HEREDITARILY STRUCTURALLY COMPLETE EXTENSIONS OF **RM**

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Abstract. This paper focuses on the structurally complete extensions of the system **R**-mingle (**RM**). The main theorem demonstrates that the set of all hereditarily structurally complete extensions of **RM** is countably infinite and forms an almost-chain, with only one branching element. As a corollary, we show that the set of structurally complete extensions of **RM** that are not hereditary is also countably infinite and forms a chain. Using algebraic methods, we provide a complete description of both sets. Furthermore, we offer a characterization of passive structural completeness among the extensions of **RM**: specifically, we prove that a quasivariety of Sugihara algebras is passively structurally complete if and only if it excludes two specific algebras. As a corollary, we give an additional characterization of quasivarieties of Sugihara algebras that are passively structurally complete but not structurally complete. We close the paper with a characterization of actively structurally complete quasivarieties of Sugihara algebras.

§1. Informal introduction. The notion of structural completeness (**SC**) was introduced by W. A. Pogorzelski in [24]. A consequence relation is **SC** if all of its admissible rules are derivable. The distinction between the admissibility and derivability of a rule lies in the extent of its applicability within a given consequence relation. An admissible rule is one that "works" on the theorems of a given logic, meaning it guarantees that its conclusion is a theorem whenever all its premises are theorems. A derivable rule, on the other hand, is a rule in the "standard sense," meaning it can be applied to an arbitrary set of formulas. Thus, roughly speaking, in a given consequence relation, φ follows from a set of premises Σ if there exists a finite subset Γ of Σ such that Γ/φ is a derivable rule in the system. Clearly, if we are working within a fixed deductive system, every derivable rule is also admissible. However, numerous counterexamples demonstrate that the converse is not generally true. Two paradigmatic examples of non-derivable admissible rules are the Harrop's rule

$$\frac{\neg\varphi \to (\psi \vee \chi)}{(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)}$$

for intuitionistic logic **Int** and the disjunctive syllogism rule DS

$$\frac{\neg\varphi \vee \psi, \varphi}{\psi}$$

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for the system of relevance logic **R**. Adding these rules to their respective systems does not affect the sets of theorems but alters the consequence relations. Thus, neither of these two logics is structurally complete (**SC**).

To emphasize the importance of the **SC** property, let us recall two major formal definitions of the notion of "a logic." The first treats it as a mere set of theorems, usually generated from an initial set of axioms via certain rules. In this approach, there is no distinction between derivable and admissible rules, as these rules are applied only to theorems. The second definition views a logic as a consequence relation—a binary relation between sets of formulas (premises) and single formulas (conclusions). In a sense, the phenomenon of structural incompleteness underscores the superiority of the second approach, as two logics can share identical sets of theorems but differ when viewed as consequence relations.[1] The distinction between derivable and admissible rules, which is only possible within the framework of consequence relations, is precisely why the second definition is more nuanced.

Hereditary structural completeness (**HSC**) is a stronger form of the standard structural completeness (**SC**). A consequence relation is said to be **HSC** if and only if it is **SC** and all of its extensions are also **SC**. In this paper, we will investigate the problem of hereditary structural completeness within extensions of the logic **R**-mingle (**RM**), which is the most well-known and widely studied extension of Belnap and Anderson's system of relevance logic **R**.

The system **RM** is obtained by adding the "mingle" axiom $p \to (p \to p)$ to **R**. Dunn established in [12] that Sugihara algebras provide an adequate algebraic semantics for **RM**. The theory of Sugihara algebras will be crucial to our investigation, as we approach the problem using the powerful framework of abstract algebraic logic. Our final theorem will demonstrate that the structure of the poset of all hereditarily structurally complete extensions of **RM** is the converse of the well-ordering ω^+ with an additional element adjoined above 1:

The problem of (hereditary) structural completeness has often been explored in the context of non-classical logics. So far, results regarding **SC** have been confined to axiomatic systems. A well-known theorem proven by Citkin in [10] characterizes **HSC** superintuitionistic logics (axiomatic extensions of **Int**) in the following manner: a given variety of Heyting algebras is **HSC** if and only if it omits five specific algebras; this is further equivalent to the statement that there are exactly five maximal non-**HSC** superintuitionistic logics. Similarly, Rybakov's theorem from [27] indicates that, in the case of axiomatic extensions of modal logic **K4**, there are twenty such algebras.

[1] If two logics have the same theorems, they also have the same admissible rules. Thus, the difference must be in a rule that is admissible in both but derivable in only one.

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While these characterizations are sophisticated and undoubtedly elegant, they can be seen as somewhat roundabout; they do not directly identify the **HSC** extensions themselves but rather highlight the algebras that must be omitted.[2] In the current work, we will provide a direct characterization of the **HSC** extensions of **RM**: we will construct the poset of the respective quasivarieties and describe the algebras that generate them. In the final part of the paper, we will also examine a weaker version of **SC**—passive structural completeness—and obtain a Citkin-style characterization for passively structurally complete quasivarieties of Sugihara algebras. In this case, only two algebras will need to be omitted.

Structural completeness has been widely investigated among substructural logics [9, 20, 23]. In particular, the problem of **SC** has been addressed for certain fragments of **RM** with Ackermann's constant **t** added to the original signature ($\mathbf{RM^t}$) in [22, 23]. The positive (negation-free) fragment of $\mathbf{RM^t}$ is **HSC** [22], while the purely implicational fragment of **RM** is not **SC** (see [23]). However, we will consider **RM** in its original signature.

The most pertinent result related to our inquiry is the theorem proven by Raftery and Świrydowicz in [19], which states that there are no **SC** axiomatic extensions of **R** other than classical propositional logic (**CPL**) and inconsistent logic. Since **RM** is an axiomatic extension of **R**, it follows that neither **RM** nor any of its axiomatic extensions—excluding **CPL** and inconsistent logic—is **SC**. In this paper, we will tackle a more general problem: we will investigate arbitrary extensions of **RM**, not limited to axiomatic ones.

§2. Preliminaries. We assume a countably infinite set of propositional variables, denoted by $\mathsf{Var} = \{p_1, p_2, p_3, \ldots\}$ (in practice, we will use letters $p, q, r, \ldots$). A language $\mathcal{L}$ is a set of symbols; specifically, $\mathcal{L} = \mathsf{Var} \cup \mathsf{Con}$, where Con is the set of logical connectives $\{\neg, \vee, \wedge, \to, \leftrightarrow\}$. As usual, $\neg$ is unary and the rest of the connectives are binary. The set of formulas $\mathcal{F}$ consists of properly constructed finite strings of symbols from $\mathcal{L}$; that is, $p, \neg\varphi, \varphi * \psi$, where $*$ is any binary connective. A uniform substitution is a map from $\mathcal{F}$ to $\mathcal{F}$ that preserves logical operations, i.e., an endomorphism of the absolutely free formula algebra. We define $\vdash \subseteq 2^{\mathcal{F}} \times \mathcal{F}$ as a consequence relation if it satisfies the following properties:

- reflexive: $\Sigma \vdash \sigma$ for any $\sigma \in \Sigma$;
- monotonic: if $\Sigma \vdash \sigma$, then $\Sigma \cup \Gamma \vdash \sigma$;
- transitive: if $\Gamma \vdash \varphi$ and for any $\gamma \in \Gamma$ we have $\Sigma \vdash \gamma$, then $\Sigma \vdash \varphi$;
- structural: if $\Sigma \vdash \varphi$, then $s(\Sigma) \vdash s(\varphi)$ for any uniform substitution s;
- finitary: $\Sigma \vdash \varphi$ if and only if $\Sigma' \vdash \varphi$ for some finite subset Σ' of Σ.

Elements of $2^{\mathcal{F}} \times \mathcal{F}$ whose first component is finite will be called rules. Instead of writing $\langle \Gamma, \varphi \rangle$, we will adopt the following notation: Γ/φ, where $\Gamma \cup \{\varphi\}$ is a finite subset of $\mathcal{F}$. If a given rule Γ/φ is a member of a consequence relation $\vdash$, then we say that Γ/φ is a derivable rule of $\vdash$. According to our definition, all consequence relations are finitary, so we can identify a given consequence relation with the set of its derivable rules. A derivable rule of a given consequence relation $\vdash$ of the form

[2] It is worth noting that due to the complexity of the lattice of superintuitionistic logics, no other description may be attainable.

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$\emptyset/\varphi$ is called a theorem of $\vdash$ and is shortly denoted by φ. Each consequence relation is a set of rules, some of which may be theorems. For two consequence relations $\vdash_0, \vdash_1 \subseteq 2^{\mathcal{F}} \times \mathcal{F}$, we say that $\vdash_1$ is an extension of $\vdash_0$ if $\vdash_0 \subseteq \vdash_1$. It is easy to see that the intersection of an arbitrary set of consequence relations is also a consequence relation: $\bigcap_{i \in I} \vdash_i$ is reflexive, monotonic, transitive, structural, and finitary, given that each $\vdash_i$ has these properties. Thus, for any set of rules X, there exists a least consequence relation $\vdash$ that contains X. In such cases, we will say that X defines $\vdash$. A rule Γ/φ is admissible in $\vdash$ if and only if, for any substitution σ, $\sigma(\varphi)$ is a theorem of $\vdash$ whenever $\sigma(\Gamma)$ is a set of theorems of $\vdash$. Furthermore, a rule Γ/φ is passive in $\vdash$ iff there is no uniform substitution σ, such that $\sigma(\Gamma)$ is a set of theorems of $\vdash$. Thus, each passive rule of $\vdash$ is vacuously admissible. A given consequence relation $\vdash$ is structurally complete (**SC**) iff each admissible rule in $\vdash$ is derivable. Furthermore, a consequence relation is hereditarily structurally complete (**HSC**) when all of its extensions are **SC**. Consequence relations that are **SC** but not **HSC** (i.e., those that have at least one extension that is not **SC**) will be called non-hereditarily structurally complete (**nHSC**).

It is natural to associate a closure operator C with a given consequence relation $\vdash$ in the following way: $\varphi \in C(X)$ if and only if $X \vdash \varphi$ for any $X \subseteq \mathcal{F}$. It has also been common to first define a structural and finitary closure operator as a primitive notion and then treat the consequence operation as a secondary entity. However, it seems more natural to talk about derivable rules in the context of consequence relations (simply as their members). Since the notions of consequence operator and consequence relation are two sides of the same coin, we decide to stick to the latter, treating it as the fundamental notion that captures the meaning of the term 'logic'.

A Horn formula, or quasi-identity as it is also called, is a first-order sentence of the form $\forall_{x_1,\dots,x_l}(\varphi_1(x_1,\dots,x_l) \approx \psi_1(x_1,\dots,x_l) \& \dots \& \varphi_n(x_1,\dots,x_l) \approx \psi_n(x_1,\dots,x_l) \Longrightarrow \varphi(x_1,\dots,x_l) \approx \psi(x_1,\dots,x_l)$. In practice, we will omit the quantifiers. Identities are quasi-identities with an empty antecedent. Let Φ be a quasi-equation and $\mathfrak{A}$ be an algebra. $\mathfrak{A} \models \Phi$ means that Φ holds in $\mathfrak{A}$. For a class of algebras, we write $\mathsf{K} \models \Phi$ to indicate that Φ holds in each member of K. We will use standard notation $\mathsf{H}, \mathsf{I}, \mathsf{S}, \mathsf{P}$, and $\mathsf{P_U}$ for the well-known algebraic closure operators defined on arbitrary classes of algebras. Thus, given a class of algebras K, $\mathsf{H}(\mathsf{K}), \mathsf{I}(\mathsf{K}), \mathsf{S}(\mathsf{K}), \mathsf{P}(\mathsf{K}), \mathsf{P_U}(\mathsf{K})$ are K's closures under homomorphic images, isomorphic copies, subalgebras, direct products, and ultraproducts, respectively. For two algebras $\mathfrak{A}$ and $\mathfrak{B}$, we will write $\mathfrak{A} \cong \mathfrak{B}$ when $\mathfrak{A} \in \mathsf{I}(\mathfrak{B})$. Given a class of algebras K, $\mathbf{F}_{\mathsf{K}}(\kappa)$ will denote the κ-generated algebra that is free in $\mathsf{ISP}(\mathsf{K})$. We will also write $\mathsf{V}(\mathsf{K})$ for $\mathsf{HSP}(\mathsf{K})$ and $\mathsf{Q}(\mathsf{K})$ for $\mathsf{ISPP_U}(\mathsf{K})$. We will use obvious notation for finite direct powers: $\mathfrak{A}^1 = \mathfrak{A}$, and $\mathfrak{A}^{n+1} = \mathfrak{A}^n \times \mathfrak{A}$. To indicate that an algebra $\mathfrak{A}$ is embeddable in (isomorphic to a subalgebra of) $\mathfrak{B}$, i.e., $\mathfrak{A} \in \mathsf{IS}(\mathfrak{B})$, we will use the following notation: $\mathfrak{A} \preceq \mathfrak{B}$. We will also write $\mathfrak{A} \prec \mathfrak{B}$ when $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{A} \ncong \mathfrak{B}$. If $\mathsf{K} = \mathsf{V}(\mathsf{K})$, then it is called a variety; if $\mathsf{K} = \mathsf{Q}(\mathsf{K})$, then it is a quasivariety. Arbitrary varieties will be referred to as $\mathcal{V}$ and quasivarieties as $\mathcal{Q}$.[3] Varieties are definable by identities, and quasivarieties by quasi-identities.

[3]Remember that Q, V stand for closure operators and $\mathcal{Q}, \mathcal{V}$ for classes of algebras. The usage will also be clear from the context, so we believe that it will not lead to misunderstandings.

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Now, we will recall some results concerning the algebraic counterpart of the structural completeness property, which were first formulated by Bergman in [5].[4] For a consequence relation, being structurally complete is equivalent to having no proper extensions with the same set of theorems. To see that, let $\vdash_0 \subsetneq \vdash_1$ be two consequence relations, where the second properly extends the first, with the additional property that they have the same theorems, i.e., $\vdash_0 \varphi$ if and only if $\vdash_1 \varphi$ for any $\varphi \in \mathcal{F}$. Hence, there is a rule Γ/ψ in $\vdash_1$ that does not belong to $\vdash_0$. However, Γ/ψ is admissible in $\vdash_0$ (otherwise, the set of theorems would be different). Hence, $\vdash_0$ is not **SC**. For the other direction, let $\vdash_0$ be such that each of its proper extensions also properly extends the set of its theorems. If there is a rule Γ/ψ that is an admissible but non-derivable rule of $\vdash_0$, then it is easy to see that $\vdash_1$, defined by the set of rules $\vdash \cup\{\langle\Gamma, \psi\rangle\}$, properly extends $\vdash_0$ and has exactly the same theorems, contradicting the initial assumption. Thus, in algebraic terms—assuming Blok–Pigozzi algebraizability, which 'reverses the order'—we have the following:

FACT 2.1. *A given quasivariety $\mathcal{Q}$ is* **SC** *if and only if, for any quasivariety $\mathcal{Q}' \subsetneq \mathcal{Q}$, it is the case that* $\mathsf{H}\mathcal{Q}' \subsetneq \mathsf{H}\mathcal{Q}$.

Thus, the **SC** quasivarieties are precisely those that are generated by their ω-generated free algebras. A given quasivariety is **HSC** if and only if all of its subquasivarieties are **SC**—analogously, a consequence relation is **HSC** if and only if it has only **SC** extensions. Not surprisingly, a given **SC** quasivariety is **nHSC** when it has at least one subquasivariety that is not **SC**.

§3. RM and Sugihara algebras. Let $\mathfrak{A}_\omega = \langle \mathbb{Z}; \wedge, \vee, \neg, \rightarrow\rangle$, where $\mathbb{Z}$ is the set of integers, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, $\neg k = - k$ and for $\rightarrow$ we have

$$k \rightarrow l = \begin{cases} - k \vee l, & \text{if } k \leq l; \\ - k \wedge l, & \text{otherwise.} \end{cases}$$

It is evident that there are two types of finite subalgebras of $\mathfrak{A}_\omega$: those that contain 0 and those that do not. The former will be termed "odd," while the latter will be referred to as "even." Consequently, we introduce the following notation for $n \geq 1$:

$$\mathfrak{A}_{2n} = \langle\{- n, \dots, - 1, 1, \dots, n\}; \wedge, \vee, \neg, \rightarrow\rangle,$$
$$\mathfrak{A}_{2n+1} = \langle\{- n, \dots, - 1, 0, 1, \dots, n\}; \wedge, \vee, \neg, \rightarrow\rangle.$$

$\mathfrak{A}_1$ is the trivial one element algebra. $\mathfrak{A}_2$ is a two element boolean algebra. An infinite proper subalgebra of $\mathfrak{A}_\omega$ will be denoted by $\mathfrak{A}_{\omega\setminus\{0\}} = \langle \mathbb{Z} \setminus \{0\}; \wedge, \vee, \neg, \rightarrow\rangle$. An algebra $\mathfrak{A}$ will be referred to as Sugihara algebra when $\mathfrak{A} \in \mathsf{V}(\mathfrak{A}_\omega) = \mathsf{Q}(\mathfrak{A}_\omega)$.

Anderson and Belnap's [4, p. 341] system of relevance logic **R** is defined by 13 axioms and two rules:

A1 $p \rightarrow p$
A2 $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$
A3 $p \rightarrow ((p \rightarrow q) \rightarrow q)$

[4] Bergman's work predates the seminal Blok–Pigozzi paper [7]. Thus, the notion of 'algebraic counterpart' of a logic, as used by him, is an intuitive one. Systematic investigations of **SC** within the framework of modern abstract algebraic logic can be found in [26].

A4 $(p \to (p \to q)) \to (p \to q)$
A5 $p \wedge q \to p$
A6 $p \wedge q \to q$
A7 $((p \to q) \wedge (p \to r)) \to (p \to q \wedge r)$
A8 $p \to p \vee q$
A9 $p \to q \vee p$
A10 $((q \to p) \wedge (r \to p)) \to (q \vee r \to p)$
A11 $p \wedge (q \vee r) \to (p \wedge q) \vee r$
A12 $(p \to \neg q) \to (q \to \neg p)$
A13 $\neg\neg p \to p$

The two rules of the system is modus ponens MP $\{p, p \to q\}/q$ and the adjunction rule AD $\{p, q\}/p \wedge q$.

The logic **R**–mingle (**RM**) is obtained by adding the "mingle axiom" $p \to (p \to p)$ to the axiomatic system of relevant logic **R**. We will consider **RM** as a consequence relation rather than merely as a set of theorems. The logic **RM** is algebraizable in the sense of [7] with respect to the quasivariety of Sugihara algebras, using the set of formulas $\{p \to q, q \to p\}$ and the equation $p \approx p \to p$. For a given finitary rule $R = \Gamma/\varphi$, where $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ we can say that Sugihara algebra $\mathfrak{A}$ satisfies R if $\mathfrak{A} \vDash \gamma_1 \approx \gamma_1 \to \gamma_1 \& \cdots \& \gamma_n \approx \gamma_n \to \gamma_n \Longrightarrow \varphi \approx \varphi \to \varphi$. In such cases, we will use the abbreviated notation $\mathfrak{A} \vDash R$ and say that $\mathfrak{A}$ satisfies the rule R. It is straightforward to observe that, to check whether R holds in $\mathfrak{A}$, we do not need to translate R into a quasi-equation; instead, we can treat $\mathfrak{A}$ as a logical matrix with the set of designated elements $\{a \in \mathsf{A} : a = a \to a\}$. We naturally define an absolute value of an element $a \in \mathfrak{A}$ as $|a| := a \to a$. We define deductive filters $F \subseteq \mathsf{A}$ as standard lattice filters such that the set of designated values is included in F; that is, $\{a \in \mathsf{A} : a = |a|\} \subseteq F$. Thus, as a consequence of algebraizability, we have an order isomorphism between the lattices of deductive filters and congruences for a given Sugihara algebra.

Now, we will gather results on Sugihara algebras from [6, 11, 12, 21], which will be crucial in the proof of the theorem.

Let us start with two important results that can be found in [6, p. 275]

THEOREM 3.1 (Blok and Dziobiak, 1985). *Sugihara algebras are locally finite, i.e., each Sugihara subalgebra generated by a finite set is finite.*

THEOREM 3.2 (Blok and Dziobiak, 1985). *Algebras of the form $\mathfrak{A}_n$, $n < \omega$ are the only—up to isomorphism—finite subdirectly irreducible algebras.*

As a consequence of these, we can restate the theorem of Dunn [12, Theorem 9, p. 9] in its purely algebraic version:

THEOREM 3.3 (Dunn, 1970). $\mathsf{V}(\mathfrak{A}_1) \subseteq \mathsf{V}(\mathfrak{A}_2) \subseteq \mathsf{V}(\mathfrak{A}_3) \subseteq \mathsf{V}(\mathfrak{A}_4) \subseteq \cdots \subseteq \mathsf{V}(\mathfrak{A}_\omega) = \mathsf{V}(\{\mathfrak{A}_n : n \in \omega\})$.

We can already observe that Sugihara algebras enjoy some nice properties. They are locally finite, their finite subdirectly irreducibles are chains, and the lattice of subvarieties of Sugihara algebras form an ω^+ well ordering. Consequently, due to the Blok–Pigozzi algebraizability, the lattice of axiomatic extensions of **RM** is the converse of ω^+ well-ordering. As will become evident below, there is also an elegant characterization of directly indecomposable Sugihara algebras.

Definition 3.4. Let $\mathfrak{A} = \langle \mathsf{A}; \wedge, \vee, \neg, \rightarrow\rangle$ be a Sugihara algebra. $\bot\mathfrak{A}\top = \langle \mathsf{A} \cup \{\bot, \top\}; \wedge, \vee, \neg, \rightarrow\rangle$, where $\bot$ and $\top$ are added as lower and upper bounds, $\wedge$ and $\vee$ are interpreted standardly in the resulting lattice, $\neg a = \neg^{\mathfrak{A}} a$ for $a \in \mathsf{A}$, $\neg\top = \bot$, $\neg\bot = \top$ and:

$$a \rightarrow b = \begin{cases} a \rightarrow^{\mathfrak{A}} b, & \text{if } a, b \in \mathsf{A}, \\ \top, & \text{if } a = \bot \text{ or } b = \top, \\ \bot, & \text{otherwise.} \end{cases}$$

We also write $\bot^{n+1}\mathfrak{A}\top^{n+1}$ to indicate $\bot\bot^n\mathfrak{A}\top^n\top$. Note that $\top, \bot$ is assumed to be disjoint from A in $\bot\mathfrak{A}\top$, so we must distinguish between the different 'tops' and 'bottoms' in algebras of the form $\bot^n\mathfrak{A}\top^n$ where $n \geq 2$. To clarify, we will use $\bot_i$ and $\top_i$ to denote the ith bottom and top, respectively. Following this notation, $\bot^n\mathfrak{A}\top^n$ becomes $\bot_n \ldots \bot_2\bot_1\mathfrak{A}\top_1\top_2 \ldots \top_n$, where $\top_i \leq \top_j$ and $\bot_j \leq \bot_i$ in terms of lattice ordering for $i \leq j$.

It is immediate to see that for any subdirectly irreducible finite Sugihara algebra $\mathfrak{A}$, $\bot\mathfrak{A}\top$ is also subdirectly irreducible, i.e., $\bot\mathfrak{A}_n\top \cong \mathfrak{A}_{n+2}$. Next, we will recall a characterization of DI Sugihara algebras [6, Corollary 2.5, p. 278].

Theorem 3.5 (Blok and Dziobiak, 1985). *A finite Sugihara algebra $\mathfrak{A}$ is directly indecomposable iff $\mathfrak{A} \cong \mathfrak{A}_2$ or $\mathfrak{A} \cong \bot\mathfrak{B}\top$, where $\mathfrak{B}$ is a finite Sugihara algebra.*

Thus, DI Sugihara algebras are simply arbitrary Sugihara algebras extended with disjoint top and bottom elements. However, the most important tool for our investigations will be critical algebras. A finite algebra $\mathfrak{A}$ is said to be critical if and only if $\mathfrak{A} \notin \mathsf{ISP}(\mathfrak{B} : \mathfrak{B} \prec \mathfrak{A})$. It is well known that every locally finite quasivariety is generated by its critical algebras (cf. [18]). Since Sugihara algebras are locally finite, we will use critical algebras as fundamental building blocks for quasivarieties. A specific description of critical Sugihara algebras was provided in [11, p. 285], and this will serve as a crucial tool in our investigations.

Theorem 3.6 (Czelakowski and Dziobiak, 1999). *If a Sugihara algebra $\mathfrak{A}$ is critical, then it is isomorphic to an algebra of one of the following four types*:

1. $\mathfrak{A}_k$;
2. $\mathfrak{A}_{2i} \times \mathfrak{A}_k$, *where* $2_i \neq k$;
3. $\bot^{p_1}\mathfrak{A}_{2k_1} \times \ldots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \ldots \top^{p_1}$, *where* $2k_n < k$ *if* k *is even, and* $k_n < k$ *and* $k_i < k_{i+1} + p_{i+1}$ *for all* $1 \leq i \leq n-1$ *whenever* $2 \leq n$;
4. $\mathfrak{A}_{2k_0} \times \mathfrak{B}$ *where* $\mathfrak{B}$ *is of type* 3 *and* $k_0 < k_1 + p_1$.

It is easy to see that algebras of the third type take the form $\bot^m\mathfrak{A}\top^m$, where $\mathfrak{A}$ is of the fourth type. To familiarize the reader with these types, we provide several examples: $\bot^2\mathfrak{A}_4 \times \mathfrak{A}_6\top^2$ is an algebra of the third type, while $\mathfrak{A}_6 \times \bot\mathfrak{A}_8 \times \mathfrak{A}_5\top$ and $\mathfrak{A}_8 \times \bot^2\mathfrak{A}_6 \times \mathfrak{A}_5\top^2$ are of the fourth type. Additionally, $\bot^{11}(\mathfrak{A}_6 \times \bot\mathfrak{A}_8 \times \mathfrak{A}_5\top)\top^{11}$ is of the third type.

On the other hand, the following algebras do not satisfy this description: $\bot^3\mathfrak{A}_3 \times \mathfrak{A}_5\top^3$—at most one algebra in the construction can be odd; $\bot\mathfrak{A}_4 \times \mathfrak{A}_4\top$—algebras within the construction cannot be the same; and $\mathfrak{A}_8 \times \bot^2\mathfrak{A}_2 \times \mathfrak{A}_3\top^2$—here, $4 \nless 1 + 2$, contrary to the description.

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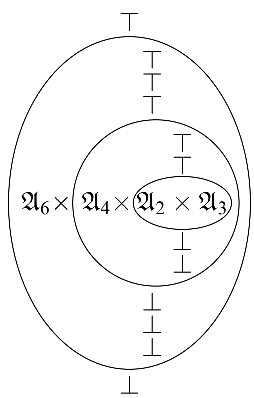
Furthermore, in the context of nested product operations, the "horizontal" notation used in third- and fourth-type algebras quickly becomes difficult to read. It is hard to determine which algebras are extended by the respective tops and bottoms. Parentheses provide some clarification, but we argue that the vertical "quasi-Hasse" notation is easier to read. For example, consider the algebra $\bot(\mathfrak{A}_6 \times (\bot^3(\mathfrak{A}_4 \times (\bot^2\mathfrak{A}_2 \times \mathfrak{A}_3\top^2))\top^3))\top$. Let us present this in a vertical notation, somewhat resembling a Hasse diagram:



The scope of the extending tops and bottoms is indicated by the nearest ellipse. Fortunately, we will later introduce a method that simplifies these nested algebras, eliminating some notational problems in the subsequent sections.

One can observe that in algebras of the third and fourth type, for any $i < n$, we have $\mathfrak{A}_{2k_i} \preceq \bot^{p_{i+1}}\mathfrak{A}_{2k_{i+1}} \times \ldots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \ldots \top^{p_{i+1}}$. While this is straightforward to verify directly, the observation also follows from three key facts:

- if $\bot\mathfrak{B}\top$ is critical, then $\mathfrak{B}$ is critical,
- if $\mathfrak{A} \times \mathfrak{B}$ is critical, then both $\mathfrak{A}$ and $\mathfrak{B}$ are critical, and
- the Lemma 3.4 from [11] which states that if $\mathfrak{A} \times \mathfrak{B}$ is critical, then it is isomorphic to $\mathfrak{A}_{2n} \times \bot\mathfrak{C}\top$ where $\mathfrak{A}_{2n}$ is embeddable in $\bot\mathfrak{C}\top$ under assumption that $\bot\mathfrak{C}\top$ is not subdirectly irreducible.

We will also use the following result from [21, p. 1250]:

THEOREM 3.7 (Krawczyk, 2022). *$\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3)$ and $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_4)$ are the only covers of $\mathsf{Q}(\mathfrak{A}_2)$ (the variety of Boolean algebras) in the lattice of subquasivarieties of $\mathsf{Q}(\mathfrak{A}_\omega)$ and $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3) \neq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_4)$.*

Thus, we know that the lattice of quasivarieties of Sugihara algebras is more complex than that of varieties: it does not form a chain, as it contains incomparable elements. As the theorem shows, such elements already appear at the very bottom of the lattice.

§4. The theorem and its proof. Our ultimate goal is to isolate the poset of all **HSC** consequence relations extending **RM**. To achieve this, we begin by describing all structurally complete quasivarieties of Sugihara algebras. Next, within the set of **SC** quasivarieties, we distinguish the **SC** quasivarieties that possess the additional

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property of hereditariness from those that do not. The fundamental theorem we aim to prove takes the following form:

THEOREM 4.1. *The set of all* **HSC** *subquasivarieties of Sugihara algebras is*

$$\{\mathsf{Q}(\mathfrak{A}_1), \mathsf{Q}(\mathfrak{A}_2), \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3)\} \cup \{\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2k}) : k \geq 2\} \cup \{\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})\}.$$

In the general algebraic setting, an approach to the problem of (hereditary) structural completeness based on projectivity has proven to be fruitful [2, 3]. However, as has already become apparent, we have a rather rich theory of Sugihara algebras at our disposal, and thus our strategy will differ. In one way or another, most of the proofs that follow will rely on the algebras from Theorem 3.6, since locally finite quasivarieties are known to be generated by their critical members [18]. However, the algebras of the third and fourth types in Theorem 3.6 are rather complex and difficult to handle. To ease the process, we will simplify certain instances of these algebras by providing their $\mathcal{Q}$-equivalent descriptions. As will become apparent later, this simplification reveals that Theorem 3.6 cannot be strengthened into a full characterization (note that it presents only a necessary condition for criticality, as it is framed as an implication). $\mathcal{Q}$-equivalence, or Horn equivalence, as we will also refer to it, is a weaker version of the standard model-theoretic notion of elementary equivalence. As the reader likely knows, or suspects, two algebras are Horn equivalent if they satisfy the same quasi-identities, rendering them indistinguishable from one another from the perspective of quasivarieties. To make things precise: by a quasi-identical theory of a given algebra $\mathfrak{A}$, we understand the set $\mathsf{Th_q}(\mathfrak{A}) := \{\varphi : \mathfrak{A} \vDash \varphi,\ \varphi \text{ is a Horn formula}\}$. The definition of Horn-equivalence has the following form:

DEFINITION 4.2 ($\mathcal{Q}$-equivalence). Let $\mathfrak{A}, \mathfrak{B}$ be algebras of the same similarity type. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{Q}$-equivalent (symbolically: $\mathfrak{A} \approx_{\mathsf{q}} \mathfrak{B}$) iff $\mathsf{Th_q}(\mathfrak{A}) = \mathsf{Th_q}(\mathfrak{B})$.

We start with a simple observation.

FACT 4.3. *For any algebras* $\mathfrak{A}, \mathfrak{B}$ *of the same type, the following statements are equivalent*:

1. $\mathfrak{A} \approx_{\mathsf{q}} \mathfrak{B}$.
2. $\mathsf{ISPP_U}(\mathfrak{A}) = \mathsf{ISPP_U}(\mathfrak{B})$.
3. *For any quasivariety* $\mathcal{Q}$, *we have* $\mathfrak{A} \in \mathcal{Q}$ *iff* $\mathfrak{B} \in \mathcal{Q}$.

The equivalence of the three statements follows directly from the fact that quasivarieties are defined by quasi-identities. In the case where both algebras are finite, we can modify the second statement to $\mathsf{ISP}(\mathfrak{A}) = \mathsf{ISP}(\mathfrak{B})$. We are now ready to prove the key "simplification" lemma. This lemma will be used to show that certain algebras of the third type are Horn-equivalent to a product of two Sugihara chains.

LEMMA 4.4. *Let* $n \in \omega$ *and* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *all be Sugihara algebras such that* $\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{C}$. *Then* $\mathfrak{A} \times \perp^n\mathfrak{B} \times \mathfrak{C}\top^n \approx_{\mathsf{q}} \mathfrak{A} \times \perp^n\mathfrak{C}\top^n$.

PROOF. Without the loss of generality, we assume that $\mathsf{A} \subseteq \mathsf{B} \subseteq \mathsf{C}$. We will show that $\mathsf{Q}(\mathfrak{A} \times \perp^n\mathfrak{C}\top^n) = \mathsf{Q}(\mathfrak{A} \times \perp^n\mathfrak{B} \times \mathfrak{C}\top^n)$.

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For the left-to-right inclusion we prove that $\mathfrak{A} \times \bot^n\mathfrak{C}\top^n \preceq \mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n$. The embedding e is given simply by

$$\begin{aligned} (a,c) &\mapsto (a,(a,c)) && \text{for } (a,c) \in \mathsf{A} \times \mathsf{C}; \\ (a,x_i) &\mapsto (a,x_i) && \text{for } (a,x_i) \in \mathsf{A} \times \{\bot_i, \top_i\}, 1 \leq i \leq n. \end{aligned}$$

Is is easy to see that such a function is indeed an injective homomorphism. Thus $\mathfrak{A} \times \bot^n\mathfrak{C}\top^n \in \mathsf{S}(\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n) \subseteq \mathsf{Q}(\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n)$, which implies $\mathsf{Q}(\mathfrak{A} \times \bot^n\mathfrak{C}\top^n) \subseteq \mathsf{Q}(\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n)$.

For the opposite inclusion, we will show that $\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n \preceq \mathfrak{A}^2 \times (\bot^n\mathfrak{C}\top^n)^2 \cong (\mathfrak{A} \times \bot^n\mathfrak{C}\top^n)^2$. Now the embedding e is defined by

$$\begin{aligned} (a,(b,c)) &\mapsto ((a,a),(b,c)) && \text{for } (a,b,c) \in \mathsf{A} \times \mathsf{B} \times \mathsf{C}; \\ (a,x_i) &\mapsto ((a,a),(x_i,x_i)) && \text{for } (a,x_i) \in \mathsf{A} \times \{\bot_i, \top_i\}, 1 \leq i \leq n. \end{aligned}$$

It is obvious that e is injective. To see that e preserves $\neg$, let $(a,b,c) \in \mathsf{A} \times \mathsf{B} \times \mathsf{C}$. $e(\neg(a,(b,c))) = e((\neg a,(\neg b,\neg c))) = ((\neg a,\neg a),(\neg b,\neg c)) = \neg((a,a),(b,c)) = \neg e((a,(b,c)))$. The case when $(a,x_i) \in \mathsf{A} \times \{\bot_i, \top_i\}$ is equally trivial by the fact that $\neg\bot_i = \top_i$ and $\neg\top_i = \bot_i$. For binary operations we check only $\to$ since $\vee, \wedge$ are trivial. Let $(a,b,c),(a',b',c') \in \mathsf{A} \times \mathsf{B} \times \mathsf{C}$. We will show two cases as examples and leave the rest for the reader since they can be shown in the same manner.

For the first example let: $e((a,(b,c))) \to e((a',(b',c'))) = ((a,a),(b,c)) \to ((a',a'),(b',c')) = ((a \to a', a \to a'),(b \to b', c \to c')) = e((a,(b,c))) \to (a',(b',c')))$.

To prove one more case: $e((a',\bot_i)) \to e((a,(b,c))) = ((a',a'),(\bot_i,\bot_i)) \to ((a,a),(b,c)) = ((a' \to a, a' \to a),(\top_i,\top_i)) = e((a' \to a, \top_i)) = e((a',\bot_i) \to (a,(b,c)))$.

Thus, we have shown $\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n \in \mathsf{SP}(\mathfrak{A} \times \bot^n\mathfrak{C}\top^n) \subseteq \mathsf{Q}(\mathfrak{A} \times \bot^n\mathfrak{C}\top^n)$. This proves the right-to-left inclusion.

As we have proven the equality of the two quasivarieties, the statement from the lemma follows immediately by the Fact 4.3. ⊣

As a consequence of the above lemma, we can assert that Theorem 3.6 cannot be strengthened to an equivalence. Given the assumption of Lemma 4.4, we have $\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n \in \mathsf{Q}(\mathfrak{A} \times \bot^n\mathfrak{C}\top^n)$, which implies that $\mathfrak{A} \times \bot^n\mathfrak{B} \times \mathfrak{C}\top^n$ is not critical. For a concrete example, consider $\mathfrak{A}_4 \times \bot\mathfrak{A}_6 \times \mathfrak{A}_8\top$, which is clearly an algebra of the fourth type. Yet, by Lemma 4.4, we have $\mathfrak{A}_4 \times \bot\mathfrak{A}_6 \times \mathfrak{A}_8\top \in \mathsf{Q}(\mathfrak{A}_4 \times \mathfrak{A}_{10})$, and $\mathfrak{A}_4 \times \mathfrak{A}_{10}$ is a proper subalgebra of $\mathfrak{A}_4 \times \bot\mathfrak{A}_6 \times \mathfrak{A}_8\top$. This shows that $\mathfrak{A}_4 \times \bot\mathfrak{A}_6 \times \mathfrak{A}_8\top$ is not critical.

However, the authors seem to acknowledge the potential incompleteness of their description, referring to it as "a certain, satisfactory-for-our-purpose description" [11, p. 281]. This description remains satisfactory for our purposes as well, as it will suffice to prove the main theorem.

It is known that for every **SC** quasivariety $\mathcal{Q}$, we have $\mathcal{Q} = \mathsf{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$. However, free algebras on finite generators are usually complicated, and ω-generated ones are even more so. Therefore, we seek an alternative description of **SC** Sugihara quasivarieties—specifically, simpler, possibly finite algebras that generate them. By Theorem 3.3, there are countably infinitely many varieties of Sugihara algebras, each generated by a single algebra of the form $\mathfrak{A}_\alpha$, where $\alpha \leq \omega$. Thus, we can partition the lattice of subquasivarieties of Sugihara algebras into a countable set of equivalence classes, with each class corresponding to the algebras that generate the same variety. In this framework, **SC** quasivarieties are precisely the minimal elements of each equivalence class, represented by $\mathsf{V}(\mathfrak{A}_\alpha)$ for $\alpha \leq \omega$.

In practice, the case where $\alpha = \omega$ turns out to be special, so we will first address the finite cases, where $n < \omega$, and then move on to the remaining infinite case.

DEFINITION 4.5. Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quasivarieties of Sugihara algebras. We define an equivalence relation on quasivarieties of Sugihara algebras putting: $\mathcal{Q}_1 \sim \mathcal{Q}_2$ iff $\mathsf{H}(\mathcal{Q}_1) = \mathsf{H}(\mathcal{Q}_2)$.

In order to determine the equivalence class of a given quasivariety, we introduce the notion of the degree of a given critical Sugihara algebra.

DEFINITION 4.6. Let $\mathfrak{A}$ be isomorphic to one of the four types of algebras from Theorem 3.6. We define the degree of $\mathfrak{A}$, $deg(\mathfrak{A})$ to be:

$$deg(\mathfrak{A}) = \begin{cases} n, & \text{if } \mathfrak{A} \cong \mathfrak{A}_n, \\ max(\{2i, k\}), & \text{if } \mathfrak{A} \cong \mathfrak{A}_{2i} \times \mathfrak{A}_k, \\ max(\{2k_n, k\}) + 2(p_n + \dots + p_1), & \text{if } \mathfrak{A} \cong \mathfrak{B}, \\ max(\{2k_n, k\}) + 2(p_n + \dots + p_1), & \text{if } \mathfrak{A} \cong \mathfrak{A}_{2k_0} \times \mathfrak{B}, \end{cases}$$

where $\mathfrak{B} = \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^{p_n} \dots \top^{p_1}$.

Observe that the notion of the degree of a given Sugihara algebra does not coincide with the standard lattice-theoretic notion of height (i.e., the cardinality of the longest chain in a lattice). To illustrate this, consider $\mathfrak{B} = \bot\mathfrak{A}_2 \times \mathfrak{A}_3\top$. By definition 4.6, $deg(\mathfrak{B}) = 5$. However, the set $\{\bot, (-1,0), (0,0), (1,0), (1,1), \top\} \subseteq \mathsf{B}$ forms a six-element subchain of $\mathfrak{B}$.

By local finiteness of Sugihara algebras, we have the following:

LEMMA 4.7. *Let $m \in \omega$. Let* K *be a class of critical algebras.* $\mathsf{Q}(\mathsf{K}) \in [\mathsf{V}(\mathfrak{A}_m)]_{/\sim}$ *iff there is a critical algebra $\mathfrak{A} \in \mathsf{K}$ such that $deg(\mathfrak{A}) = m$ and there is no critical algebra $\mathfrak{B}$ in* K *such that $deg(\mathfrak{B}) > m$.*

PROOF. First we will show that for any critical algebra $\mathfrak{A}$ s.t. $deg(\mathfrak{A}) = m$, we have $\mathsf{V}(\mathfrak{A}) = \mathsf{V}(\mathfrak{A}_m)$. The cases when $\mathfrak{A}$ is of the first and second type (see Theorem 3.6) are trivial. Hence assume that $\mathfrak{A} \cong \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_1}$. We will show that 1) $\mathfrak{A}_m \in \mathsf{H}(\mathfrak{A})$ and 2) $\mathfrak{A} \in \mathsf{V}(\mathfrak{A}_m)$.

For 1) we will proceed inductively. Let $n = 1$. Thus $\mathfrak{A} \cong \bot^{p_1}\mathfrak{A}_{2k_0} \times \mathfrak{A}_k\top^{p_1}$. Let $z = max\{2k_0, k\}$. Obviously $\mathfrak{A}_z \in \mathsf{H}(\mathfrak{A}_{2k_0} \times \mathfrak{A}_k)$. It is equally easy to see that $\mathfrak{A}_m \cong \bot^{p_1}\mathfrak{A}_z\top^{p_1} \in \mathsf{H}(\bot^{p_1}\mathfrak{A}_{2k_0} \times \mathfrak{A}_k\top^{p_1})$. Now let $\mathfrak{A} \cong \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_1}$, where $n \geq 2$. By inductive hypothesis $\mathfrak{A}_{m-p_1} \in \mathsf{H}(\bot^{p_2}\mathfrak{A}_{2k_2} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_2})$. Thus $\mathfrak{A}_{m-p_1} \in \mathsf{H}(\mathfrak{A}_{2k_1} \times \bot^{p_2}\mathfrak{A}_{2k_2} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_2})$, since $\mathfrak{A}_{2k_1}$ is embeddable in $\bot^{p_2}\mathfrak{A}_{2k_2} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_2}$. But then it is easy to see that $\mathfrak{A}_m \cong \bot^{p_1}\mathfrak{A}_{m-p_1}\top^{p_n} \in \mathsf{H}(\bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k\top^{p_n} \dots \top^{p_1})$.

For 2) notice that for any $\mathfrak{B} \in \mathsf{V}(\mathfrak{A}_l)$ it is the case that $\bot\mathfrak{B}\top \in \mathsf{V}(\bot\mathfrak{A}_l\top)$, because if $\mathfrak{B}$ is a subdirect product of $\{\mathfrak{A}_{l_i} : i \in I, l_i \leq l\}$, then $\bot\mathfrak{B}\top$ is a subdirect product of $\{\bot\mathfrak{A}_{l_i}\top : i \in I, l_i \leq l\}$. Also, if $\mathfrak{B} \in \mathsf{V}(\mathfrak{A}_l)$ and $\mathfrak{C} \preceq \mathfrak{B}$, then $\mathfrak{B} \times \mathfrak{C} \in \mathsf{V}(\mathfrak{A}_l)$.

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We start from the fact that $\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \in \mathsf{V}(\mathfrak{A}_z)$, where $z = max(\{2k_n, k\})$. Then we combine two previous observations using them alternately to get 2).

The result for the remaining fourth type of algebras follows immediately from the previous case.

Now, to prove the lemma, let K be a class of critical algebras, and $\mathsf{Q}(\mathsf{K}) \in [\mathsf{V}(\mathfrak{A}_m)]_{/\sim}$. Hence $\mathsf{K} \subseteq \mathsf{Q}(\mathsf{K}) \subseteq \mathsf{HQ}(\mathsf{K}) = \mathsf{V}(\mathfrak{A}_m)$. If there is $k > m$, $\mathfrak{B} \in \mathsf{K}$ such that $deg(\mathfrak{B}) = k$, then $\mathfrak{A}_k \in \mathsf{H}(\mathfrak{B}) \subseteq \mathsf{V}(\mathfrak{A}_m)$ which cannot be the case, since it contradicts Theorem 3.3. If, on the other hand, for each $\mathfrak{B} \in \mathsf{K}$, $deg(\mathfrak{B}) < m$, then let $z = max(\{deg(\mathfrak{B}) : \mathfrak{B} \in \mathsf{K}\})$. Obviously $z < m$, so $\mathsf{Q}(\mathsf{K}) \sim \mathsf{V}(\mathfrak{A}_z)$ which contradicts the assumption. This gives us the right-hand side of the equivalence from the lemma.

For the other direction, assume that there is a critical algebra $\mathfrak{A} \in \mathsf{K}$ such that $deg(\mathfrak{A}) = m$ and there is no critical algebra $\mathfrak{B}$ in K such that $deg(\mathfrak{B}) > m$. From the first conjunct we know that $\mathsf{V}(\mathfrak{A}_m) \subseteq \mathsf{HQ}(\mathsf{K})$. From the second conjunct we get $\mathsf{K} \subseteq \mathsf{V}(\mathfrak{A}_m)$. Thus $\mathsf{Q}(\mathsf{K}) \sim \mathsf{V}(\mathfrak{A}_m)$. ⊣

In Lemma 4.7, we addressed all finite cases, i.e., the members of the equivalence classes $[\mathsf{V}(\mathfrak{A}_m)]/\sim$ for $m \in \omega$. We now turn to the infinite case, involving the members of $[\mathfrak{A}_\omega]/\sim$.

LEMMA 4.8. *Let* K *be a class of critical algebras.* $\mathsf{Q}(\mathsf{K}) \in [\mathsf{V}(\mathfrak{A}_\omega)]_{/\sim}$ *iff for each* $n \in \omega$ *there is a critical algebra* $\mathfrak{A} \in \mathsf{K}$ *such that* $deg(\mathfrak{A}) \geq n$.

PROOF. For the left to right direction assume that there is $n \in \omega$ s.t. there is no $m \geq n$, s.t. $deg(\mathfrak{A}) = m$ and $\mathfrak{A} \in \mathsf{K}$. But then by Lemma 4.7, we get $\mathsf{Q}(\mathsf{K}) \subseteq \mathsf{V}(\mathfrak{A}_m)$ which means that $\mathsf{Q}(\mathsf{K}) \notin [\mathsf{V}(\mathfrak{A}_\omega)]_{/\sim}$.

For the right to left assume that for each $n \in \omega$ there is a critical algebra $\mathfrak{A} \in \mathsf{K}$ such that $deg(\mathfrak{A}) \geq n$. Then, using the fact that for each critical $\mathfrak{B}$, s.t. $deg(\mathfrak{B}) = m$ we have $\mathfrak{A}_m \in \mathsf{H}(\mathfrak{B})$, we obtain that for each n there is $m \geq n$ s.t. $\mathfrak{A}_m \in \mathsf{H}(\mathsf{K})$. But also $\mathfrak{A}_n \in \mathsf{HS}(\mathfrak{A}_m)$ for each $n \leq m$. This way we have shown that for each n, $\mathfrak{A}_n \in \mathsf{V}(\mathsf{K})$, which proves the Lemma. ⊣

Lemmas 4.7 and 4.8 establish an equivalent condition for membership in a given equivalence class, which ultimately depends on the degree of critical algebras. Based on this result, we will characterize each **SC** quasivariety as being generated by a single (in most cases finite) algebra.

LEMMA 4.9. *For any* $m \geq 2$, $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_m)$ *is* **SC**.

PROOF. It is enough to show that for any non-trivial quasivariety $\mathcal{Q}$ of Sugihara algebras such that: $\mathcal{Q} \in [\mathsf{V}(\mathfrak{A}_m)]_{/\sim}$, $(m \geq 2)$, we have $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_m) \subseteq \mathcal{Q}$. By Lemma 4.7, there is a critical $\mathfrak{A} \in \mathcal{Q}$ s.t. $deg(\mathfrak{A}) = m$. Obviously $\mathfrak{A}_2 \in \mathcal{Q}$. By Theorem 3.6, there are four cases to consider.

If $\mathfrak{A} \cong \mathfrak{A}_m$, then obviously $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathcal{Q}$.

If $\mathfrak{A} \cong \mathfrak{A}_j \times \mathfrak{A}_m$ where $j < m$ and at least one of these numbers is even, then $\mathfrak{A}_2 \times \mathfrak{A}_m \preceq \mathfrak{A}_j \times \mathfrak{A}_m$, so $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathcal{Q}$.

Let $\mathfrak{A} \cong \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^{p_n} \dots \top^{p_1}$. Obviously $\mathfrak{A}_2 \times \mathfrak{A} \in \mathcal{Q}$. We have $\mathfrak{A}_2 \preceq \mathfrak{A}_{2k_1}$ and $\mathfrak{A}_{2k_i} \preceq \bot^{p_{i+1}}\mathfrak{A}_{2k_{i+1}} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^{p_n} \dots \top^{p_{i+1}}$ for each $i < n$, so we can apply Lemma 4.4 $n-1$ times and get $\mathfrak{A}_2 \times \mathfrak{A} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^l$, where $l = p_1 + \dots + p_n$. Obviously $\mathfrak{A}_2$ is embeddable in both $\mathfrak{A}_{2k_n}$ and $\mathfrak{A}_k$. If $2k_n > k$, then

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$m = 2(k_n + l)$ and $\mathfrak{A}_2 \times \perp^l \mathfrak{A}_2 \times \mathfrak{A}_{2k_n} \top^l \preceq \mathfrak{A}_2 \times \perp^l \mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^l$, so $\mathfrak{A}_2 \times \perp^l \mathfrak{A}_2 \times \mathfrak{A}_{2k_n} \top^l \in \mathcal{Q}$ and we can apply Lemma 4.4 one more time: $\mathfrak{A}_2 \times \perp^l \mathfrak{A}_2 \times \mathfrak{A}_{2k_n} \top^l \approx_{\mathsf{q}} \mathfrak{A}_2 \times \perp^l \mathfrak{A}_{2k_n} \top^l \cong \mathfrak{A}_2 \times \mathfrak{A}_m$. Hence, $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathcal{Q}$. If, on the other hand, $2k_n < k$, then 4.4 can be applied immediately, since $\mathfrak{A}_{2k_n} \preceq \mathfrak{A}_k$, and thus we get $\mathfrak{A}_2 \times \mathfrak{A} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \perp^l \mathfrak{A}_k \top^l \cong \mathfrak{A}_2 \times \mathfrak{A}_m$. Again, we obtain $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathcal{Q}$.

Finally, if $\mathfrak{A}$ is of the fourth type, that is $\mathfrak{A} \cong \mathfrak{A}_{2k_0} \times \mathfrak{B}$, where $\mathfrak{B}$ is of the third type, then $\mathfrak{A}_2 \times \mathfrak{B} \preceq \mathfrak{A}$ and we can repeat the reasoning from the previous case. ⊣

Once again, we address the infinite case separately.

LEMMA 4.10. $\mathsf{Q}(\{\mathfrak{A}_2 \times \mathfrak{A}_{2m} : m \in \omega\}) = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega \setminus \{0\}})$ *is* **SC**.

PROOF. First we prove equality, then we will move on to structural completeness. Inclusion from the left to right is trivial. To see that $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega \setminus \{0\}}) \subseteq \mathsf{Q}(\{\mathfrak{A}_2 \times \mathfrak{A}_{2m} : m \in \omega\})$ let U be a non-principal ultrafilter over ω. It can be proven that $\mathfrak{A}_2 \times \mathfrak{A}_{\omega \setminus \{0\}}$ is embeddable in $(\Pi_{n \in \omega}(\mathfrak{A}_2 \times \mathfrak{A}_{2n}))_{/U}$. To see that, first define a function f from $\{-1, 1\} \times \mathbb{Z} \setminus \{0\}$ into the direct product: $\Pi_{n \in \omega}(\{-1, 1\} \times \{-n, \dots, -1, 1, \dots, n\})$ by

$$f((x, n))(m) = \begin{cases} (1, 1), & \text{if } m < n, \\ (x, n), & \text{otherwise.} \end{cases}$$

Now e given by $e((x, n)) = [f((x, n))]_{/U}$ can be easily seen to be an embedding of $\mathfrak{A}_2 \times \mathfrak{A}_{\omega \setminus \{0\}}$ into $\Pi_{n \in \omega}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})_{/U}$.

Let $\mathsf{Q}(\mathsf{K}) \in [\mathsf{V}(\mathfrak{A}_\omega)]_{/\sim}$, where K is a class of critical algebras. By Lemma 4.8 for each n there is $\mathfrak{A} \in \mathsf{K}$ such that $deg(\mathfrak{A}) \geq n$. By the reasoning from 4.9 we know that there is $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathsf{Q}(\mathfrak{A})$, where $n \leq m = deg(\mathfrak{A})$. Thus $\mathfrak{A}_2 \times \mathfrak{A}_m \in \mathsf{Q}(\mathsf{K})$. Also for any $2j = l \leq m$, $\mathfrak{A}_2 \times \mathfrak{A}_l \preceq \mathfrak{A}_2 \times \mathfrak{A}_m$, which means that $\mathsf{Q}(\{\mathfrak{A}_2 \times \mathfrak{A}_{2m} : m \in \omega\}) \subseteq \mathsf{Q}(\mathsf{K})$. ⊣

By combining Lemmas 4.9 and 4.10, we can now state that the set of all **SC** quasivarieties of Sugihara algebras is $\{\mathsf{Q}(\mathfrak{A}_1), \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega \setminus \{0\}})\} \cup \{\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_n) : 2 \leq n \in \omega\}$. As will become apparent later, this subposet of the entire lattice of quasivarieties of Sugihara algebras is illustrated in Figure 1. We are now ready to eliminate some of the **SC** quasivarieties from Figure 1 as non-hereditary.

LEMMA 4.11. *For each* $n \geq 2$, *it is the case that* $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n+1})$ *is not* **HSC**.

PROOF. By Theorem 3.7, we know that $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3, \mathfrak{A}_2 \times \mathfrak{A}_4)$ is not **SC**, since $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3, \mathfrak{A}_2 \times \mathfrak{A}_4) \supsetneq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_4)$ and $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3, \mathfrak{A}_2 \times \mathfrak{A}_4) \sim \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_4)$. Also, $\mathfrak{A}_2 \times \mathfrak{A}_3, \mathfrak{A}_2 \times \mathfrak{A}_4 \preceq \mathfrak{A}_2 \times \mathfrak{A}_{2n+1}$ for $n \geq 2$ which means that $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3, \mathfrak{A}_2 \times \mathfrak{A}_4) \subsetneq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n+1})$ and consequently falsifies the heredity of **SC** for $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n+1})$. ⊣

We have spotted infinitely many quasivarieties which are **nHSC**. To prove that the remaining **SC** extensions are hereditary, let us define three special rules.

$$\{p, \neg p\}/q \tag{NP}$$

$$\{p, \neg p \vee q\}/q \tag{DS}$$

$$\{p, q, \neg(p \to q)\}/r \tag{R}$$

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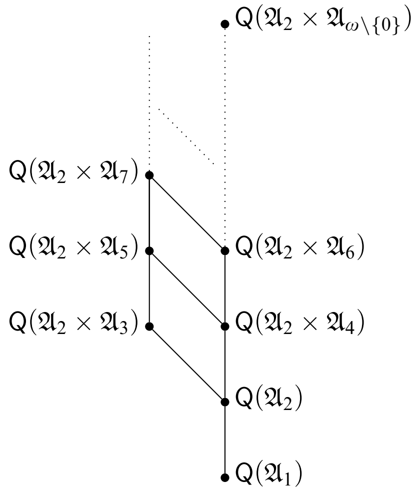

FIGURE 1. Structurally complete quasivarieties of Sugihara algebras.

We have a non-paraconsistency rule $\mathtt{NP}$, disjunctive syllogism $\mathtt{DS}$ and a special rule $\mathtt{R}$. Due to algebraizability, these rules are equivalent to respective quasi-identities—here, we chose the 'logical notation' over the 'algebraic' one for the sake of readability.

LEMMA 4.12. *It is the case that*:

$$\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \vDash \mathtt{NP}, \mathtt{DS}, \mathtt{R}$$
$$\mathfrak{A}_3 \nvDash \mathtt{NP}$$
$$\mathfrak{A}_4 \nvDash \mathtt{R}$$
$$\mathfrak{A}_2 \times \mathfrak{A}_3 \nvDash \mathtt{DS}$$

PROOF. Designated values (positive elements) on $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}$ are elements of the form $(1, k)$ where $k \geq 1$. It is easy to see that for any interpretation i s.t. $i(p) = (1, k)$, it must be that case that $i(\neg p) = (-1, -k)$ and thus the premises of $\mathtt{NP}$ are not satisfiable, which means $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \vDash \mathtt{NP}$. On the other hand, if we take the interpretation i s.t. $i(p) = 0$ and $i(q) = -1$ on $\mathfrak{A}_3$, we see that $\mathfrak{A}_3 \nvDash \mathtt{NP}$.

Now, if we take any interpretation such that $i(p) = (1, j)$ and $i(q) = (1, j)$, for $j, k \geq 1$, we immediately see that $i(p \to q) = (1, j \to k)$ and thus $i(\neg(p \to q)) = (-1, \neg(j \to k))$, which shows that the set of premises of $\mathtt{R}$ are not satisfiable in $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}$ and consequently $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \vDash \mathtt{R}$. Now let $i(p) = 2, i(q) = 1, i(r) = -1$ in $\mathfrak{A}_4$. Hence $i(\neg(p \to q)) = -(2 \to 1) = --2 = 2$. This interpretation falsifies $\mathtt{R}$ in $\mathfrak{A}_4$, i.e., $\mathfrak{A}_4 \nvDash \mathtt{R}$.

Finally, let $i(p) = (1, k)$ where $k \geq 1$. Hence $i(\neg p) = (-1, -k)$ which means that in order for $i(\neg p \vee q)$ to be positive, it must be the case that $i(q) = (1, j)$ for some $j \geq 1$. This gives us $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \vDash \mathtt{DS}$. Lastly we show that $\mathfrak{A}_2 \times \mathfrak{A}_3 \nvDash \mathtt{DS}$: take i s.t. $i(p) = (1, 0)$, $i(q) = (1, -1)$. ⊣

LEMMA 4.13. *Let $\mathcal{Q}$ be a quasivariety of Sugihara algebras and $n \in \omega$. If $\mathcal{Q} \subseteq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})$ and $\mathcal{Q} \neq \mathsf{Q}(\mathfrak{A}_1)$, then $\mathcal{Q} = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2m})$, where $m \leq n$.*

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Proof. First observe that $\mathfrak{A}_2 \times \mathfrak{A}_{2n} \preceq \mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}$, so $\mathfrak{A}_2 \times \mathfrak{A}_{2n} \vDash \mathtt{NP}, \mathtt{DS}, \mathtt{R}$. Let $\mathfrak{A} \in \mathcal{Q} \subseteq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})$ be critical. We can assume that $\mathfrak{A}$ is not a Boolean algebra, since $\mathsf{Q}(\mathfrak{A}_2) = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_2)$. If $\mathfrak{A} \cong \mathfrak{A}_k$, then either $\mathfrak{A}_3 \preceq \mathfrak{A}$ or $\mathfrak{A}_4 \preceq \mathfrak{A}$. By Lemma 4.12, it means that either $\mathfrak{A} \nvDash \mathtt{NP}$ or $\mathfrak{A} \nvDash \mathtt{R}$. Hence $\mathfrak{A} \notin \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})$, so it must be the case that $\mathfrak{A} \ncong \mathfrak{A}_k$ for any k. If $\mathfrak{A} \cong \mathfrak{A}_{2k} \times \mathfrak{A}_m$ and $\mathfrak{A} \ncong \mathfrak{A}_2 \times \mathfrak{A}_j$ for any j, then $m > 2$ and $k > 1$, so either $\mathfrak{A}_4 \preceq \mathfrak{A}$ (when m is even) or $\mathfrak{A}_2 \times \mathfrak{A}_3 \preceq \mathfrak{A}$ (when m is odd), hence either $\mathfrak{A} \nvDash \mathtt{R}$ or $\mathfrak{A} \nvDash \mathtt{NP}$ which by Lemma 4.12 cannot be the case. If $\mathfrak{A} \cong \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^{p_n} \dots \top^{p_1}$, then—due to the fact that for any non-trivial finite $\mathfrak{B}$, it is the case that $\mathfrak{A}_4 \preceq \bot\mathfrak{B}\top$—we get $\mathfrak{A}_4 \preceq \mathfrak{A}$ which, as we have previously showed, cannot be the case. If $\mathfrak{A} \cong \mathfrak{A}_{2k_0} \times \bot^{p_1}\mathfrak{A}_{2k_1} \times \dots \bot^{p_n}\mathfrak{A}_{2k_n} \times \mathfrak{A}_k \top^{p_n} \dots \top^{p_1}$, then either $k_0 > 1$, or $k_0 = 1$. If $k_0 > 1$, then $\mathfrak{A}_4 \preceq \mathfrak{A}$—contradiction. If $k_0 = 1$, then by Lemma 4.4, $\mathfrak{A} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k_n} \times \mathfrak{A}_n \top^l$, where $l = p_1 + \dots + p_n$. If n is odd, then it can be easily seen that $\mathfrak{A}_2 \times \mathfrak{A}_3 \preceq \mathfrak{A}_2 \times \bot^l\mathfrak{A}_2 \times \mathfrak{A}_3\top^l \preceq \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k_n} \times \mathfrak{A}_n\top^l$, hence again $\mathfrak{A} \nvDash \mathtt{DS}$—contradiction. If $n = 2m$, then $\mathfrak{A}_2 \preceq \mathfrak{A}_{2k_0} \preceq \mathfrak{A}_n$ which by Lemma 4.4 gives us $\mathfrak{A} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \mathfrak{A}_{2(m+l)}$.

This way, we have shown that all critical Sugihara algebras from $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_n)$ are Horn-equivalent to algebras of the form $\mathfrak{A}_2 \times \mathfrak{A}_{2m}$, which proves the lemma. ⊣

Lemma 4.14. *If $\mathcal{Q}$ is a non-trivial quasivariety of Sugihara algebras and $\mathcal{Q} \subsetneq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})$, then $\mathcal{Q} = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})$ for some $n \in \omega$.*

Proof. Let $\mathcal{Q} \subseteq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})$ be a quasivariety and $\mathcal{Q} \supseteq \mathsf{K}$ be its class of critical algebras. Obviously $\mathcal{Q} = \mathsf{Q}(\mathsf{K})$. By previous reasoning, it must be the case that members of K are of the form $\mathfrak{A}_2 \times \mathfrak{A}_{2n}$. Since $\mathcal{Q}$ is properly included in $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})$ it must be the case that $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \notin \mathcal{Q}$, which proves the Lemma. ⊣

The only remaining step for the main result to follow is to prove the following:

Lemma 4.15. $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3)$ *is* **HSC**.

Proof. By Lemma 4.9, we already know that $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3)$ is **SC**. By Theorem 3.7, we know that it is **HSC** as its only subquasivarieties are the Boolean algebras and the trivial class. ⊣

The fundamental Theorem 4.1 follows from Lemmas 4.9–4.11 and 4.13–4.15. Additionally, we have an immediate corollary stating that the quasivarieties from Lemma 4.11 are the only ones that are **nHSC**. Formally, from Lemmas 4.9–4.11, and Theorem 4.1, we derive the following corollary:

Corollary 4.16. *The set of all* **nHSC** *quasivarieties of Sugihara algebras is* $\{\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2k+1}) : k \geq 2\}$.

The content of Theorem 4.1, along with Corollary 4.16, is depicted in Figure 2.

§5. Passive structural completeness. Finally, let us note that there is a weaker version of the **SC** property discussed in the literature. This is the concept of overflow, or passive structural completeness, as formulated in [30]. A logical system (or a quasivariety) is said to be overflow complete if all of its passive admissible rules are derivable. It has been shown that a quasivariety is overflow complete if and only if all of its non-trivial members satisfy exactly the same positive sentences [30, Fact 2, p. 68]. Recall that a positive first-order sentence is one constructed using

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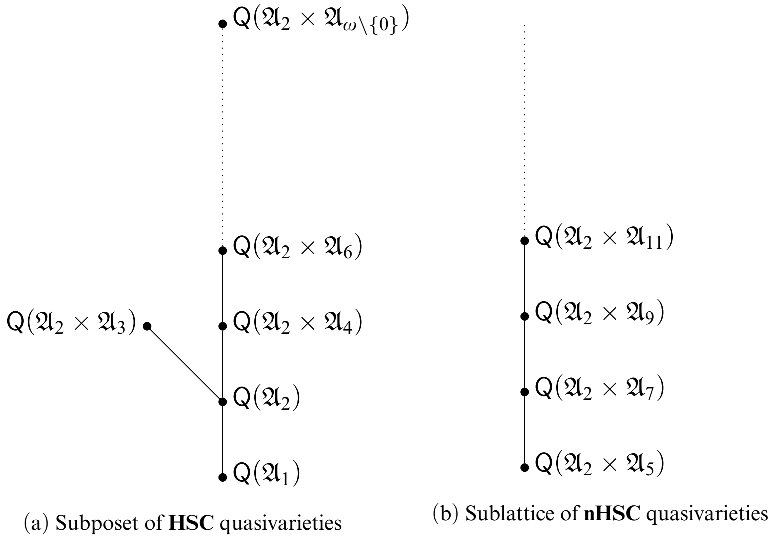

FIGURE 2. Structurally complete quasivarieties of Sugihara algebras.

the existential quantifier $\exists$, the falsum constant $\mathbf{f}$, and the connectives of conjunction & and disjunction $\veebar$. Using the previously proven lemmas, along with some new ones, we can provide a Citkin-style characterization of overflow complete extensions of **RM** in terms of two ‘omitting’ algebras.

In order to prove the theorem, we shall first recall the original Wroński characterization of overflow completeness, and then modify it for our purposes. Standardly, by $\mathsf{Th}(\mathsf{K})$ we will understand a first-order theory of a class of structures K.

FACT 5.1 (Wroński, 2009). *Let $\mathcal{Q}^+$ be a class of all non-trivial members of a quasivariety $\mathcal{Q}$. The following conditions are equivalent*:

1. *$\mathcal{Q}$ is overflow complete.*
2. $\mathsf{Th}(\mathcal{Q}^+) \cap \{\varphi, \neg\varphi\} \neq \emptyset$, *for every positive sentence φ.*

We can slightly weaken the assumption of the above characterization.

FACT 5.2. *Let $\mathcal{Q} = \mathsf{Q}(\mathsf{K})$, where K does not contain a trivial algebra. Let $\mathsf{K}^* = \mathsf{K} \cup \{\mathbf{F}_{\mathcal{Q}}(\omega)\}$. The following conditions are equivalent*:

1. *$\mathcal{Q}$ is overflow complete.*
2. $\mathsf{Th}(\mathsf{K}^*) \cap \{\varphi, \neg\varphi\} \neq \emptyset$, *for every positive sentence φ.*

PROOF. One direction follows immediately from the original characterization. The second implication is just a matter of rewriting Wroński's original proof—the weaker assumption is just enough. For the sake of self-containment let us include it here. Assume 2. Let R be a passive rule which is equivalent to a quasi-identity $\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n \Longrightarrow x \approx y$. Further, let R be admissible for $\mathcal{Q} = \mathsf{Q}(\mathsf{K})$. This means that $\mathbf{F}_{\mathcal{Q}}(\omega) \nvDash \exists_{x_1,\ldots,x_k}(\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n)$, where $x_1, \ldots, x_k$ exhaust

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all of the variables from the conjunction of identities. From assumption, we get $\mathsf{K} \not\models \exists_{x_1,\dots,x_k}(\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n)$ which means that R is derivable. $\dashv$

In the first formulation, proving overflow completeness comes down to showing positive equivalence of all non-trivial members of a given quasivariety. In the second variant, we can restrict ourselves to the generating class plus the free algebra on infinitely many free generators.

LEMMA 5.3. *For any $n, m \in \omega$, $\mathfrak{A}_2 \times \mathfrak{A}_n$ is positively equivalent to $\mathfrak{A}_2 \times \mathfrak{A}_m$.*

PROOF. Without the loss of generality assume that $n \leq m$. If both n and m are even or if m is odd, then $\mathfrak{A}_2 \times \mathfrak{A}_n \in \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_m)$. But by Lemma 4.9 $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_m)$ is **SC**. Thus, it is also overflow complete which by Fact 5.1 means that all of its non-trivial members are positively equivalent. The remaining case is when n is odd and m is even. But then $\mathfrak{A}_2 \times \mathfrak{A}_n, \mathfrak{A}_2 \times \mathfrak{A}_m \in \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{m+1})$ and we can repeat the reasoning. $\dashv$

LEMMA 5.4. *Let $\mathcal{Q}$ be a quasivariety of Sugihara algebras, n a natural number. $\mathbf{F}_{\mathcal{Q}}(\omega)$ is positively equivalent to $\mathfrak{A}_2 \times \mathfrak{A}_n$.*

PROOF. Since $\mathcal{Q} \sim \mathcal{V} = \mathsf{V}(\mathfrak{A}_\alpha)$ for some ordinal $\alpha \leq \omega$, it must be the case that $\mathbf{F}_{\mathcal{Q}}(\omega) = \mathbf{F}_{\mathcal{V}}(\omega)$. If α is a natural number k, then $\mathbf{F}_{\mathcal{V}}(\omega) \in \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_k)$ and $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_k)$ is **SC**, so also overflow complete and thus by Fact 5.1 $\mathfrak{A}_2 \times \mathfrak{A}_k$ is positively equivalent to $\mathbf{F}_{\mathcal{V}}(\omega)$. But by Lemma 5.3 $\mathfrak{A}_2 \times \mathfrak{A}_k$ is positively equivalent to $\mathfrak{A}_2 \times \mathfrak{A}_n$, so the result follows. If, on the other hand, $\alpha = \omega$, then $\mathbf{F}_{\mathcal{V}}(\omega) \in \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}) = \mathsf{Q}(\{\mathfrak{A}_2 \times \mathfrak{A}_{2n} : n \geq 0\})$ which is **SC** and we repeat the reasoning. $\dashv$

Now we can move on to the proof of the theorem:

THEOREM 5.5. *A subquasivariety of Sugihara algebras $\mathcal{Q}$ is overflow complete iff $\mathfrak{A}_3 \notin \mathcal{Q}$ and $\mathfrak{A}_4 \notin \mathcal{Q}$.*

PROOF. For the left to right direction assume that either $\mathfrak{A}_4$ or $\mathfrak{A}_3$ belong to $\mathcal{Q}$. By the Lemma 4.10 it must be the case that $\mathfrak{A}_2 \times \mathfrak{A}_n \in \mathcal{Q}$ for some $n \leq \omega$. If $\mathfrak{A}_3 \in \mathcal{Q}$, then define $\varphi := \exists x(x = \neg x)$ and[5] note that $\mathfrak{A}_3 \Vdash \varphi$ while $\mathfrak{A}_2 \times \mathfrak{A}_n \nVdash \varphi$. If $\mathfrak{A}_4 \in \mathcal{Q}$, then let $\psi := \exists x \exists y\,(|x| \approx x \,\&\, |y| \approx y \,\&\, |\neg(x \to y)| = \neg(x \to y))$. Observe that $\mathfrak{A}_4 \Vdash \psi$ and $\mathfrak{A}_2 \times \mathfrak{A}_n \nVdash \psi$. This means that $\mathcal{Q}$ is not overflow complete.

For the opposite direction assume that $\mathfrak{A}_3 \notin \mathcal{Q}$ and $\mathfrak{A}_4 \notin \mathcal{Q}$. We will show that all of its critical algebras are of the form $\mathfrak{A}_2 \times \mathfrak{A}_n$ for some $n \in \omega$. Obviously $\mathcal{Q}$ cannot contain any chains longer than $\mathfrak{A}_2$. Any algebra of the second type from the Theorem 3.6 must be of the form $\mathfrak{A}_2 \times \mathfrak{A}_n$ (otherwise $\mathfrak{A}_4 \in \mathcal{Q}$). If $\mathfrak{B}$ is a non-trivial finite algebra, then $\mathfrak{A}_4 \preceq \bot\mathfrak{B}\top$, so any algebra of the third type cannot be included in $\mathcal{Q}$. If an algebra is of the fourth type, then by the already established reasoning (Lemmas 4.4 and 4.9) it has to be Horn-equivalent to an algebra of the form $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k} \times \mathfrak{A}_j\top^l$ (otherwise $\mathfrak{A}_4 \in \mathcal{Q}$). If $2k \leq j$, then Lemma 4.4 applies again and we get an equivalent algebra $\mathfrak{A}_2 \times \mathfrak{A}_{j+2l}$. If $j < 2l$ and j is even, then the situation is completely analogous. Lastly, if j is odd and $j < 2k$, then both $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k}\top^l$ and $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_j\top^l$ are subalgebras of $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k} \times \mathfrak{A}_j\top^l$. For the first embedding take $e_1(i,a) = (i,(a,0))$, if $a \in \mathfrak{A}_{2k}$ and

[5] '$\neg$' is a symbol for an algebraic operation here—not a negation connective, so φ is positive.

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$e_1(i,x) = (i,x)$, if $x \in \{\bot_1, \dots, \bot_l, \top_1, \dots, \top_l\}$. For the embeddability of the second algebra observe that by Lemma 4.4 $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_j\top^l \preceq \mathfrak{A}_2 \times \bot^l\mathfrak{A}_2 \times \mathfrak{A}_j\top^l \preceq \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k} \times \mathfrak{A}_j\top^l$. But then $\mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k} \times \mathfrak{A}_j\top^l \preceq \mathfrak{A}_2 \times \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k}\top^l \times \bot^l\mathfrak{A}_j\top^l \cong \mathfrak{A}_2 \times \bot^l\mathfrak{A}_{2k}\top^l \times \mathfrak{A}_2 \times \bot^l\mathfrak{A}_j\top^l$ Thus $\mathcal{Q}$ is generated by the algebras of the form $\mathfrak{A}_2 \times \mathfrak{A}_n$, so we can apply Lemmas 5.3 and 5.4 and Fact 5.2 to conclude the proof. $\dashv$

On the basis of our main Theorems 4.1 and 5.5, we can characterize those quasivarieties of Sugihara algebras which are overflow complete but not **SC**.

COROLLARY 5.6. *A quasivariety $\mathcal{Q}$ of Sugihara algebras is overflow complete but not* **SC** *iff either*:

(i) $\mathcal{Q} = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n+1}, \mathfrak{A}_2 \times \mathfrak{A}_{2m})$ *for some* $m > n$, *or*
(ii) $\mathcal{Q} = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}, \mathfrak{A}_2 \times \mathfrak{A}_{2n+1})$ *for some* n, *or*
(iii) $\mathcal{Q} = \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega})$.

§6. Active structural completeness. Another variant of **SC** is active or almost structural completeness (**ASC**), a concept recently introduced and algebraically developed in [13]. This notion can be seen as complementary to **PSC**. A consequence relation is said to be **ASC** if its only admissible non-derivable rules are passive.

We will adopt a strategy similar to the one used previously: first, we will modify the original algebraic characterization of **ASC** to align with the techniques established in the proof of the main theorem. Then, we will apply these methods to characterize all **ASC** subquasivarieties of Sugihara algebras and, subsequently, to isolate those that are **ASC** but not **SC**.

Among the many equivalent algebraic properties for a quasivariety to be **ASC**, we recall the condition most useful for our purposes [13, Theorem 3.1, p. 532]:

THEOREM 6.1 (Dzik and Stronkowski, 2016). *A quasivariety $\mathcal{Q}$ is* **ASC** *iff for any* $\mathfrak{A} \in \mathcal{Q}$, $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(\omega) \in \mathsf{ISPP_U}(\mathbf{F}_{\mathcal{Q}}(\omega))$.

Once again, we will modify the theorem in a manner similar to the approach taken for **PSC**: restricting $\mathfrak{A}$ to be a critical algebra and replacing the free algebra with the product of a two-element Boolean algebra and a characteristic chain. This leads to the following modified version:

LEMMA 6.2. *Let $\mathcal{Q}$ be a quasivariety of Sugihara algebras such that $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_n)$ for some $n \in \omega$. Then $\mathcal{Q}$ is* **ASC** *iff for any critical $\mathfrak{B} \in \mathcal{Q}$, $\mathfrak{B} \times \mathfrak{A}_2 \times \mathfrak{A}_n \in \mathsf{ISPP_U}(\mathfrak{A}_2 \times \mathfrak{A}_n)$. If $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_\omega)$, then $\mathcal{Q}$ is* **ASC** *iff for any critical $\mathfrak{B} \in \mathcal{Q}$, $\mathfrak{B} \times \mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \in \mathsf{ISPP_U}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})$.*

PROOF. Let $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_n)$.

Assume that $\mathcal{Q}$ is **ASC**. Let $q := \varphi \Rightarrow s(x_1, \dots, x_n) \approx t(x_1, \dots, x_n)$ where $\varphi := s_0(x_1, \dots, x_n) \approx t_0(x_1, \dots, x_n) \,\&\, \dots \,\&\, s_k(x_1, \dots, x_n) \approx t_k(x_1, \dots, x_n)$ be such that $\mathbf{F}_{\mathcal{Q}}(\omega) \vDash q$. Let $\mathfrak{B} \in \mathcal{Q}$. If q is passive, then $\mathbf{F}_{\mathcal{Q}}(\omega) \nvDash \exists x_1, \dots, x_n\varphi$. By Lemmas 5.3 and 5.4 we get $\mathfrak{A}_2 \times \mathfrak{A}_n \nvDash \exists x_1, \dots, x_n\varphi$. But then also $\mathfrak{B} \times \mathfrak{A}_2 \times \mathfrak{A}_n \nvDash \exists x_1, \dots, x_n\varphi$ which vacuously means $\mathfrak{B} \times \mathfrak{A}_2 \times \mathfrak{A}_n \vDash q$. Now, assume that q is not passive. But then q is derivable in $\mathcal{Q}$, so $\mathfrak{B} \vDash q$ and also $\mathfrak{B} \times \mathfrak{A}_2 \times \mathfrak{A}_n \vDash q$.

For the other direction, let $q := (s_0(\overline{x}) \approx t_0(\overline{x}) \,\&\, \dots \,\&\, s_k(\overline{x}) \approx t_k(\overline{x})) \Rightarrow s(\overline{x}) \approx t(\overline{x})$ where $\overline{x} = x_1, \dots, x_n$ be an admissible rule in $\mathcal{Q}$. Thus, $\mathbf{F}_{\mathcal{Q}}(\omega) \vDash q$. By

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Lemma 4.9, this means that $\mathfrak{A}_2 \times \mathfrak{A}_n \vDash q$. Suppose further that there is a critical $\mathfrak{B} \in \mathcal{Q}$ such that $\mathfrak{B} \nvDash q$. Then there are $b_1, \dots, b_n \in \mathsf{B}$ such that $s_0(b_1, \dots, b_n) = t_0(b_1, \dots, b_n)$,...,$s_k(b_1, \dots, b_n) = t_k(b_1, \dots, b_n)$ and $s(b_1, \dots, b_n) \neq t(b_1, \dots, b_n)$. Let $\varphi := (s_0(\overline{x}) \approx t_0(\overline{x}) \,\&\, \dots \,\&\, s_k(\overline{x}) \approx t_k(\overline{x}))$. Assume for reductio that $\mathbf{F}_{\mathcal{Q}}(\omega) \vDash \exists\overline{x}\varphi$. By Lemmas 5.3 and 5.4 we get that $\mathfrak{A}_2 \times \mathfrak{A}_n \vDash \exists\overline{x}\varphi$. Let $a_1, \dots, a_n \in \mathfrak{A}_2 \times \mathfrak{A}_n$ be such that $s_0(a_1, \dots, a_n) = t_0(a_1, \dots, a_n)$,...,$s_k(a_1, \dots, a_n) = t_k(a_1, \dots, a_n)$. But then $s_0((a_1, b_1), \dots, (a_n, b_n)) = t_0((a_1, b_1), \dots, (a_n, b_n))$,...,$s_k((a_1, b_1), \dots, (a_n, b_n)) = t_k((a_1, b_1), \dots, (a_n, b_n))$, but $s((a_1, b_1), \dots, (a_n, b_n)) \neq t((a_1, b_1), \dots, (a_n, b_n))$. This means that $\mathfrak{A} \times \mathfrak{A}_2 \times \mathfrak{A}_n \nvDash q$—contradiction. Thus $\mathbf{F}_{\mathcal{Q}}(\omega) \nvDash \exists\overline{x}\varphi$ which means that q is passive, so $\mathcal{Q}$ is **ASC**.

The case where $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_\omega)$ is covered similarly—just substitute $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}}$ for $\mathfrak{A}_2 \times \mathfrak{A}_n$ in the above proof. ⊣

THEOREM 6.3. *Let $\mathcal{Q}$ be a quasivariety of Sugihara algebras. $\mathcal{Q}$ is* **ASC** *iff either*: $\mathfrak{A}_2 \times \mathfrak{A}_3 \notin \mathcal{Q}$ *or* $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_{2n+1})$ *for some* $n \in \omega$.

PROOF. For one direction, assume that $\mathcal{Q} \not\sim (\mathfrak{A}_{2n+1})$ and $\mathfrak{A}_2 \times \mathfrak{A}_3 \in \mathcal{Q}$. Thus, either i) $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_{2n})$, where $n \geq 2$ (if $n = 1$, then it must be the case that $\mathfrak{A}_2 \times \mathfrak{A}_3 \notin \mathcal{Q}$), or ii) $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_\omega)$. But then $\mathfrak{A}_2 \times \mathfrak{A}_3 \preceq \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_2 \times \mathfrak{A}_{2n} \notin \mathcal{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n})$ which by Lemma 6.2 means that $\mathcal{Q}$ is not **ASC**. If $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_\omega)$, then $\mathfrak{A}_2 \times \mathfrak{A}_3 \preceq \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}} \notin \mathcal{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus\{0\}})$.

For the opposite direction, first assume that $\mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_{2n+1})$. We will show that for any critical algebra $\mathfrak{B}$ of $\mathcal{Q}$ it is the case that $\mathfrak{A}_2 \times \mathfrak{B} \times \mathfrak{A}_{2n+1} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \mathfrak{A}_{2n+1}$. By Lemma 6.2, this will be enough to show that $\mathcal{Q}$ is **ASC**. Let $\mathfrak{B} \in \mathcal{Q}$ be critical. By Lemma 4.7, its degree must be less or equal to $2n + 1$. Thus we can keep applying the reasoning from the simplification Lemma 4.4 to obtain Horn equivalence.

In case $\mathcal{Q} \not\sim \mathsf{V}(\mathfrak{A}_{2n+1})$, we have that $\mathfrak{A}_2 \times \mathfrak{A}_3 \notin \mathcal{Q}$. But this means that for any critical $\mathfrak{B} \in \mathcal{Q}$ it must be the case that $deg(\mathfrak{B})$ is even (otherwise $\mathfrak{A}_2 \times \mathfrak{A}_3 \preceq \mathfrak{A}_2 \times \mathfrak{B}$). But then by our key simplification Lemma 4.4 we have $\mathfrak{A}_2 \times \mathfrak{B} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \mathfrak{A}_{2m}$ for some m. Furthermore, $\mathfrak{A}_2 \times \mathfrak{B} \times \mathfrak{A}_{2n} \approx_{\mathsf{q}} \mathfrak{A}_2 \times \mathfrak{A}_{2n}$ for any $n > m$ which proves the theorem. ⊣

We have an immediate corollary:

COROLLARY 6.4. *$\mathcal{Q}$ is* **ASC** *but not* **SC** *iff either*:

(i) $\mathfrak{A}_3 \in \mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_{2n+1})$ *for some* $n \in \omega$, *or*
(ii) $\mathfrak{A}_4 \in \mathcal{Q} \sim \mathsf{V}(\mathfrak{A}_{2n+1})$ *for some* $n \in \omega$, *or*
(iii) $\mathfrak{A}_4 \in \mathcal{Q}$ *and* $\mathfrak{A}_2 \times \mathfrak{A}_3 \notin \mathcal{Q}$.

§7. Conclusion. Let us conclude with several remarks. First, observe that if it were not for $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_3)$, the poset of **RM**'s **HSC** extensions would be a chain—a chain isomorphic to **RM**'s axiomatic extensions. The covering relation in the **HSC** poset of Figure 2a is the restricted covering relation within the entire lattice of Sugihara quasivarieties. In contrast, the **nHSC** quasivarieties form a chain; however, the covering relation in its lattice, shown in Figure 2b, is not a restriction of the covering relation in the entire lattice of Sugihara quasivarieties. To see this, note that we have $\mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n+1}) \supsetneq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n}, \mathfrak{A}_2 \times \mathfrak{A}_{2n-1}) \supsetneq \mathsf{Q}(\mathfrak{A}_2 \times \mathfrak{A}_{2n-1})$.

Next, let us examine the three rules from Lemma 4.12 more closely. All of these rules are admissible in **RM**—we have shown that they are derivable for $\mathsf{ISPP_U}(\mathfrak{A}_2 \times$

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$\mathfrak{A}_{\omega\setminus\{0\}}$), and the logic associated with this quasivariety is **RM**'s **SC** extension with the same set of theorems (admissible rules).[6] The first and third rules are similar in a certain way: arbitrary formulas follow from their premises, so proving their non-derivability in a given Sugihara matrix involves finding an interpretation under which the set of premises is satisfiable. Failure of NP satisfaction has commonly been adopted as a definition of paraconsistency. The R rule, on the other hand, is a new rule introduced specifically to separate algebras containing the 4-element Sugihara subchain from the **HSC** quasivarieties. A somewhat similar rule can be found in Tokarz's work [28, p. 66]. Tokarz's rule takes the form $\neg((p \to p) \leftrightarrow (q \to q)/r$. This rule can also be observed to hold in $\mathfrak{A}_2 \times \mathfrak{A}_{\omega\setminus 0}$ but not in $\mathfrak{A}_4$. Lastly, the only non-passive rule from Lemma 4.12 is the disjunctive syllogism DS, also referred to as the (γ) rule in [4]. The structural incompleteness of **R** has long been known, stemming from the nonderivability of DS.

Finally, to put our result in a broader context, let us once again mention $\mathbf{RM^t}$ and its positive fragment. As it turns out, odd Sugihara monoids, which serve as adequate semantics for $\mathbf{RM^t}$, are categorically equivalent to relative Stone algebras, which, in turn, provide adequate algebraic semantics for positive fragment Gödel–Dummett logic (for representation theorems for Sugihara monoids, see [15–17]). The lattice of subvarieties of linear Stone algebras forms an ω^+ well-ordering, just like the subvarieties of Sugihara algebras. Gödel–Dummett logic is known to be hereditarily structurally complete (HSC) [14, 25], which, in algebraic terms, implies that any subquasivariety of Stone algebras is, in fact, a variety. By categorical equivalence, this property also holds for odd Sugihara monoids.

More generally, it is interesting to note that having isomorphic lattices of subvarieties does not necessarily yield any similarity in the structure of respective lattices of subquasivarieties. It is known that the structure of subvarieties of p-algebras (implication-free reducts of Heyting algebras with intuitionistic negation) is also that of ω^+ well-ordering, yet the lattice of its subquasivarieties is uncountable [1, 29]. The cardinality of the set of subquasivarieties of Sugihara algebras remain unknown. It has been established that the number of subquasivarieties of a quasivariety generated by a finite set of finite Sugihara algebras has to be finite [6]. Thus, the cardinality of the whole lattice must be determined by the size of the class $[\mathsf{V}(\mathfrak{A}_\omega)]_\sim$.

There are two clear conclusions to be drawn from these final remarks. First, modifying the original signature of the logic or its corresponding algebras (as in this case, by adding the constant **t** or removing negation) can significantly alter the structure of the extensions of a given logic. Secondly, an isomorphism between the lattices of subvarieties of two classes of algebras does not necessarily imply any similarity in the structure of their lattices of subquasivarieties.

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[6] Thus, this is a corollary of our results from the main section. Instead of our argument, one could also use the admissibility algorithm from [8], which was specifically devised for testing admissibility within Sugihara algebras.

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