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ON TRANSFORMATION AND OSCILLATION OF LINEAR DIFFERENTIAL SYSTEMS

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1. Introduction. In this paper we study second order linear differential systems. We examine the relationship between oscillation of n-dimensional systems and certain associated m-dimensional systems, where $m \leqq n$. Several theorems are presented which unify and encompass in the linear case a number of results from the literature. In particular, we present a transformation which extends an oscillation theorem due to Allegretto and Erbe [**1**], and a comparison theorem due to Kreith [**9**], and explains some work of Howard [**7**].

We shall be concerned with the differential system

$$(1.1) \qquad l[u] \equiv [R(t)u' + Q(t)u]' - [Q^*(t)u' - P(t)u] = 0,$$

where each of the $n \times n$ matrix functions $R(t)$, $P(t)$, $Q(t)$ has complex valued entries which are continuous on a given subinterval I of the real line and $R(t)$ and $P(t)$ are hermitian on I. A *solution* of (1.1) on I is an n-dimensional vector function $u(t)$, for which $u(t)$, and $v(t) = R(t)u'(t) + Q(t)u(t)$ are continuously differentiable on I, and (1.1) holds on I.

Definition. The system (1.1) is said to be *oscillatory* on the interval $[a, \infty)$ if for each $\alpha \geqq a$ there is a $\beta > \alpha$ and a solution u of (1.1) defined on $[a, \infty)$ such that $u(\alpha) = 0$, $v(\alpha) \neq 0$ and $u(\beta) = 0$.

Corresponding to the vector system (1.1), we have the matrix differential system

$$(1.1\text{m}) \quad l[U] \equiv [R(t)U' + Q(t)U]' - [Q^*U' - P(t)U] = 0,$$

with a *solution*, an $n \times s$ matrix, $U(t)$, $V(t) = R(t)U'(t) + Q(t)U(t)$, defined in a manner analogous to the earlier definition, $1 \leqq s \leqq n$. The relationship between (1.1) and (1.1m) is that U is a solution of (1.1m) if and only if $u(t) = U(t)\gamma$ is a solution of (1.1) for all s-vectors γ.

From time to time we shall require $R(t)$ to be nonsingular and/or positive definite $R > 0$, but we wish to remark that at the present time no such assumptions are being made, also we do *not* make a general assumption of positiveness on $P(t)$.

2. Oscillation. In this section we present an oscillation theorem which we will show is the foundation of a number of recent results in the literature. First, we shall have need of some preliminary definitions and theorems.

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Two solutions $u_1(t)$, $u_2(t)$ of (1.1) on I are said to be *isotropic* [**3**, Chapter 2] or (mutually) *conjoined* [**11**, Chapter 7], if $v_2^*(t)u_1(t) - u_2^*(t)v_1(t)$, which is constant on I, has the value zero. Let $U(t)$ be an $n \times n$ solution of (1.1m) on I, if the column vectors of $U(t)$ are mutually conjoined, i.e., $V^*U - U^*V = 0$ on I, and the column vectors of the $2n \times n$ matrix $(U(t); V(t))$, (the first n rows is U, the next n rows is V), are linearly independent on I then $U(t)$ is called a *conjoined basis* for (1.1m).

For a given compact interval $[a, b]$ let $D[a, b] = D_n[a, b]$ denote the set of all n-vector functions η, which are absolutely continuous on $[a, b]$ and for which $R\eta' + Q\eta$ is Lebesgue square integrable on $[a, b]$. The subclass of $D[a, b]$ on which $\eta(a) = \eta(b) = 0$ will be denoted by $D_0[a, b]$. For $\eta \in D[a, b]$ we shall denote by $J[\eta; a, b]$ the functional

$$J[\eta; a, b] = \int_a^b \{\eta^{*\prime}[R\eta' + Q\eta] + \eta^*[Q^*\eta' - P\eta]\}dt.$$

We remark that an $n \times s$ matrix, $1 \leqq s \leqq n$, may be inserted in this functional in place of η. The following is a standard identity involving $J[\eta; a, b]$ and follows from an integration by parts.

LEMMA 2.1. *If $\eta \in D[a, b]$ and $R\eta' + Q\eta$ is absolutely continuous on $[a, b]$, then*

$$J[\eta; a, b] = \eta^*(R\eta' + Q\eta)\Big|_a^b - \int_a^b \eta^* l[\eta]dt.$$

In particular, if $l[\eta] = 0$ and $\eta(a) = \eta(b) = 0$ then $J[\eta; a, b] = 0$.

We shall show that recent results on oscillation are based on the following theorem.

THEOREM 2.2. *Let $R(t) > 0$ on $[a, \infty)$. Then* (1.1) *is oscillatory on $[a, \infty)$ if and only if there exists a sequence of intervals $[\alpha_k, \beta_k]$, $a \leqq \alpha_k < \beta_k$, with $\alpha_k \to \infty$ with k, and nontrivial functions η_k in $D_0[\alpha_k, \beta_k]$ such that $J[\eta_k; \alpha_k, \beta_k] \leqq 0$.*

Proof. It is well-known that $J[\eta; a, b]$ is positive definite on $D_0[a, b]$ if and only if there exists a conjoined basis $U(t)$ which is nonsingular on $[a, b]$. Assume the conditions hold and (1.1) is not oscillatory then for some $\alpha \geqq a$, the conjoined basis $U(t)$ satisfying $U(\alpha) = 0$, $V(\alpha) = E_n$ (the $n \times n$ identity matrix) is nonsingular for all $t > \alpha$. Let $\alpha_k > \alpha$ then $J[\eta; \alpha_k, \beta_k]$ is positive definite on $D_0[\alpha_k, \beta_k]$, contradicting the existence of η_k. The converse is immediate in view of Lemma 2.1.

3. Transformation and oscillation. In this section we shall present a transformation of (1.1) and several theorems on oscillation.

THEOREM 3.1. *Let $H(t)$ be an $n \times r$ matrix function, $1 \leqq r \leqq n$, such that*

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H and $RH' + QH$ are absolutely continuous on I and

$$(3.1) \qquad (RH' + QH)^*H = H^*(RH' + QH), \quad \text{on } I$$

and put $\mathscr{R}[t] = H^*RH, \mathscr{P}(t) = H^*l[H]$. If

$$(3.2) \qquad \mathscr{L}[X] = (\mathscr{R}(t)X')' + \mathscr{P}(t)X$$

for X an $r \times s$ matrix function, $1 \leqq s \leqq r$, with X and RHX' absolutely continuous *on I, then*

$$(3.3) \qquad H^*l[U] = \mathscr{L}[X]$$

where $U = HX$.

Furthermore,

$$(3.4) \qquad J[U; a, b] = (HX)^*(RH' + QH)X \Big|_a^b + \int_a^b \{X^{*\prime}\mathscr{R}X' - X^*\mathscr{P}X\}dt.$$

Proof.
$$\begin{aligned}\mathscr{L}[X] &= H^{*\prime}(RHX') + H^*(RHX')' \\ &\quad + H^*(RH' + QH)'X - H^*(Q^*H' - PH)X \\ &= H^*[(RH' + QH)'X + (RH' + QH)X' + (RHX')'] \\ &\quad - H^*(RH' + QH)X' + H^{*\prime}RHX' - H^*(Q^*H' - PH)X \\ &= H^*[(RH' + QH)X + RHX']' - (RH + QH)^*HX' \\ &\quad + H^{*\prime}RHX' - H^*(Q^*H' - PH)X \\ &= H^*l[U].\end{aligned}$$

It follows from Lemma 2.1 and (3.3) that

$$J[U; a, b] = (HX)^*[R(HX)' + Q(HX)] \Big|_a^b - \int_a^b X^*\mathscr{L}[X]dt.$$

Now an integration by parts of the last integral yields the desired conclusion.

The following two corollaries follow from the theorem in a straightforward manner using the definitions given earlier.

COROLLARY 3.2. *If $H(t)$ is an $n \times n$ nonsingular matrix satisfying the hypothesis of the theorem, then X is a conjoined basis for $\mathscr{L}[X] = 0$ on I if and only if $U = HX$ is a conjoined basis for* (1.1m) *on I.*

COROLLARY 3.3. *If $H(t)$ is an $n \times n$ nonsingular matrix satisfying the hypothesis of the theorem, then the system* (1.1) *is oscillatory on $[a, \infty)$ if and only if the system $\mathscr{L}[X] = 0$ is oscillatory on $[a, \infty)$.*

An interesting application of Corollary 3.3 occurs when, in (1.1), $R \equiv E_n$, $Q \equiv 0$ and $\int_t^\infty P(s)ds \equiv S(t)$ exists (finitely). One takes H to be the fundamental solution of the system $H' = 2S(t)H$, then $\mathscr{R} = H^*H > 0$ and $\mathscr{P} = H^*[4S^2(t) - P(t)]H$. It follows that if $4[\int_t^\infty P(s)ds]^2 \leqq P(t)$ then $\mathscr{L}[X]$ is nonoscillatory and hence $U'' + P(t)U = 0$ is nonoscillatory (the result itself is known and goes back to Wintner). The setting of this example is studied in [**5**].

The next theorem is the kind of result which is of primary interest in this paper. The result obtains oscillation of (1.1) on considering a related (in this case, scalar) equation. Analogous results appear in Simons [**13**].

For $A(t)$ an $n \times n$ matrix, hermitian and continuous on $[a, \infty)$, we shall let $\lambda_1(A(t)), \ldots, \lambda_n(A(t))$ be the characteristic values of $A(t)$, with the notation so chosen that $\lambda_1(A(t)) \leqq \ldots \leqq \lambda_n(A(t))$ for each t in $[a, \infty)$. The functions $\lambda_i(A(t))$ are continuous.

THEOREM 3.4. *Let $R > 0$, and q_{ii} be real valued and continuously differentiable for each i, $Q = (q_{ij})$. If the scalar differential equation*

$$((\text{tr }(R(t)))y')' + (\text{tr }(Q'(t) + P(t)))y = 0 \tag{3.5}$$

is oscillatory on $[a, \infty)$, then (1.1) is oscillatory on $[a, \infty)$. In particular, if $(\lambda_n(R(t))y')' + \lambda_1(Q'(t) + P(t))y = 0$ is oscillatory, then (1.1) is oscillatory.

Proof. If we denote the J-functional associated with (3.5) by $J_s[\eta; a, b]$, then Theorem 2.2 implies that there exists a sequence of intervals $[\alpha_k, \beta_k]$ with $\alpha_k \to \infty$ and nontrivial scalar functions η_k in $D_0[\alpha_k, \beta_k]$ such that $J_s[\eta_k; \alpha_k, \beta_k] \leqq 0$. In Theorem 3.1 take H to be the n-vector all of whose components are zero except the i-th which is 1, and take X to be η_k, then for $\nu_i = H\eta_k$, it follows from (3.4) that

$$J[\nu_i; \alpha_k, \beta_k] = \int_{\alpha_k}^{\beta_k} \{\eta_k' r_{ii} \eta_k' - \eta_k(q_{ii}' + p_{ii})\eta_k\} dt, \tag{3.6}$$

here $R(t) = (r_{ij}(t))$, etc. If one sums on i in (3.6) the right hand side becomes $J_s[\eta_k; \alpha_k, \beta_k]$, which is less than or equal to zero, thus for some i, $J[\nu_i; \alpha_k, \beta_k] \leqq 0$. The first statement now follows from Theorem 2.2.

The second statement follows from the Sturm Comparison Theorem and the fact that for a hermitian matrix $A(t)$, $n\lambda_n(A) \geqq \text{tr }(A)$, and $n\lambda_1(A) \leqq \text{tr }(A)$.

The following corollary utilizes scalar criteria due to Hartman (see, e.g., [**6**, XI, Theorem 7.3]). Eliason [**4**] obtains a related theorem using entirely different methods.

COROLLARY 3.5. *Let $Q \equiv 0$, $R \equiv E_n$. If*

$$\lambda_n\left[\frac{1}{T}\int^T \left(\int^t P(s)ds\right)dt\right] \to +\infty \quad \text{as } T \to +\infty$$

and

$$\liminf \lambda_1\left[\frac{1}{T}\int^T \left(\int^t P(s)ds\right)dt\right] > -\infty \quad \text{as } T \to +\infty$$

then (1.1) is oscillatory. In particular if $P \geqq 0$ and $\int^\infty ||P||ds = +\infty$ then (1.1) is oscillatory.

The corollary follows from the theorem and Hartman's theorem since the hypothesis insures that

$$\frac{1}{T}\int^T\left(\int^t \frac{\operatorname{tr}(P(s))}{n}ds\right)dt \to +\infty \quad \text{as } T \to +\infty$$

and hence (3.5) is oscillatory.

We remark that the second statement of the corollary is not true if the nonnegative definiteness of P is dropped. To see this choose $P(t) = \operatorname{diag}(\sin t/4t, -\sin t/4t)$ then $\|P(t)\| = |\sin t|/4t$. The scalar equations $u'' \pm (\sin t/4t)u = 0$ are nonoscillatory [**15**] and thus the (diagonal) system (1.1) is nonoscillatory.

The following theorem extends the result of Allegretto and Erbe [**1**], and others, in the case under consideration here.

THEOREM 3.6. *Let $R(t) > 0$ on $[a, \infty)$, and let $H(t)$ be an $n \times r$ matrix function, $1 \leqq r \leqq n$, such that H and $RH' + QH$ are absolutely continuous on $[a, \infty)$ and (3.1) holds on $[a, \infty)$. If the system*

$$\mathscr{L}[x] = (\mathscr{R}x')' + \mathscr{P}x = 0,$$

$\mathscr{R} = H^*RH$, $\mathscr{P} = H^*l[H]$, is oscillatory on $[a, \infty)$, then the system (1.1) is oscillatory on $[a, \infty)$.

Proof. Let $\alpha \geqq a$. Since $\mathscr{L}[x] = 0$ is oscillatory there is a $\beta > \alpha$ and an r-dimensional vector solution $x(t)$ of $\mathscr{L}[x] = 0$ such that $x(\alpha) = 0$, $x(\beta) = 0$ and $\mathscr{R}(\alpha)x'(\alpha) \not\equiv 0$. Now take $X(t)$ of Theorem 3.1 to be $x(t)$ and put $\eta(t) = Hx$, then $\eta(\alpha) = \eta(\beta) = 0$, and (3.4) plus Lemma 2.1 yield

$$J[\eta; \alpha, \beta] = \int_\alpha^\beta \{x^{*\prime}\mathscr{R}x - x^*\mathscr{P}x\}dt = 0.$$

Furthermore, $\eta(t) \not\equiv 0$, for if so, then $0 \equiv \eta'(t) = H'x + Hx'$ and thus $\mathscr{R}(\alpha)x'(\alpha) = (H^*RHx')(\alpha) = -(H^*RH'x)(\alpha) = 0$, a contradiction. The conclusion of the theorem now follows from Theorem 2.2.

Allegretto and Erbe have shown that a number of interesting results follow from theorems such as the previous one. The corollary below shows that to determine oscillation of (1.1) one may consider certain subsystems. If a one dimensional such subsystem is used one obtains the result first proved by Swanson [**18**] (see also Barrett [**2**]), that if a "diagonal equation" oscillates then the original system is oscillatory. It follows that all scalar oscillation criteria yield, in a trivial manner, oscillation criteria for systems (1.1), for example, one can take H to be an $n \times 1$ vector and the resulting transformed system is scalar.

For a given $n \times n$ matrix $A = (a_{ij})$, let A_γ be the $r \times r$ submatrix of A, $1 \leqq r \leqq n$ obtained by deleting each row and column of A except row and column $i_1, i_2, \ldots, i_r$, where $1 \leqq i_1 \leqq \ldots \leqq i_r \leqq n$, here $\gamma = (i_1, \ldots, i_r)$. If one chooses $H_{(\gamma)}$ to be the $n \times r$ submatrix of the $n \times n$ identity matrix E, obtained by deleting each column of E except column $i_1, \ldots, i_r$, then $A_\gamma = H_{(\gamma)}^* A H_{(\gamma)}$.

COROLLARY 3.7. *Let $R > 0$ and Q be continuously differentiable. If for some γ, $Q_\gamma^* = Q_\gamma$, and the system*

$$(R_\gamma(t)u')' + (Q_\gamma'(t) + P_\gamma(t))u = 0$$

is oscillatory, then (1.1) *is oscillatory.*

This statement follows directly from the theorem on choosing H to be $H_{(\gamma)}$.

In [**7**], H. C. Howard discusses oscillation of systems of the form (1.1) with $Q \equiv 0$, by introducing and placing conditions on new functions which are related to R and P. Theorem 3.1 makes it possible to understand the relationship between these new functions and the original R and P. If one takes $H(t) = \sqrt{g(t)E}$, here $g(t)$ is a positive scalar function for which g, and Rg' are C^1, then Howard's functions are obtained as the coefficients of the system (3.2), in particular,

$$\begin{aligned} \mathscr{R}(t) &= gR(t) \\ \mathscr{P}(t) &= gP(t) - (1/4)R(t)(g')^2 g^{-1} + (1/2)(R(t)g')'. \end{aligned} \tag{3.7}$$

For $g(t) = t$, one obtains the Kneser-type conditions, see, e.g., [**11**].

With the help of the Fite-Wintner-Leighton theorem [**10**], the following theorem encompasses, in the linear case, Howard's original theorem, its extension by Kartsatos [**8**], and the Noussair and Swanson theorem.

THEOREM 3.8. *Let $R > 0$ and continuously differentiable, $Q \equiv 0$. If for some γ, the scalar equation*

$$((\mathrm{tr}\,(\mathscr{R}_\gamma(t)))y')' + (\mathrm{tr}\,(\mathscr{P}_\gamma(t)))y = 0$$

$\mathscr{R}$ and $\mathscr{P}$ as in (3.7), *is oscillatory, then* (1.1) *is oscillatory.*

Proof. By Theorem 3.4

$$(\mathscr{R}_\gamma y')' + \mathscr{P}_\gamma y = 0$$

is oscillatory and thus by Corollary 3.7, (1.1) is oscillatory.

We wish to remark that a natural choice for $H(t)$, when $R(t) > 0$, is the inverse of the positive definite square root of $R(t)$, for then $\mathscr{R}(t)$ becomes the identity matrix.

4. Comparison and oscillation. In the previous section many of the results are obtained via an indirect comparison of two systems. In this section we present a more direct comparison theorem which utilizes the H-transformation.

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Consider two systems of the form (1.1),

$$l_i[u] = [R_i(t)u' + Q_i(t)u]' - [Q_i^*(t)u' - P_i(t)u] = 0, \quad i = 1, 2,$$

where the conditions on R_i, Q_i, and P_i are the same as those on (1.1).

The following result is a significant extension of the theorem in Kreith [**9**] for the case under consideration here and encompasses the standard Sturm type comparison theorem.

THEOREM 4.1. *Let $R_2 > 0$ on $[a, \infty)$. Let $H(t)$ be an $n \times r$ matrix function, $1 \leqq r \leqq n$, such that H and $R_iH' + Q_iH$ are absolutely continuous on $[a, \infty)$, $(R_iH' + Q_iH)^*H = H^*(R_iH' + Q_iH)$, and $\mathscr{R}_i(t) = H^*R_iH$, $\mathscr{P}_i(t) = H^*l_i[H]$ and $\mathscr{L}_i[X] = (\mathscr{R}_iX')' + \mathscr{P}_iX$ for $i = 1, 2$. If $\mathscr{L}_1[x] = 0$ is oscillatory on $[a, \infty)$ and if there exists a sequence of real numbers (α_k), $\alpha_k \to \infty$ such that for each $k = 1, 2, \ldots$*

$$\int_{\alpha_k}^{\tau} \{\eta^{*\prime}H^*(R_1 - R_2)H\eta' + \eta^*H^*(l_2 - l_1)[H]\eta\}dt$$

is nonnegative for all $\eta \in D_0[\alpha_k, \tau] \subseteq D_r[a, b]$, $\tau > \alpha_k$, then $l_2[u] = 0$ is oscillatory on $[a, \infty)$.

Proof. We shall verify that the condition of Theorem 2.2 is satisfied. For a given α_k, $\mathscr{L}_1[x]$ oscillatory implies the existence of a solution x of $\mathscr{L}_1[x] = 0$ and a number $\beta_k > \alpha_k$ such that $x(\alpha_k) = x(\beta_k) = 0$, $(\mathscr{R}_1x')(\alpha_k) \neq 0$. Now

$$\begin{aligned} 0 &\leqq \int_{\alpha_k}^{\beta_k} \{x^{*\prime}H^*(R_1 - R_2)Hx' + x^*H^*(l_2 - l_1)[H]x\}dt \\ &= \int_{\alpha_k}^{\beta_k} \{x^{*\prime}\mathscr{R}_1x' - x^*\mathscr{P}_1x\}dt - \int_{\alpha_k}^{\beta_k} \{x^{*\prime}\mathscr{R}_2x' - x^*\mathscr{P}_2x\}dt \\ &\equiv \mathscr{J}_1[x; \alpha_k, \beta_k] - \mathscr{J}_2[x; \alpha_k, \beta_k]. \end{aligned}$$

But $\mathscr{J}_1[x; \alpha_k, \beta_k] = 0$ by Lemma 2.1 so $\mathscr{J}_2[x; \alpha_k, \beta_k] \leqq 0$ and since, by (3.4), $\mathscr{J}_2[x; \alpha_k, \beta_k] = J_2[Hx; \alpha_k, \beta_k]$, it follows that $J_2[Hx; \alpha_k, \beta_k] \leqq 0$.

The following corollary is a trivial consequence of the theorem obtained on taking H to be E, the $n \times n$ identity matrix, and observing that $R(t) - \lambda_1(R(t))E$ and $\lambda_n(P(t))E - P(t)$ are nonnegative definite matrices.

COROLLARY 4.2. *Let $R(t) > 0$ and $Q(t) \equiv 0$. If (1.1) is oscillatory then the scalar equation*

$$((\lambda_1(R(t)))y')' + (\lambda_n(P(t)))y = 0$$

is oscillatory.

We remark that the converse of this theorem is not true as can be seen from the example following Corollary 3.5.

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