otimes_R M \to E$ defined via $m[f, n] = (m, f)n$ for $m, n \in M, f \in M^*$. One then has $[f, m]g = f(m, g)$ for any $m \in M, f, g \in M^*$; and note also that $(M, f) = 0, f \in M^*$, implies that $f = 0$. We set $F(M) = [M^*, M]$ = image of $[\ ,\ ]$ in $E(M)$, a two-sided ideal of $E(M)$.

2. In this section we show that the only dualizable abelian group with a commutative ring of endomorphisms is the infinite cyclic group. The next lemma provides the main step.

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LEMMA 2.1. *An R-module $M = M_1 \oplus M_2$ has a commutative endomorphism ring if and only if each M_i does and $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$.*

Proof. First observe that if $\mathrm{Hom}_R(M, M)$ is commutative, then every endomorphic image of M is fully invariant; for given $\alpha, \beta \in \mathrm{Hom}_R(M, M)$, $M\alpha\beta = M\beta\alpha \subseteq M\alpha$. Let π_i be the natural projection of M onto M_i, $i = 1, 2$. Then $\mathrm{Hom}_R(M_i, M_i) \cong \pi_i \mathrm{Hom}_R(M, M)\pi_i$, and so $\mathrm{Hom}_R(M_i, M_i)$ is commutative. Also for $i \neq j$ and $\varphi \in \mathrm{Hom}_R(M_i, M_j)$, φ can be extended to an endomorphism of M by defining φ to be zero on M_j. This violates the fully invariant property of M_i, unless $\varphi = 0$.

The proof of the converse is straightforward, and is left to the reader.

THEOREM 2.2. *For a dualizable module M over a Dedekind domain R, $E(M)$ is commutative if and only if M is isomorphic to an ideal of R.*

Proof. Assume that $E(M)$ is commutative with $M^* \neq 0$. Taking $0 \neq f \in M^*$ we have an exact sequence

$$0 \to K \to M \xrightarrow{f} I \to 0$$

with $K = \ker f$ and $I = Mf$. Since R is Dedekind, $M \cong I \oplus K$. By the previous lemma, $\mathrm{Hom}(I, K) = 0$. Since $I \oplus I \cong R \oplus I^2$ [**2**, p. 150], it follows that $K \cong \mathrm{Hom}_R(R, K) \cong$ (into) $\mathrm{Hom}_R(I \oplus I, K) \cong \mathrm{Hom}_R(I, K) \oplus \mathrm{Hom}_R(I, K) = 0$. Thus $K = 0$ and $M \cong I$. The endomorphism ring of an ideal is certainly commutative.

COROLLARY 2.3. *The only dualizable abelian group with a commutative endomorphism ring is the infinite cyclic group.*

3. Recall that a ring R is *semiprime* if it has no nilpotent left or right ideals; equivalently if every left (right) ideal has zero intersection with its left(right) annihilator. R is *prime* if the left (right) annihilator of a left (right) ideal is zero.

In the bracket notation, for a left R-module M to be torsionless means that $m = 0$ whenever $(m, M^*) = 0$. We will often use the following observation.

LEMMA 3.1. *Let M be a torsionless left module over a semiprime ring R. Then for any $m \in M$, $[M^*, m] = 0$ implies that $m = 0$.*

Proof. $[M^*, m] = 0$ implies that $0 = (M, [M^*, m]M^*) = (M, M^*(m, M^*)) = (M, M^*)(m, M^*)$. Hence $(m, M^*) \subseteq (M, M^*) \cap$ right ann$(M, M^*) = 0$. Since M is torsionless, $m = 0$.

Now suppose that M is any left module over a semiprime ring R. Then for $f \in M^*$, $[f, M] = 0$ implies that $0 = (M, [f, M]M^*) = (M, f(M, M^*)) = (M, f)(M, M^*)$, so that $(M, f) \subseteq (M, M^*) \cap$ left ann$(M, M^*) = 0$. Thus $f = 0$. From this it follows that $M^* \neq 0$ if and only if $F(M) \neq 0$.

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Henceforth, M will always denote a left R-module. We will say that a ring S is a right (left) *order* in an overring T if for every $0 \neq t \in T$,

$$tS \cap S \neq 0 \quad (St \cap S \neq 0).$$

PROPOSITION 3.2. *If R is semiprime (prime) and M is torsionless, then*
(1) *$E(M)$ is semiprime (prime), and*
(2) *$F(M)$ is a left and right order in $E(M)$.*

Proof. (1) is [**6**, Proposition 1.2(i)]. To prove (2), let $0 \neq \alpha \in E(M)$. Since $F(M)$ is a two-sided ideal in $E(M)$, it suffices to show that $0 \neq F(M)\alpha$ and $0 \neq \alpha F(M)$. If $0 = F(M)\alpha = [M^*, M]\alpha = [M^*, M\alpha]$, then

$$0 = (M, [M^*, M\alpha]M^*) = (M, M^*(M\alpha, M^*)) = (M, M^*)(M\alpha, M^*),$$

and so $(M\alpha, M^*) = 0$, from which it follows that $M\alpha = 0$ contradicting $\alpha \neq 0$. A similar computation establishes that $\alpha F(M) \neq 0$.

The next theorem establishes the fact that in the setting of torsionless modules commutativity of $F(M)$ implies commutativity of $E(M)$. We will later extend this to a larger class of modules (Corollary 5.2).

THEOREM 3.3. *If R is semiprime and M is torsionless, then $E(M)$ is commutative if $F(M)$ is.*

The proof follows from the following elementary observation, together with the previous proposition.

LEMMA 3.4. *Suppose that R is a prime (semiprime) ring and L is a non-zero left ideal of R (which is a left order in R). Then R is commutative if L is.*

Proof. Assume that L is commutative. First observe that left ann$(L) = 0$. (For, left ann$(L) \cap L = 0$ and L is a left order in R.) Next let $r \in R$, $x \in L$ be given. Then for any $y \in L$, $(rx - xr)y = r(xy) - x(ry) = r(yx) - (ry)x = 0$ since $ry \in L$. Since $y \in L$ was arbitrary, $rx - xr \in$ left ann$(L) = 0$. Hence L is contained in the centre of R.

Next, suppose that $r, s \in R$, and let $x \in L$. Then $(rs - sr)x = r(sx) - s(rx) = (sx)r - s(xr) = 0$ since $sx \in L$. Hence $rs - sr \in$ left ann$(L) = 0$. r and s being arbitrary, the proof is complete.

COROLLARY 3.5. *If R is a ring without zero divisors and L is a non-zero commutative left ideal, then R is commutative.*

4. Our aim in this section is to show that if M is a torsionless module with $F(M)$ commutative, then M is a uniform module. We recall that a module is *uniform* if any two non-zero submodules have non-zero intersection. The next lemma provides a crucial step.

LEMMA 4.1. *Suppose that R is a semiprime ring and that $F(M)$ is commutative. Then given $m, n \in M$ with $(m, M^*)(n, M^*) \neq 0$, it follows that $Rm \cap Rn \neq 0$; more precisely, $(M, M^*)(m, M^*)n = (M, M^*)(n, M^*)m \neq 0$.*

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Proof. Since $F(M)$ is commutative, $[M^*, m][M^*, n] = [M^*, n][M^*, m]$. Hence $M[M^*, m][M^*, n] = M[M^*, n][M^*, m]$ from which one obtains $(M, M^*)(m, M^*)n = (M, M^*)(n, M^*)m$. If this were zero, then

$$(M, M^*)(m, M^*)(n, M^*) = 0$$

whence $(m, M^*)(n, M^*) \subseteq (M, M^*) \cap \text{right ann}(M, M^*) = 0$, and so we would have $(m, M^*)(n, M^*) = 0$. Therefore $(m, M^*)(n, M^*) \neq 0$ implies that $0 \neq (M, M^*)(m, M^*)n = (M, M^*)(n, M^*)m \subseteq Rn \cap Rm$.

PROPOSITION 4.2. *Suppose that R is a prime ring and $F(M)$ is commutative. Then $M/K(M)$ is a uniform torsionless module (which is non-zero if and only if M is dualizable).*

Proof. We have only to show that $M/K(M)$ is uniform. Thus suppose that $m, n \in M \setminus K(M)$. We must prove that $Rm \cap Rn \not\subseteq K(M)$. By our choice of m and n, (m, M^*) and (n, M^*) are non-zero right ideals. Since R is a prime ring, $(m, M^*)(n, M^*) \neq 0$. By Lemma 4.1,

$$0 \neq (M, M^*)(m, M^*)n \subseteq Rm \cap Rn.$$

If $Rm \cap Rn \subseteq K(M)$, then $(M, M^*)(m, M^*)(n, M^*) = 0$ from which it follows that $(m, M^*)(n, M^*) = 0$, a contradiction. Therefore

$$Rm \cap Rn \not\subseteq K(M).$$

We remark that the lemma is valid for M a module over a semiprime ring provided that $(m, M^*)(n, M^*) \neq 0$ for all $m, n \in M \setminus K(M)$. For this was just the point at which the proof required primeness of R.

COROLLARY 4.3. *Suppose that R is a prime ring and M is a torsionless module with $F(M)$ commutative. Then M is uniform.*

5. By Proposition 4.2 the existence of a dualizable module M over a prime ring for which $F(M)$ is commutative implies the existence of a uniform torsionless module. In this section we show that the endomorphism ring of this uniform module is also commutative.

For any R-module M it is easy to see that one has an exact sequence

$$0 \to \text{Hom}(M, K(M)) \to E(M) \to E(M/K(M)).$$

This follows directly from the fact that $K(M)$ is a fully invariant submodule of M, so that endomorphisms of M induce endomorphisms of $M/K(M)$ in a natural manner. In general the map $E(M) \to E(M/K(M))$ is not surjective. For $F(M)$, however, the corresponding map is indeed surjective.

LEMMA 5.1. *The restriction of the above sequence to $F(M)$ yields an exact sequence*

$$0 \to F(M, K(M)) \to F(M) \to F(M/K(M)) \to 0,$$

where $F(M, K(M)) = \text{Hom}(M, K(M)) \cap F(M)$.

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Proof. We show how to lift a homomorphism in $F(M/K(M))$. Say

$$\Sigma[\bar{f}_i, \bar{m}_i] \in F(M/K(M)),$$

where $\bar{f}_i \in (M/K(M))^*$, $\bar{m}_i = m_i + K(M) \in M/K(M)$.

Then letting π be the natural homomorphism of M onto $M/K(M)$, $f_i = \pi\bar{f}_i \in M^*$. We leave it to the reader to check that $\Sigma[f_i, m_i]$ is mapped onto $\Sigma[\bar{f}_i, \bar{m}_i]$.

This lemma provides generalizations of Proposition 3.2 and Theorem 3.3. Note that a non-zero module M with $\mathrm{Hom}(M, K(M)) = 0$ is dualizable.

COROLLARY 5.2. *Suppose that R is a semiprime (prime) ring and M is an R-module with $\mathrm{Hom}_R(M, K(M)) = 0$. Then*

(i) *$E(M)$ is semiprime (prime),*

(ii) *$F(M)$ is a left and right order in $E(M)$,*

(iii) *$E(M)$ is commutative if $F(M)$ is.*

Proof. By Proposition 3.2, $F(M/K(M))$ is a right and left order in $E(M/K(M))$. From the exact sequences above, $\mathrm{Hom}_R(M, K(M)) = 0$ implies that

$$F(M/K(M)) \xrightarrow{\mu} F(M) \subseteq E(M) \xrightarrow{\nu} E(M/K(M)),$$

where the map ν is a monomorphism and μ is an isomorphism ($\mu = \nu^{-1}$ with restricted domain). The corollary follows from this together with the fact that $F(M/K(M))$ is itself a semiprime (prime) ring.

PROPOSITION 5.3. *If R is a prime ring and M a module with $F(M)$ non-zero and commutative, then $M/K(M)$ is a uniform torsionless module whose endomorphism ring is a commutative integral domain.*

Proof. By Proposition 4.2, $M/K(M)$ is a non-zero uniform torsionless module. Lemma 5.1 assures us that $F(M/K(M))$ is commutative. Thus by the results in § 3, $E(M/K(M))$ is a commutative prime ring, hence an integral domain.

6. For brevity we call a ring R which possesses a module M for which $F(M)$ is non-zero and commutative a (left) CT-*ring*. In this section we investigate the structure of prime CT-rings.

THEOREM 6.1. *A prime* CT-*ring possesses a uniform left ideal whose endomorphism ring is a commutative integral domain.*

Proof. By Proposition 5.3, R has a uniform torsionless module U whose endomorphism ring S is a commutative integral domain. We show that U is isomorphic to a left ideal of R.

Let f be any non-zero element of U^*, and set $V = \ker f$. If $V \neq 0$, then $[f, V]$ is a non-zero subset of S. For $[f, V] = 0$ implies $0 = U[f, V] =$

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$(U,f)V = (Uf)V$; but V is a faithful R-module, and so $Uf = 0$ contradicting $f \neq 0$. Next, $U[f, V]^2 = (U,f)(V,f)V = (Uf)(Vf)V = 0$, and so $[f, V]^2 = 0$. Since S is a domain, $[f, V] = 0$, a contradiction. We conclude that $V = 0$, and hence that f is an isomorphism of U onto a left ideal of R.

COROLLARY 6.2. *The* CT-*rings without zero divisors are commutative integral domains.*

Proof. Let R be a CT-ring without zero divisors, U the left ideal of R given by Theorem 6.1. Since right multiplication by distinct elements of U gives distinct R-endomorphisms of U, we can regard U as a subring of $\text{Hom}_R(U, U)$. Thus U is commutative. By Lemma 3.5 so is R.

Remark. The proof of Theorem 6.1 shows in fact that over a prime ring a uniform torsionless module with a commutative endomorphism ring is isomorphic to a left ideal.

In view of Corollary 4.3, it then follows that over a prime CT-ring, the torsionless modules with commutative endomorphism rings are isomorphic to (uniform) left ideals.

THEOREM 6.3. *A prime ring R is a* CT-*ring if and only if it is a right order in a ring of column-finite matrices over a commutative field.*

Proof. Suppose that R is a prime CT-ring, and let U be a uniform left ideal with $S = E(U)$ a commutative integral domain. First note that U is a torsionless right S-module. For given $0 \neq u \in U$, $[U^*, u] \neq 0$ by Lemma 3.1, and so there exists $f \in U^*$ with $[f, u] \neq 0$. Define $\varphi: U_S \to S_S$ via $\varphi(x) = [f, x]$ for any $x \in U$. Since φ is a homomorphism of right S-modules and $\varphi(u) \neq 0$, U_S is torsionless.

Let K be the quotient field of S, $T = \text{Hom}_S(U_S, U_S)$,

$$K_\lambda = \text{Hom}_K(U \otimes_S K, U \otimes_S K);$$

thus K_λ is the ring of column-finite $\lambda \times \lambda$ matrices over K acting as left operators on $U \otimes_S K$, and λ is the dimension of $U \otimes_S K$ as a K-vector space. Using the fact that U_S is torsionless, one proceeds through an elementary argument (see [**6**, the proof of Theorem 2.2(ii) as well as Lemma 1.1]) to show that T is a right order in K_λ.

Since U is a faithful left R-module, we can assume that $R = R^l \subseteq T$ under the identification $r \leftrightarrow r^l =$ left multiplication by $r \in R$. We now show that R is a right order in T. Given $0 \neq \tau \in T$, then for any $u, v \in U$, $(\tau u^l)v = \tau(uv) = (\tau u)v = (\tau u)^l v$ since τ is an S-homomorphism and $\tau u \in U$. Thus $\tau u^l - (\tau u)^l$ annihilates U. But U is a faithful left T-module, and so $\tau u^l = (\tau u)^l \in U^l$. Thus U^l is a left ideal in the prime ring T, and

$$0 \neq \tau U^l \subseteq U^l \cap \tau U^l \subseteq R^l \cap \tau R^l.$$

This shows that R is a right order in T.

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It remains to prove that R is a right order in K_λ. Let $Z_{K_\lambda}(K_\lambda)$ denote the right singular ideal of K_λ. (We refer the reader to [**3**] for the definition and basic properties of the singular submodule.) Since K_λ is a regular ring, $Z_{K_\lambda}(K_\lambda) = 0$. In this situation the relationship of "right order" becomes transitive [**3**, Lemma 11.2(3)]. Thus R is a right order in K_λ.

Conversely, suppose that R is a prime ring which is a right order in K_λ with K a commutative field. Let $Z_R(K_\lambda)$ and $Z_R(R)$ denote the singular submodule of K_λ as a right R-module and the right singular ideal of R, respectively. Since R is a right order in K_λ and $Z_{K_\lambda}(K_\lambda) = 0$, it follows that $Z_R(R) = 0$. Since $Z_R(K_\lambda) \cap R = Z_R(R)$, $Z_R(K_\lambda) = 0$. Finally, given any $\kappa \in K_\lambda$, set $(R:\kappa) = \{r \in R | \kappa r \in R\}$; $(R:\kappa)$ is an essential right ideal of R.

Let $e = e_{11} \in K_\lambda$ be the idempotent element which has 1 in the (1,1)-position, 0 elsewhere. We claim that Re is a torsionless left R-module. (For the remainder of this proof, Re is the R-module generated by e and containing e.) For given $0 \neq ae \in Re$, set $I = (R:ae) \cap (R:e)$; I is an essential right ideal of R. Since $Z_R(K_\lambda) = 0$, $0 \neq aeb$ for some $b \in I$. Thus the function given by $re \mapsto reb$ defines an R-homomorphism from Re to R which is non-zero on ae.

By Proposition 3.2 it follows that $E(Re) = \mathrm{Hom}_R(Re, Re)$ is a prime ring. Set $J = (R:e)$. Since R is semiprime, $eJ \neq 0$ implies that $(eJ)^2 \neq 0$; in particular, $eJe \neq 0$. Note that right multiplications by distinct elements of eJe give distinct elements of $E(Re)$. (For if $a, b \in J$ and $re(eae) = re(ebe)$ for every $r \in R$, then $Re(a-b)e = 0$. Then $Re(a-b)eJ = 0$, and since R is prime and $e(a-b)eJ \subseteq (eJ)^2 \subseteq R$, we have $e(a-b)eJ = 0$. Since $Z_R(K_\lambda) = 0$, $e(a-b)e = 0$.) Hence we may regard eJe as a subring of $E(Re)$.

Finally, given any $\varphi \in E(Re)$, and any $a \in J$, $re \in Re$, we have $re((eae)\varphi) = (reae)\varphi = rea(e\varphi) = rea(s_\varphi e)$ for some $s_\varphi e \in Re$. Hence $(eae)\varphi = e(as_\varphi)e \in eJe$, proving that eJe is a non-zero right ideal in $E(Re)$. Since $eJe \subseteq eK_\lambda e \cong K$, eJe is commutative. But then by Lemma 3.4 for prime rings, $E(Re)$ is commutative. Thus R is a CT-ring.

There are right orders of matrix rings which are CT-rings but not prime. For example, the ring R of 2×2 lower triangular matrices over a field F is certainly a right (and left) order in the full ring of 2×2 matrices over F. And R certainly has a left ideal with a commutative endomorphism ring (isomorphic to F, in fact). But R is not even semiprime. Our final result covers such examples.

THEOREM 6.4. *Let T be a ring of matrices over a field K, and R a subring of T. If $\{e_{\mu\nu} | 1 \leqq \mu, \nu \leqq \Lambda\}$ (Λ an ordinal number) is a set of matrix units for T with $e_{11} \in R$, then R is a left and a right* CT-*ring.*

Proof. The proof is almost trivial. For if $e_{11} \in R$, then $Re_{11} \subseteq R$; thus $R(1 - e_{11}) \subseteq R$, and hence $R = Re_{11} \oplus R(1 - e_{11})$. Under these circumstances,

$$E(Re_{11}) = \mathrm{Hom}_R(Re_{11}, Re_{11}) \cong e_{11}Re_{11} \subseteq e_{11}Te_{11} \cong K.$$

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Thus R is a left CT-ring. The same argument can be applied to $e_{11}R$ to show that R is a right CT-ring.

Added in proof.

We conclude this paper with a statement on left-right symmetry.

THEOREM 6.5. *A prime left* CT-*ring is a right* CT-*ring.*

Proof. Let R be a prime left CT-ring, U a uniform left ideal of R with a commutative endomorphism ring S. Consider $U^* = \text{Hom}_R(U, R)$, and S-R-bimodule which is torsionless as a right R-module.

First note that U^* is a faithful left S-module; for if $\alpha \in S$ and $\alpha U^* = 0$, then $0 = [\alpha U^*, U] = \alpha[U^*, U]$ so that $\alpha = 0$. Next set $T = \text{Hom}_R(U^*, U^*)$; we can regard $[U^*, U]$ as a subring of T. Let $\tau \in T, f \in U^*, u \in U$. Then for any $g \in U^*$, $(\tau[f, u])g = \tau([f, u]g) = \tau(f(u, g)) = (\tau f)(u, g) = [\tau f, u]g$ since $(u, g) \in R$. Thus, $[U^*, U]$ is a left ideal of T. T is prime by Proposition 3.2; and Lemma 3.4 then ensures that T is commutative.

Remark. Since a prime CT-ring has been seen to have zero singular ideal and uniform left ideals, it is actually "densely" embedded in a full ring of linear transformations (see, for example, S. Amitsur, *Rings of quotients and Morita contexts*, not yet published).

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