*Adv. Appl. Prob.* **21**, 470–472 (1989)
*Printed in N. Ireland*


# LETTERS TO THE EDITOR

## ON THE ASYMPTOTIC BEHAVIOUR OF THE EXTINCTION TIME OF THE SIMPLE BRANCHING PROCESS

ANTHONY G. PAKES,* *The University of Western Australia*

**Abstract**

The time to extinction of a subcritical Galton–Watson branching process and the time of last mutation of its infinite-alleles version are maxima of independent random variables having an upper tail of geometric type, and hence they are not attracted to any extreme value distribution. It is shown that Anderson's asymptotic results for maxima of discrete variates are applicable, and this rectifies a false assertion made in respect to the infinite-alleles simple branching process.

MAXIMA; EXTREME VALUE DISTRIBUTION

Let $L$ be the time of last mutation in the infinite-alleles version of the simple branching process as described by Griffiths and Pakes (1988). The underlying branching process is denoted by $\{Z_n : n \geqq 0\}$ and $f(.)$ is its offspring probability generating function (p.g.f.), $\beta(.)$ is the p.g.f. of the total number of births and $u$ is the probability that a newly born individual mutates to a new allele. It was shown in Lemma 3.1 of Griffiths and Pakes (1988) that the distribution function of $L$ is given by

$$P(L \leqq n \mid Z_0 = i) = [f_n(\delta)]^i,$$

where $\delta = \beta(1-u)$ and $f_n(.)$ is the $n$-fold functional interate of $f(.)$. Hence $L$ has the same distribution as the maximum of $i$ independent copies of $L$ evaluated for a family tree having a single ancestor. If $\delta = 0$ we recover the distribution function of the time $T$ to extinction of $\{Z_n\}$.

We shall be concerned with the subcritical case $m = E_1(Z_1) < 1$. In this case the conditional distribution of $Z_n$, given $T > n$, has a weak limit whose p.g.f. is denoted by $Q(.)$; see Athreya and Ney (1972). Let $\zeta = \log m^{-1}$ and $\psi(.)$ be the inverse of $Q(.)$. Griffiths and Pakes (1988) purported to prove (Theorem 3.1(ii)) that

$$\lim_{i\to\infty} P_i(i\psi(\exp - \zeta L) \leqq x) = 1 - \exp[-x(1 - Q(\delta))].$$

*This is not true* (*mea culpa*).

When $\lambda = E(Z_1 \log^+ Z_1) < \infty$ then $c = Q'(1-) < \infty$ and then the asserted result could be rewritten to say that a centring sequence $b(i)$ exists such that $\{L - b(i)\}$ converges in law to the Gumbel extreme value distribution. However, since

$$P_1(L > n)/P_1(L > n+1) = (1 - f_n(\delta))/(1 - f_{n+1}(\delta)) \to 1/m, \tag{1}$$

the well-known necessary condition for attraction to the Gumbel distribution is violated; see

Received 8 December 1988.
* Postal address: Department of Mathematics, The University of Western Australia, Nedlands, WA 6009, Australia.



https://doi.org/10.2307/1427170 Published online by Cambridge University Press

Resnick (1987), p. 45. The error in the Griffiths and Pakes proof occurs where they let $i$ and $n$ be such that $i(1-f_n(0))\to x$; this is not possible for all $x$.

The situation can be partially salvaged because (1) is necessary and sufficient for Anderson's (1970) Theorem 2 to be applicable. Recall that $c>1$ and $c<\infty$ iff $\lambda<\infty$.

*Theorem* 1. *If $m<1$ there is an increasing sequence of constants $\{b(i)\}$ satisfying*

$$b(i)-\zeta^{-1}\log i \qquad (i\to\infty) \tag{2}$$

*and*

$$\exp(-e^{-\zeta(x-1)})\leqq \liminf_{i\to\infty} P_i(L-b(i)\leqq x)\leqq \limsup_{i\to\infty} P_i(L-b(i)\leqq x)\leqq \exp(-e^{-\zeta x}).$$

*When $\lambda<\infty$ then*

$$b(i)=\zeta^{-1}[\log i+\log((1-Q(\delta))/c)]. \tag{3}$$

*Proof.* Since $1/m=\exp(-\zeta)$ the limit assertion follows from Anderson's theorem and (1). It remains to derive the above expressions for $b(i)$. Define $h:\mathbb{N}_+\to\mathbb{R}_+$ by $h(n)=-\log P_1(L>n)$ and the linear interpolation $h_c:\mathbb{R}_+\to\mathbb{R}_+$ by

$$h_c(x)=h([x])+\langle x\rangle(h(1+[x])-h([x]))$$

where $\langle x\rangle=x-[x]$ is the fractional part of $x$. Then $b(i)$ is determined by

$$h_c(b(i))=\log i. \tag{4}$$

As shown by Seneta (1974), proof of Theorem 2,

$$1-f_n(\delta)=(1-Q(\delta))e^{-\zeta n}L(e^{-\zeta n})$$

where $L(.)$ is non-decreasing, slowly varying at the origin and $L(0+)=1/c$. Then, with $\Lambda(n)=-\log L(e^{-\zeta n})$,

$$h(n)=\zeta n-\log(1-Q(\delta))+\Lambda(n).$$

Consequently

$$h_c(x)=\zeta x-\log(1-Q(\delta))+\Lambda([x])+\langle x\rangle(\Lambda(1+[x])-\Lambda([x]))$$

and in principle $b(i)$ can be determined by solving (4).

Clearly $b(i)\to\infty$ and hence $\Lambda([b(i)])/b(i)\to 0$. The estimate (2) now follows. When $\lambda<\infty$ then $\Lambda([b(i)])\to\log c$ and it follows that $b(i)$ is given by the right-hand side of (3) apart from terms which are $o(1)$ as $i\to\infty$. Omitting such terms does not disturb the limit assertion and hence we may take $b(i)$ according to (3). The proof is complete.

We remark that the theorem applies to the extinction time $T$ provided that in (3) we omit the term $Q(\delta)$. Consequently, when $m<1$, the extinction time distribution does *not* lie in the domain of attraction of an extreme value distribution, though it does so when $m=1$ and some further moment conditions are satisfied. In fact, if $m=1$ and $f(s)-s=(1-s)^{1+\alpha}L(1-s)$, where $0<\alpha\leqq 1$ and $L(.)$ is slowly varying at the origin, it is easy to use Gnedenko's characterisation of the domain of attraction of the extreme value distribution $G_2(x)=\exp(-x^{-\alpha})$ (Resnick (1987), p. 54) and Slack's (1968) determination of the rate of decay of $1-f_n(s)(n\to\infty)$ to show that if $a(i)=i^\alpha/\alpha L(1/i)$ then

$$\lim_{i\to\infty} P_i(L\leqq a(i)x)=G_2(x)$$

and this also holds with $T$ replacing $L$. The case where the offspring distribution has a finite variance is covered in Griffiths and Pakes (1988), Theorem 3.1(i). In the case of the continuous-time Markov branching process with $m<1$ and $\lambda<\infty$ the extinction time distribution is attracted to the Gumbel distribution. These things must surely be well known, although they seem not to have been put into print—but see Pakes (1989) for the continuous-time case.

From Theorem 4 of Anderson (1980) we obtain the following local counterpart of Theorem 1.

*Theorem* 2. *For any sequence of integers* $k(i)$,

$$\lim_{i\to\infty} P_i(L = k(i)) - [G_1(x_i) - G_1(x_i - \zeta)] = 0$$

*where* $x_i = \zeta(k_i(i) - b(i))$ *and* $G_1(x) = \exp(-e^{-x})$ *is the Gumbel distribution function.*

## References

ANDERSON, C. W. (1970) Extreme value theory for a class of discrete distributions with applications to some stochastic processes. *J. Appl. Prob.* **7,** 99–113.

ANDERSON, C. W. (1980) Local limit theorems for the maxima of discrete random variables. *Math. Proc. Camb. Phil. Soc.* **88,** 161–165.

ATHREYA, K. B. and NEY, P. E. (1972) *Branching Processes.* Springer-Verlag, Berlin.

GRIFFITHS, R. C. and PAKES, A. G. (1988) An infinite-alleles version of the simple branching process. *Adv. Appl. Prob.* **20,** 489–524.

PAKES, A. G. (1989) Asymptotic results for the extinction time of Markov branching processes allowing emigration, I. Random walk decrements. *Adv. Appl. Prob.* **21,** 243–269.

RESNICK, S. I. (1987) *Extreme Values, Regular Variation, and Point Processes.* Springer-Verlag, New York.

SENETA, E. (1974) Regularly varying functions in the theory of simple branching processes. *Adv. Appl. Prob.* **6,** 408–420.

SLACK, R. S. (1968) A branching process with mean one and possibly infinite variance. *Z. Wahrscheinlichkeitsth.* **9,** 139–145.

https://doi.org/10.2307/1427170 Published online by Cambridge University Press