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FINITELY EMBEDDED MODULES OVER GROUP RINGS

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Let R be a ring and X a right R-module (all rings have identities and all modules are unitary). The intersection of all non-zero submodules of X is denoted by $\mu(X)$. The module X is called *monolithic* if and only if $\mu(X)\neq 0$ and in this case $\mu(X)$ is an essential simple submodule of X. (Recall that a submodule Y of X is essential if and only if $Y\cap A\neq 0$ for every non-zero submodule A of X.) It is well known that a module X is monolithic if and only if there is a simple right R-module U such that X is a submodule of the injective hull $E(U)$ of U. If x is a non-zero element of an arbitrary right R-module X then by Zorn's Lemma there is a submodule Y_x of X maximal with the property $x\notin Y_x$. It can easily be checked that X/Y_x is monolithic and $\cap Y_x=0$, where the intersection is taken over all non-zero elements x of X.

Vámos (**15**) defined a right R-module X to be *finitely embedded* in case the injective hull of X is a finite direct sum $E(S_1)\oplus E(S_2)\oplus\cdots\oplus E(S_n)$ of injective hulls of simple right R-modules S_i, $1\leq i\leq n$. He proved that a right R-module X is finitely embedded if and only if its socle is a finitely generated essential submodule, and he showed that this is equivalent to X having the property that for every collection of submodules Y_λ, $\lambda\in\Lambda$, of X with $\cap_\Lambda Y_\lambda=0$ there exists a finite subset Λ' of Λ such that $\cap_{\Lambda'} Y_\lambda=0$. He also proved that a module X is Artinian if and only if every homomorphic image of X is finitely embedded.

A ring R is a *Hilbert ring* if and only if R is right Noetherian and the Jacobson radical of every homomorphic image of R is nilpotent. A simple homomorphic image of a ring is called a *capital* of that ring. A field is called *absolute* if and only if it is an algebraic extension of a finite field. A ring R is called a CHACA ring if and only if R is a commutative Hilbert ring such that every capital of R is an absolute field.

A group G is *polycyclic* if and only if there is a series

$$1=G_0\lhd G_1\lhd\cdots\lhd G_{n-1}\lhd G_n=G$$

in which the factors G_i/G_{i-1}, $1\leq i\leq n$, are all cyclic. It is well known that a finitely generated nilpotent group is polycyclic. If $\mathfrak{X}$ and $\mathfrak{Y}$ are group classes then a group H is called an $\mathfrak{X}$-by-$\mathfrak{Y}$ group if and only if there is a normal subgroup N of H such that N lies in $\mathfrak{X}$ and H/N in $\mathfrak{Y}$.

Let J be a CHACA ring, G be a polycyclic-by-finite group and X be a finitely generated monolithic right JG-module. If $U=\mu(X)$ then there is a maximal ideal M of J such that U is finite dimensional over the field J/M by a theorem of Roseblade (**11**, Corollary A). It follows by methods of Jategaonkar (**6**) that for some positive integer k, X is a finitely generated module over the ring J/M^k and in particular X has finite length. The question arises whether monolithic JG-modules are Artinian and in this note we prove the following result.

Theorem A. *Let J be a commutative Hilbert ring and G be a finitely generated nilpotent-by-finite group. Let X be a finitely embedded JG-module such that the socle of X is finitely generated as a J-module. Then X is Artinian.*

This result together with Roseblade's theorem mentioned above combine to give immediately:

Corollary A. *Let J be a CHACA ring and G a finitely generated nilpotent-by-finite group. Then every finitely embedded JG-module is Artinian.*

Let K be any field, G be a finitely generated nilpotent-by-finite group and X be a finitely embedded right KG-module. If K is absolute then X is Artinian by Corollary A. Also it is not hard to see that if G is Abelian-by-finite then X is Artinian (Corollary 2.2). Otherwise the problem of whether or not X is Artinian reduces to considering finitely embedded modules over capitals of KH where H is a finitely generated nilpotent normal subgroup of finite index in G. In §1 we show that if every capital of KH is Artinian then every finitely embedded KH-module is Artinian and in §2 we show how this extends to finitely embedded KG-modules. Unfortunately, this takes us no further because of the next result.

Theorem B. *Let K be a field and G be a finitely generated nilpotent-by-finite group such that every capital of KG is Artinian. Then either K is absolute or G is Abelian-by-finite.*

Snider (**14**) proved that if K is a field and G is a polycyclic group such that every primitive ideal of the group ring KG is a maximal ideal then K is absolute or G is nilpotent-by-finite. We prove that if J is a ring and G is a group such that every primitive homomorphic image of the group ring JG is Artinian then every primitive homomorphic image of JH is Artinian for every normal subgroup H of finite index in G (Lemma 3.3). But it is well known that if P is a primitive ideal of a ring R and the ring R/P is Artinian then R/P is simple and hence P is a maximal ideal. Thus combining these results with Theorem B we can extend Theorem B as follows.

Theorem B′. *Let K be a field and G be a polycyclic-by-finite group such that every primitive homomorphic image of KG is Artinian. Then either K is absolute or G is Abelian-by-finite.*

Now suppose that J is a commutative Artinian ring and G is a finite group. If U is a simple right JG-module then Rosenberg and Zelinsky (**13**, Theorem 3) proved that the injective hull $E(U)$ has finite length, and so in particular is finitely generated. We shall prove the following result.

Theorem C. *Let J be a CHACA ring and G be a polycyclic-by-finite group such* *that either J is not Artinian or G is infinite. Then there does not exist a non-zero finitely generated injective JG-module which is faithful for J.*

I am grateful to the referee for bringing (**14**) to my attention. There is some overlap with (**14**). In particular, Theorem 3.2 is essentially the same as (**14**, Lemma 1).

1. Polycentral rings

Let R be a ring. An ideal I of R is said to have a *centralising set of generators* if and only if there is a finite set of elements c_i, $1 \leq i \leq n$, of I and a chain

$$O = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{n-1} \subseteq I_n = I$$

of ideals I_j of R such that I_j is generated by $c_1, c_2, \ldots, c_j$ and $c_j + I_{j-1}$ is a central element of the ring R/I_{j-1} for all $1 \leq j \leq n$. If every ideal of a ring R has a centralising set of generators then we call the ring R *polycentral*.

Let X be a right R-module, R any ring. If S is a non-empty subset of R then the annihilator of S in X will be denoted by $\operatorname{ann}_X(S)$; thus $\operatorname{ann}_X(S) = \{x \in X : xS = 0\}$. If $S = \{c\}$ then we shall denote $\operatorname{ann}_X(S)$ simply by $\operatorname{ann}_X(c)$. If T is a non-empty subset of X then the annihilator of T in R is $\operatorname{ann}_R(T) = \{r \in R : Tr = 0\}$.

Lemma 1.1. *Let R be a ring and X a right R-module which contains an Artinian submodule Y. Let c be a central element of R such that $\operatorname{ann}_X(c)$ is Artinian. Then $\{x \in X : xc \in Y\}$ is an Artinian submodule of X.*

Proof. It is clear that $A = \{x \in X : xc \in Y\}$ is a submodule of X. Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a descending chain of submodules of A. If $B_i = A_i \cap \operatorname{ann}_X(c)$ for each positive integer i then $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ is a descending chain of submodules of $\operatorname{ann}_X(c)$ and $A_1c \supseteq A_2c \supseteq A_3c \supseteq \cdots$ is a descending chain of submodules of Y. Since $\operatorname{ann}_X(c)$ and Y are both Artinian there exists a positive integer n such that $B_n = B_{n+1} = B_{n+2} = \cdots$ and $A_nc = A_{n+1}c = A_{n+2}c = \cdots$. It is a simple check to show that $A_n = A_{n+1} = A_{n+2} = \cdots$. It follows that A is Artinian.

Lemma 1.2. *Let R be a ring and X a finitely embedded right R-module which is not Artinian. Then there exists an Artinian submodule Y of X such that X/Y is not finitely embedded but X/A is finitely embedded for every proper submodule A of Y.*

Proof. Let $\mathcal{S}$ be the collection of submodules C of X such that X/C is not finitely embedded. Since X is not Artinian it follows by (**15**, Proposition 2*) that $\mathcal{S}$ is non-empty. Let $\{S_\lambda : \lambda \in \Lambda\}$ be a chain of submodules in $\mathcal{S}$ and $S = \cap_\Lambda S_\lambda$. If S does not lie in $\mathcal{S}$ then X/S is finitely embedded and hence $\cap_\Lambda (S_\lambda/S) = 0$ implies that $S_\lambda/S = 0$ for some λ in Λ by (**15**, Proposition 1*), a contradiction. It follows that S belongs to $\mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a minimal member Y. If A is a proper submodule of Y then X/A is finitely embedded by the choice of Y. By (**15**, Proposition 3*) it follows that Y/A is finitely embedded for every proper submodule A of Y and hence by (**15**, Proposition 2*) Y is Artinian.

Corollary 1.3. *Let R be a ring and c be a central element of R. Let X be a c-torsion right R-module such that* $\mathrm{ann}_X(c)$ *is Artinian. Then X is Artinian.*

Proof. For every non-zero element x of X there exists a positive integer k such that $xc^k = 0$, $xc^{k-1} \neq 0$, and hence $xc^{k-1} \in \mathrm{ann}_X(c)$. Thus $\mathrm{ann}_X(c)$ is an essential submodule of X. Since $\mathrm{ann}_X(c)$ is Artinian it follows that X is finitely embedded by (**15**, Lemma 1). If X is not Artinian then there exists an Artinian submodule Y of X such that X/Y is not finitely embedded (Lemma 1.2). By Lemma 1.1 $\mathrm{ann}_{X/Y}(c)$ is Artinian and thus X/Y is finitely embedded by the first part of the proof. This contradiction gives the result.

Lemma 1.4. *Let R be a ring and I an ideal with a centralising set of generators. Let X be an I-torsion right R-module such that* $\mathrm{ann}_X(I)$ *is Artinian. Then X is Artinian.*

Proof. Let $c_1, c_2, \ldots, c_n$ be a centralising set of generators of I. If $n = 1$ then the result follows by Corollary 1.3. If $n > 1$ let $\bar{R}$ denote the ring $R/(c_1R)$, $\bar{I}$ the ideal $I/(c_1R)$ and Y the submodule $\mathrm{ann}_X(c_1)$. Clearly Y is an $\bar{I}$-torsion right $\bar{R}$-module such that $\mathrm{ann}_Y(\bar{I})$ is Artinian. By induction on n, Y is Artinian and by Corollary 1.3 X is Artinian.

Let R be a ring and I an ideal of R with a centralising set of generators. By (**9**, 2.7) I has the *AR property*; that is, for every submodule Y of a Noetherian right R-module X there exists a positive integer n such that $Y \cap XI^n \subseteq YI$. It can easily be checked that if the ring R is right Noetherian then a right R-module X is I-torsion if and only if $\mathrm{ann}_X(I)$ is an essential submodule of X. If R is any ring and X is a monolithic right R-module with $U = \mu(X)$ then we shall denote the primitive ideal $\mathrm{ann}_R(U)$ by $\pi(X)$. With these remarks and notation we see that Lemma 1.4 immediately gives the following result.

Theorem 1.5. *Let R be a right Noetherian ring and I be an ideal of R such that I has a centralising set of generators. Let X be a monolithic right R-module such that $I \subseteq \pi(X)$ and* $\mathrm{ann}_X(I)$ *is Artinian. Then X is Artinian.*

Let R be a right Noetherian ring and X be a monolithic right R-module such that P is the primitive ideal $\pi(X)$. If the ring R/P is Artinian then $\mathrm{ann}_X(P) = \mu(X)$ since $\mathrm{ann}_X(P)$ is semisimple and monolithic in this case. Thus $\mathrm{ann}_X(P)$ is Artinian if R/P is Artinian. For example, if R is a commutative Noetherian ring then Theorem 1.5 reduces to the theorem of Matlis (**8**, Proposition 3) which states that finitely embedded R-modules are Artinian and the proof is more elementary than that given by Matlis.

Now let K be a field and G be a finitely generated nilpotent group. If R is the group ring KG then R is polycentral by (**12**, Theorem A). Let X be a monolithic right R-module. If $P = \pi(X)$ then P is a maximal ideal of R by (**16**, Theorem 3) and $\mathrm{ann}_X(P)$ is a monolithic right R/P-module.

2. Proof of Theorem A

Let R be a ring and $\mathfrak{X}$ a class of right R-modules. A right R-module X is called a poly-$\mathfrak{X}$ module if and only if there exists a chain

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1} \supseteq X_n = 0$$

of submodules X_i of X such that the factor module X_{i-1}/X_i lies in $\mathfrak{X}$ for each $1 \leqslant i \leqslant n$.

Lemma 2.1. *Let J be a ring and H a normal subgroup of finite index in a group G. Let X be a monolithic right JG-module. Then considered as a JH-module X can be embedded in a finite direct sum of poly-monolithic right JH-modules.*

Proof. Let R be the group ring JG and S the group ring JH. Let U be the simple R-submodule of X. Suppose that $G = \bigcup_{i=1}^{t} x_iH$ for some positive integer t and elements x_i, $1 \leqslant i \leqslant t$, of G. If u is a non-zero element of U then $U = uR = \Sigma_{i=1}^{t} ux_iS$ and hence U is a finitely generated right S-module. Let V be a maximal S-submodule of U. Then $\bigcap_{i=1}^{t} Vx_i$ is an R-submodule of U and hence $\bigcap_{i=1}^{t} Vx_i = 0$. This gives a natural S-monomorphism $U \to \bigoplus_{i=1}^{t} (U/Vx_i)$.

Thus there exist a positive integer n and simple right S-submodules U_i, $1 \leqslant i \leqslant n$, of U such that $U = \bigoplus_{i=1}^{n} U_i$. Let us now define a chain $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1} \supseteq X_n$ of S-submodules X_i of X by demanding that X_i be an S-submodule of X_{i-1} maximal with respect to the conditions $\bigoplus_{j=i+1}^{n} U_j \subseteq X_i$ and $U_i \cap X_i = 0$, $1 \leqslant i \leqslant n$. Clearly for each $1 \leqslant i \leqslant n$ the right S-module X_{i-1}/X_i is monolithic with $\mu(X_{i-1}/X_i) \cong U_i$. Moreover $X_n \cap U = 0$. It follows that if $Y = \bigcap_{i=1}^{t} X_nx_i$ then Y is an R-submodule of X with $Y \cap U = 0$ and hence $Y = 0$. For each $1 \leqslant i \leqslant t$, X/X_nx_i is a poly-monolithic right S-module. In addition there is a natural S-monomorphism $X \to \bigoplus_{i=1}^{t} (X/X_nx_i)$. This completes the proof.

Corollary 2.2. *Let J be a commutative Noetherian ring and G be a finitely generated Abelian-by-finite group. Then every finitely embedded JG-module is Artinian.*

Proof. The group G has a finitely generated Abelian normal subgroup A of finite index. The group ring JA is a commutative Noetherian ring and by (**8**, Proposition 3) (or Theorem 1.5) every finitely embedded JA-module is Artinian. The result follows by Lemma 2.1.

Let J be a ring and H be a normal subgroup of a group G. We shall denote by ωH the right ideal of the group ring JG generated by the elements $1 - h$ with h in H. It can easily be checked that ωH is the kernel of the canonical homomorphism of the group ring JG onto the group ring $J(G/H)$, and thus ωH is a two-sided ideal of JG.

Proof of Theorem A. Without loss of generality we can suppose that X is a monolithic right JG-module and, by Lemma 2.1, that G is finitely generated and nilpotent. Let $P = \pi(X)$ and $Q = P \cap J$. By (**11**, Corollary C3) Q is a maximal ideal of

J. If R is the group ring JG then R is right Noetherian by (**4**, Theorem 1) and the ideal QR of R is generated by central elements of R. By Theorem 1.5 we can suppose without loss of generality that $Q = 0$. Thus it is sufficient to prove the result when J is a field.

Let H be the normal subgroup $G \cap (1+P)$ of G. A theorem of Mal'cev (**7**, Theorem 1) shows that G/H is Abelian-by-finite. Since $\mathrm{ann}_X(\omega H)$ is a monolithic module over the ring $JG/\omega H$ and since $JG/\omega H$ is isomorphic to the group ring $J(G/H)$ it follows by Corollary 2.2 that $\mathrm{ann}_X(\omega H)$ is Artinian. Clearly $\mathrm{ann}_X(P) \subseteq \mathrm{ann}_X(\omega H)$ and hence $\mathrm{ann}_X(P)$ is Artinian. By (**12**, Theorem A) P has a centralising set of generators. Finally by Theorem 1.5 X is Artinian.

3. Primitive images of group rings

Let K be a field and G be a group. We are concerned in this section with primitive images of the group ring KG. Let us begin by observing that we can easily handle the situation when K is a large field. Let K be an algebraically closed field such that the cardinality of the set K is greater than that of the set G. Let P be a primitive ideal of KG. If the ring KG/P is Artinian then there is a K-division algebra D and a positive integer n such that $KG/P \cong D_n$, the complete ring of $n \times n$ matrices with entries in D. Now $\dim_K D \leq \dim_K D_n \leq \dim_K KG = |G| < |K|$. Since K is algebraically closed the proof of (**10**, Theorem 3) shows that $D = K$. Thus $\dim_K(KG/P)$ is finite. Thus, if G is polycyclic and every primitive homomorphic image of KG is Artinian then by (**5**, Theorem 3.3) G is Abelian-by-finite.

We next extend a result of Zalesskii (**16**, Theorem 3) which states that if K is a field and G a finitely generated nilpotent group then every primitive ideal of KG is a maximal ideal. We need the following lemma.

Lemma 3.1. *Let J be a right Noetherian ring and H be a normal subgroup of finite index in a polycyclic-by-finite group G. Let $P \supset Q$ be prime ideals of JG. Then $P \cap JH \supset Q \cap JH$.*

Proof. By (**4**, Theorem 1) JG is a right Noetherian ring and hence JG/Q is a prime right Noetherian ring. By (**3**, Theorem 10) the ideal P/Q contains a regular element $c + Q$ of JG/Q. Since JG is a finitely generated right JH-module and JH is a right Noetherian ring there exists a least positive integer k such that

$$S + cS + \cdots + c^k S + Q = S + cS + \cdots + c^k S + c^{k+1} S + Q,$$

where S is the group ring JH. It follows that there exist elements a_i, $0 \leq i \leq k$, of S such that $a_0 + ca_1 + \cdots + c^k a_k + c^{k+1}$ belongs to Q. If $a_0 \in Q$ then $c(a_1 + \cdots + c^{k-1}a_k + c^k) \in Q$ and hence $a_1 + \cdots + c^{k-1}a_k + c^k \in Q$, which contradicts the choice of k. It follows that $a_0 \notin Q$. However $a_0 \in P$ because $c \in P$ and we conclude that $P \cap S \supset Q \cap S$.

Theorem 3.2. *Let J be a commutative Hilbert ring and G a finitely generated nilpotent-by-finite group. Then every primitive ideal of JG is a maximal ideal.*

Proof. By (**11**, Corollary C3) we can suppose without loss of generality that J is a field which we shall denote by K. Let P be a primitive ideal of KG and X a simple right KG-module such that $P=\mathrm{ann}_{KG}(X)$. If H is a normal nilpotent subgroup of finite index in G then H is finitely generated. Let $|G:H|=n$ and $T=\{t_1=1,t_2,\ldots,t_n\}$ a transversal to the cosets of H in G. By the proof of Lemma 2.1 there exists a maximal right KH-submodule Y of X such that $\cap_{i=1}^n Yt_i=0$ and a KH-monomorphism $X\to\oplus_{i=1}^n(X/Yt_i)$. If $Q=\mathrm{ann}_{KH}(X/Y)$ then $P\cap KH\subset Q$ and $\cap_{i=1}^n Q^{t_i}\subseteq P\cap KH$. Thus $P\cap KH=\cap_{i=1}^n Q^{t_i}$ where of course Q is a primitive ideal of KH. By (**16**, Theorem 3) Q is a maximal ideal of KH. Now let M be a maximal ideal of KG such that $P\subseteq M$. Then $\cap_{i=1}^n Q^{t_i}\subseteq M\cap KH$ and hence $Q^t\subseteq M\cap KH$ for some t in T. Since Q^t is a maximal ideal it follows that $Q^t=M\cap KH$ and hence $P\cap KH=M\cap KH$. By Lemma 3.1 we obtain the desired conclusion $P=M$.

Lemma 3.3. *Let J be a ring and G be a group such that every primitive homomorphic image of JG is Artinian. Let H be a normal subgroup of finite index in G. Then every primitive image of JH is Artinian.*

Proof. Let S denote the ring JH and R the ring JG. Let P be a primitive ideal of S and M a maximal right ideal of S such that $P=\mathrm{ann}_S(S/M)$. Since MR is a proper ideal of R it follows that there exists a maximal right ideal M_1 of R such that $MR\subseteq M_1$ and hence $M=M_1\cap S$. If $Q=\mathrm{ann}_R(R/M_1)$ then Q is a primitive ideal of R and by hypothesis the ring R/Q is Artinian. Moreover $Q\cap S\subseteq P$. There exists a positive integer n and elements x_i, $1\le i\le n$, of G such that $T=\{x_1,x_2,\ldots,x_n\}$ is a transversal to the cosets of H in G. Let $\bar S=(S+Q)/Q$ and $\bar x_i=x_i+Q$, $1\le i\le n$. Then clearly $\bar x_i\bar S=\bar S\bar x_i$, $1\le i\le n$, and $R/Q=\bar x_1\bar S+\bar x_2\bar S+\cdots+\bar x_n\bar S$. By (**2**, Theorem 4) $\bar S$ is Artinian and hence $S/(Q\cap S)$ is Artinian. We conclude that S/P is Artinian, as required.

Let S be a simple ring and α an automorphism of S. Let X be an indeterminate and let $R=S[X,X^{-1},\alpha]$ denote the ring of polynomials $\Sigma_{i=s}^t\sigma_iX^i$ where $s\le t$ are integers and $\sigma_i\in S$, $s\le i\le t$, multiplication being given by $X\sigma=\sigma^\alpha X$, $\sigma\in S$. We shall require the following fact about R.

Lemma 3.4. *If the above ring R contains a proper ideal M such that R/M is Artinian then S is Artinian.*

Proof. Since R/M is Artinian it is clear that $M\neq 0$. Let $M_1=S\cap(M+SX+SX^2+\cdots)$. Then M_1 is a non-zero ideal of S and hence M_1 contains the identity 1 of S. That is, there exist elements σ_i of S such that $1+\Sigma_{i=1}^m\sigma_iX^i\in M$ and we infer that $X^{-m}+\Sigma_{i=1}^m\sigma_iX^{i-m}\in M$. Similarly there exist elements σ'_i of S such that $X^n+\Sigma_{i=1}^n\sigma'_iX^{n-i}\in M$. Since $M\cap S=0$ it follows by (**2**, Theorem 4) (see the proof of Lemma 3.3) that S is Artinian.

If H is a normal subgroup of a group G and J is a ring then an ideal I of the group ring JH is called *G-invariant* if and only if $I^x=x^{-1}Ix\subseteq I$ for all x in G.

Corollary 3.5. *Let J be a ring and G be a group such that every capital of JG is Artinian. Let H be a normal subgroup of G such that G/H is infinite cyclic and let P be a G-invariant maximal ideal of JH. Then the ring JH/P is Artinian.*

Proof. Let S be the ring JH and R the ring JG. Then PR is a proper ideal of R and hence $PR \subseteq M$ for some maximal ideal M of R. It follows that $P = M \cap S$. Let G/H be generated by the coset xH and let α be the automorphism of $\bar{S} = S/P$ induced by conjugation by x. Then clearly $R/PR \cong \bar{S}[X, X^{-1}, \alpha]$. By Lemma 3.4, S is Artinian.

Lemma 3.6. *Let K be a field and G be a finitely generated nilpotent-by-finite group such that every capital of KG is Artinian. Let H be a normal subgroup of G such that G/H is infinite cyclic. Then every capital of KH is Artinian.*

Proof. Let S be the ring KH and R the ring KG. Let P be a maximal ideal of S and $N = \{x \in G : P^x = P\}$. Then N is a normal subgroup of G and $H \subseteq N$. If $H \neq N$ then N has finite index in G and by Theorem 3.2 and Lemma 3.3 every capital of KN is Artinian. Corollary 3.5 applied to the group N gives S/P is Artinian. Now suppose that $H = N$. Let M be a maximal right ideal of S such that $P = \mathrm{ann}_S(S/M)$ and M_1 a maximal right ideal of R such that $M = M_1 \cap S$ (see the proof of Lemma 3.3). Let $A = R/M_1$ and $B = (S + M_1)/M_1$. If $Q = \mathrm{ann}_R(A)$ then Q is a maximal ideal of R by Theorem 3.2 and the ring R/Q is Artinian by hypothesis.

Let $E_1 \supseteq E_2 \supseteq E_3 \cdots$ be a descending chain of right ideals of S such that each right ideal E_i contains P. Then $E_1R \supseteq E_2R \supseteq E_3R \supseteq \cdots$ is a descending chain of right ideals of R and there exists a positive integer n such that $E_nR + Q = E_{n+1}R + Q = E_{n+2}R + Q = \cdots$. Let $e \in E_n$. If G/H is generated by the coset Hx then it is clear that $E_{n+1}R = \sum_{i=-\infty}^{\infty} E_{n+1}x^i$ and hence there exist integers $s \leq t$ and elements f_i, $s \leq i \leq t$, of E_{n+1} and q of Q such that $e = \sum_{i=s}^{t} f_i x^i + q$. Then $be = \sum_{i=s}^{t} bf_i x^i$ for each element b of B. But by (**11**, Lemma 3) $A = \bigoplus_{i=-\infty}^{\infty} Bx^i$ and it follows that $b(e - f_0) = 0$ for each element b of B. Hence $e - f_0 \in P$ and $e \in E_{n+1}$. It follows that $E_n = E_{n+1} = E_{n+2} = \cdots$ and the ring S/P is Artinian, as required.

Lemma 3.7. *Let K be a field and G be a polycyclic-by-finite group such that for every subnormal subgroup H of G every primitive homomorphic image of KH is Artinian. Then either K is absolute or G is Abelian-by-finite.*

Proof. Suppose K is not absolute and G is not Abelian-by-finite. By adapting Hall's proof of (**5**, Theorem 3.3) we can suppose without loss of generality that G is generated by a free Abelian normal subgroup A and an element z. Let X be Hall's simple right KG-module with basis $\{v_m : m \in Z\}$ (see (**5**, p. 616)), and note that $v_m z = v_{m+1}$ for all m. If $R = KG$ and $P = \mathrm{ann}_R(X)$ then P is a primitive ideal of R and we claim that the ring R/P is not Artinian. Consider the chain of left ideals $R(1-z) + P \supseteq R(1-z^2) + P \supseteq R(1-z^4) + P \supseteq \cdots$ and suppose that n is a positive integer such that $R(1-z^n) + P = R(1-z^{2n}) + P$. Then $X(1-z^n) = X(1-z^{2n})$. In particular

$$v_0(1-z^n) = \left(\sum_{i=s}^{t} k_i v_i\right)(1-z^{2n})$$

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for some integers $s \leqslant t$ and elements k_i, $s \leqslant i \leqslant t$, of K with k_s, $k_t \neq 0$. That is,

$$v_0 - v_n = \sum_{i=s}^{t} k_i v_i - \sum_{i=s}^{t} k_i v_{i+2n},$$

which is impossible. Thus R/P is not Artinian, as required.

Proof of Theorem B. Let K be a field and G be a finitely generated nilpotent-by-finite group such that every capital of KG is Artinian. If H is a subnormal sub-group of G then there is a chain $H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that each factor H_i/H_{i-1}, $1 \leqslant i \leqslant n$, is either finite or cyclic. By Theorem 3.2 and Lemmas 3.3 and 3.6 every capital of KH is Artinian. Theorem 3.2 and Lemma 3.7 combine to complete the proof.

4. Proof of Theorem C

It is a well known fact that if c is a regular element of a ring R and X is an injective right R-module then $X = Xc$.

Suppose that J is a CHACA ring and G is a polycyclic-by-finite group such that there exists a non-zero finitely generated injective right JG-module X. Our aim is to prove that $\operatorname{ann}_J(X)$ contains a finite product of maximal ideals and G is finite. It is well known that because JG is a right Noetherian ring X is a finite direct sum of indecomposable injective right JG-modules. Thus without loss of generality we can suppose that X is indecomposable.

Firstly we prove that X is Artinian. If P is maximal in $\{\operatorname{ann}_{JG}(Y): Y$ is a non-zero submodule of $X\}$ then P is a prime ideal of JG. Suppose that P is not a primitive ideal and let S be the prime right Noetherian ring JG/P. If $A = \operatorname{ann}_X(P)$ then A is a finitely generated injective right S-module. If a is a non-zero element of A then there exists a maximal right ideal M of S such that $\operatorname{ann}_S(a) \subseteq M$. Let Q be the primitive ideal $\operatorname{ann}_S(S/M)$ of S. By (**6**, Theorem 6′) there exists an ideal I of S such that $I \subseteq Q$, I has the AR property and the ring S/I is right Artinian. Since I is a non-zero ideal of S it follows that I contains a regular element by (**3**, Theorem 10). Thus by the above remark $A = AI$. There exists a positive integer n such that $aR \cap AI^n \subseteq aI$ and hence $a(1-i) = 0$ for some element i of I, and this contradicts $I \subseteq M$. Thus P is a primitive ideal of JG. By (**11**, Corollary A) JG/P is an Artinian ring and it follows that X has non-zero socle. Therefore X is monolithic and by (6, Theorem 6′) X is Artinian.

Clearly $P = \pi(X)$ and by (**11**, Corollary C3) $P \cap J$ is a maximal ideal of J. By (**9**, 2.7) there exists a positive integer m such that $(P \cap J)^m \subseteq \operatorname{ann}_J(X)$. Moreover, X has a composition series and by (**11**, Corollary A) the set H of all elements g of G such that $X(1-g) = 0$ is a normal subgroup of finite index in G. If an element h of H has infinite order then $1-h$ is a regular element of JG and hence $X = X(1-h) = 0$. Thus H is periodic and hence finite. Therefore G is finite and this completes the proof of Theorem C.

Added in proof:

Corollary A was proved independently for the case when J is the ring of integers by R. L. Snider (Injective hulls of simple modules over group rings, in *Ring Theory* edited by S. K. Jain and K. E. Eldridge (Dekker, 1977), pp. 223–226).

REFERENCES

(1) I. G. CONNELL, On the group ring, *Canad. J. Math.* **15** (1963), 650–685.

(2) E. FORMANEK and A. V. JATEGAONKAR, Subrings of Noetherian rings, *Proc. Amer. Math. Soc.* **46** (1974), 181–186.

(3) A. W. GOLDIE, The structure of prime rings under ascending chain conditions, *Proc. London Math. Soc.* (3)**8** (1958), 589–608.

(4) P. HALL, Finiteness conditions for soluble groups, *Proc. London Math. Soc.* (3)**4** (1954), 419–436.

(5) P. HALL, On the finiteness of certain soluble groups, *Proc. London Math. Soc.* (3)**9** (1959), 595–622.

(6) A. V. JATEGAONKAR, Integral group rings of polycyclic-by-finite groups, *J. Pure Appl. Algebra* **4** (1974), 337–343.

(7) A. I. MAL'CEV, On certain classes of infinite soluble groups, *Mat. Sb.* **28** (1951), 567–588 (*Amer. Math. Soc. Translations* (2)**2** (1956), 1–21).

(8) E. MATLIS, Modules with descending chain condition, *Trans. Amer. Math. Soc.* **97** (1960), 495–508.

(9) Y. NOUAZÉ and P. GABRIEL, Ideaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente, *J. Algebra* **6** (1967), 77–99.

(10) D. S. PASSMAN, Primitive group rings, *Pacific J. Math.* **47** (1973), 499–506.

(11) J. E. ROSEBLADE, Group rings of polycyclic groups, *J. Pure Appl. Algebra* **3** (1973), 307–328.

(12) J. E. ROSEBLADE and P. F. SMITH, A note on hypercentral group rings, *J. London Math. Soc.* (2)**13** (1976), 183–190.

(13) A. ROSENBERG and D. ZELINSKY, Finiteness of the injective hull, *Math. Z.* **70** (1959), 372–380.

(14) R. L. SNIDER, Primitive ideals in group rings of polycyclic groups, *Proc. Amer. Math. Soc.* **57** (1976), 8–10.

(15) P. VÁMOS, The dual of the notion of 'finitely generated', *J. London Math. Soc.* **43** (1968), 643–646.

(16) A. E. ZALESSKII, Irreducible representations of finitely generated nilpotent torsion-free groups, *Math. Notes* **9** (1971), 117–123.

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