BULL. AUSTRAL. MATH. SOC. 20B05, 20BXX
VOL. 67 (2003) [249–256]

IMPROVEMENT ON THE BOUNDS OF PERMUTATION GROUPS WITH BOUNDED MOVEMENT

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Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. Then we define the movement of G as, $m :=$ move(G) $:= \sup_\Gamma\{|\Gamma^g \setminus \Gamma| \mid g \in G\}$. Let p be a prime, $p \geqslant 5$. If G is not a 2-group and p is the least odd prime dividing $|G|$, then we show that $n := |\Omega| \leqslant 4m - p + 3$.

Moreover, if we suppose that the permutation group induced by G on each orbit is not a 2-group then we improve the last bound of n and for an infinite family of groups the bound is attained.

1. INTRODUCTION

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for each subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g \in G$, we define the *movement* of Γ as move$(\Gamma) = \max_{g\in G} |\Gamma^g \setminus \Gamma|$. If move$(\Gamma) \leqslant m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* and the *movement* of G is defined as the maximum of move(Γ) over all subsets Γ, that is,

$$m := \text{move}(G) := \sup\{|\Gamma^g \setminus \Gamma| \mid \Gamma \subseteq \Omega, g \in G\}$$

This notion was introduced in [4]. By [4, Theorem 1], if G has movement m, then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m. In particular, it was proved that each G-orbit has length at most $3m$ and $n := |\Omega| \leqslant 5m - 2$. In [1] it was shown that $n = 5m - 2$ if and only if $n = 3$ and G is transitive. But in [3], this bound was refined further and it was shown that $n \leqslant (9m - 3)/2$. Moreover, if $n = (9m - 3)/2$ then either $n = 3$ and $G = S_3$ or G is an elementary Abelian 3-group and all its orbits have length 3. Now suppose that G is not a 2-group, and let p $(\geqslant 5)$ be the least odd prime dividing the $|G|$. Then by [4, Lemma 2,2], $n \leqslant (9m - 3)/2$. In this paper we aim to improve the bound for any group with above conditions as follows:

THEOREM 1.1. *Let p be a prime, $p \geqslant 5$, let m be a positive integer, and let G be a permutation group on a set Ω with movement m such that G has no fixed points in Ω. If G is not a 2-group and p is the least odd prime dividing $|G|$, then $n := |\Omega| \leqslant 4m-p+3$.*

Received 24th July, 2002

We do not know whether the bound in Theorem 1.1 is attained. But according to hypothesis in Theorem 1.1, and with the assumption that G is not a 2-group on each orbit, then we have the following result such that the bound is attained for an infinite family of groups. We denote by $P \ltimes K$ a semi-direct product $K.P$ with normal subgroup K.

THEOREM 1.2. *Let G be a permutation group on a set Ω with movement m such that G has no fixed points in Ω. If G is not a 2-group on any orbit, and $p \geqslant 5$ is the least odd prime dividing $|G|$, then the following hold:*

1. *$n := |\Omega| \leqslant (p/p-1)\Big(\lfloor(2mp/p-1)\rfloor - 1\Big)$ (for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x);*
2. *moreover, if $|\Omega| = (p/p-1)\Big(\lfloor(2mp/p-1)\rfloor - 1\Big)$ then either $n = p$, $m = (p-1)/2$, and $G :\leqslant Z_{2^{a_0}} \ltimes Z_p$ where $a_0 = \max\{a \mid 2^a \text{ divides } p-1\}$, that is, 2^{a_0} is the 2-part of $(p-1)$, or for some positive integer d, $G \cong Z_p^d$, $m = p^{d-1}(p-1)/2$ and all its orbits have length p.*

2. EXAMPLES AND PRELIMINARIES

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t nontrivial cycles of lengths $l_1, \dots, l_t$ say. We might represent g as

$$g = (a_1 a_2 \dots a_{l_1})(b_1 b_2 \dots b_{l_2}) \dots (z_1 z_2 \dots z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example, we could choose $\Gamma(g) = \{a_2, a_4, \dots, b_2, b_4, \dots, z_2, z_4, \dots\}$. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind we say that $\Gamma(g)$ *consists of every second point of every cycle of g*. From the definition of $\Gamma(g)$ we see that

$$\left|\Gamma(g)^g \backslash \Gamma(g)\right| = \left|\Gamma(g)\right| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \backslash \Gamma|$ for an arbitrary subset Γ of Ω.

LEMMA 2.1 ([2, Lemma 2.1].) *Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \backslash \Gamma| \leqslant \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$ where l_i is the length of the i^{th} cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

Now we shall show that there certainly is an infinite family of groups for which equality in Theorem 1.2(1) holds, for any prime p.

EXAMPLE 2.2. Let d be a positive integer, let $G := Z_p^d$, let $t := (p^d - 1)/(p-1)$, and let $H_1, \ldots, H_t$ be an enumeration of the subgroups of index p in G. Define Ω_i to be the coset space of H_i in G and $\Omega = \Omega_1 \cup \ldots \cup \Omega_t$. If $g \in G - \{1\}$, then g lies in $(p^{d-1} - 1)/(p-1)$ of the groups H_i and therefore acts on Ω as a permutation with $(p/p-1)(p^{d-1} - 1)$ fixed points and p^{d-1} orbits of length p. It follows from Lemma 2.1 that move$(G) = m = p^{d-1}(p-1)/2$. Thus $\lfloor (2mp/p-1) \rfloor = p^d$ and

$$n = |\Omega| = p.t = \frac{p}{p-1}(p^d - 1) = \frac{p}{p-1}\left(\left\lfloor \frac{2mp}{p-1} \right\rfloor - 1 \right).$$

Now we have the following lemma which is a classification of all transitive permutation groups G of degree p where p is the least odd prime dividing $|G|$.

LEMMA 2.3. *Let G be a transitive permutation group on a set Ω of size p, where $p \geqslant 5$ is the least odd prime dividing $|G|$. Then $G = Z_{2^a} \ltimes Z_p$, where $a \geqslant 0$, and $2^a \mid (p-1)$.*

PROOF: Let G be a transitive permutation group on a set Ω of size p. Then G is isomorphic to a transitive subgroup of S_p and so p is the largest prime divisor of $|G|$. Since p is also the least odd prime dividing $|G|$, we have $|G| = p.2^a$ for some $a \geqslant 0$. By Burnside's "pq theorem" (see [**6**, Theorem 2, 10, 17]) G is soluble, and hence by a theorem of Galois [**6**, Theorem 3.6.1], G is isomorphic to a subgroup of the group $AGL(1,p)$ of affine transformations of a finite field consisting of p elements. Thus $G = Z_{2^a} \ltimes Z_p$ as asserted. ∎

COROLLARY 2.4. *Let G be a permutation group on a set Ω, and suppose that Δ is a G-orbit of length p in Ω where p is the least odd prime dividing $|G|$. Then the induced permutation group G^Δ is $G = Z_{2^a} \ltimes Z_p$ where $0 \leqslant a \leqslant a_0$, and 2^{a_0} is the 2-part of $(p-1)$.*

3. THE MAXIMUM DEGREE OF BOUNDED MOVEMENT GROUPS

Suppose that $G \leqslant Sym(\Omega)$ and that G is not a 2-group and move$(G) = m$, and such that $p \geqslant 5$ is the least odd prime dividing the $|G|$. In this section we find an upper bound for $|\Omega|$ that is a linear function of m.

To prove the main theorems, we introduce the following notation.

$r_p :=$ number of G-orbits of length p on which G acts as Z_p;

$r_p'(a) :=$ number of G-orbits of length p on which G acts as $G = Z_{2^a} \ltimes Z_p$ with $1 \leqslant a \leqslant a_0$; and set $r_p' := \sum_{a=1}^{a_0} r_p'(a)$;

$\Phi :=$ union of G-orbits of lengths 2^b, where $1 \leqslant b \leqslant \log_2 p$; and u is the number of orbits in Φ.

$s :=$ number of G-orbits of length $> p$.

https://doi.org/10.1017/S0004972700033712 Published online by Cambridge University Press

The orbits are labelled accordingly. Thus $\Omega_1, \dots, \Omega_{r_p}$ are those of length p on which G acts as Z_p; $\Omega_{r_p+1}, \dots, \Omega_{r_p+r'_p}$ are those of length p on which G acts as $G = Z_{2^a} \ltimes Z_p$ for some $a \geqslant 1$; $\Omega_{r_p+r'_p+1}, \dots, \Omega_{r_p+r'_p+u}$ are those of length 2^b where $1 \leqslant b \leqslant \log_2 p$; *et cetera.* Define $t := r_p + r'_p + u + s$, $t_1 := r_p + r'_p + u$, and $t_0 = r_p + r'_p$. So t is the total number of G-orbits.

For $1 \leqslant i \leqslant t_0$ define K_i to be the kernel of the action of G on Ω_i and for $g \in G$ define $k(g)$ to be the number of i in that range for which g is not in K_i. For $g \in G$ and a G-invariant set Δ we denote by $\mathrm{fix}_\Delta(g) = \{\alpha \in \Delta \mid \alpha^g = \alpha\}$ and $\mathrm{supp}_\Delta(g) = \{\alpha \in \Delta \mid \alpha^g \neq \alpha\}$ the set of fixed points of g in Δ and the support of g in Δ, respectively (so that $|\mathrm{fix}_\Delta(g)| + |\mathrm{supp}_\Delta(g)| = |\Delta|$), and define $\mathrm{odd}_\Delta(g) :=$ the number non-trivial cycles of g in Δ that have odd length.

Lemma 3.1. *With the above notation, let* $\Delta := \bigcup_{i=t_1+1}^{t} \Omega_i$ *and let* $g \in G$. *Then*

$$\frac{p-1}{2}k(g) + \frac{1}{2}|\mathrm{supp}_\Phi(g)| + \frac{1}{2}\Big(|\mathrm{supp}_\Delta(g)| - \mathrm{odd}_\Delta(g)\Big) \leqslant m.$$

Proof: For each i such that $1 \leqslant i \leqslant t_0$ and g is not in K_i, since $|\Omega_i| = p$ then g^{Ω_i} is a p-cycle or a 2-element with one fixed point and we may choose a subset Γ_i of $(p-1)/2$ points of Ω_i such that $\Gamma_i^g \cap \Gamma_i = \emptyset$. Let Γ_0 be the set of chosen points from all the Γ_i for $1 \leqslant i \leqslant t_0$, and so by definition $\Gamma_0^g \cap \Gamma_0 = \emptyset$.

For each of the non-trivial cycles $(b_1 \dots b_{2l})$ and $(a_1 a_2 \dots a_k)$ of g in Φ and Δ respectively, adjoin the points $b_1, b_3, \dots, b_{2l-1}$ and also $a_1, a_3, \dots, a_{k'}$ to Γ_0, where k' is odd and $k-2 \leqslant k' \leqslant k-1$. Let Γ be the resulting set. It has been constructed so that $\Gamma^g \cap \Gamma = \emptyset$. Therefore $|\Gamma| \leqslant m$. Since

$$|\Gamma| = \frac{p-1}{2}k(g) + \frac{1}{2}|\mathrm{supp}_\Phi(g)| + \frac{1}{2}\Big(|\mathrm{supp}_\Delta(g)| - \mathrm{odd}_\Delta(g)\Big),$$

we have the stated inequality. ∎

To prove Theorem 1.1 we first prove the following lemma.

Lemma 3.2:

$$\left(\sum_{a=0}^{a_0} \frac{p-1}{2} . \Big(1 - \frac{1}{2^a p}\Big) r'_p(a) + \frac{|\Phi| - u}{2} + \frac{p-1}{2p}\big(|\Delta| - s\big)\right) < m,$$

where for $a = 0$ *the number* $r'_p(a)$ *is* r_p.

Proof: Suppose that $1 \leqslant i \leqslant t_0$. Then the group induced by G on Ω_i is $G = Z_{2^a} \ltimes Z_p$ for some $a \geqslant 0$, such that $2^a \mid (p-1)$, and since $|G : K_i| = 2^a p$, there are

$$|G| - |K_i| = (2^a p - 1)|K_i|$$

elements g which act nontrivially on Ω_i. It follows that

$$\sum_{g \in G} \frac{p-1}{2} k(g) = \frac{p-1}{2} \sum_{a=0}^{a_0} \Big(\frac{2^a p - 1}{2^a p}|G|\Big) r'_p(a)$$

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where $r'_p(0) := r_p$.

For $t_0 + 1 \leqslant i \leqslant t_1$, the group induced by G on Ω_i is a 2-group. The union of these sets Ω_i is Φ, and since by Burnside's Lemma [**5**, Theorem 3.26] the average number of fixed points of elements of G in Φ is the number u of G-orbits in Φ, we have

$$\sum_{g\in G}\frac{1}{2}|\mathrm{supp}_\Phi(g)| = \frac{1}{2}\sum_{g\in G}\Bigl(|\Phi| - \bigl|\operatorname*{fix}_{\Phi}(g)\bigr|\Bigr) = \frac{1}{2}|\Phi|.|G| - \frac{|G|}{2}.u.$$

Similarly,

$$\sum_{g\in G}\frac{1}{2}|\mathrm{supp}_\Delta(g)| = \frac{1}{2}|\Delta|.|G| - \frac{s|G|}{2}$$

and since $\mathrm{odd}_\Delta(g) \leqslant (1/p)\bigl|\mathrm{supp}_\Delta(g)\bigr|$, we have

$$\sum_{g\in G}\frac{1}{2}\Bigl(\bigl|\mathrm{supp}_\Delta(g)\bigr| - \mathrm{odd}_\Delta(g)\Bigr) \geqslant \frac{p-1}{2p}\bigl(|\Delta|.|G| - s|G|\bigr).$$

Thus adding the inequality of lemma 3.1 over all $g \in G$, we obtain

$$m|G| > |G|\Biggl(\sum_{a=0}^{a_0}\frac{p-1}{2}.\Bigl(1 - \frac{1}{2^a p}\Bigr)r'_p(a) + \frac{|\Phi| - u}{2} + \frac{p-1}{2p}\bigl(|\Delta| - s\bigr)\Biggr),$$

where the strict inequality recognises the fact the inequality of Lemma 3.1 is strict for the identity element of $|G|$. This completes the proof of Lemma 3.2. ∎

Now

$$n = \Bigl(\sum_{a=0}^{a_0} r'_p(a)\Bigr)p + |\Phi| + |\Delta|.$$

Also we have $|\Phi| \geqslant 2u$, and so

$$\frac{|\Phi| - u}{2} \geqslant \frac{|\Phi|}{4}.$$

Thus the inequality in Lemma 3.2 implies that

$$\begin{aligned} m &> \frac{n}{4} + \sum_{a=0}^{a_0} r'_p(a)\Bigl(\frac{p-1}{2} - \frac{p-1}{2^{a+1}p} - \frac{p}{4}\Bigr) + |\Delta|\Bigl(\frac{p-1}{2p} - \frac{1}{4}\Bigr) - \frac{p-1}{2p}s \\ &= \frac{n}{4} + \sum_{a=0}^{a_1} r'_p(a)\Bigl(\frac{p-2}{4} - \frac{p-1}{2^{a+1}p}\Bigr) + |\Delta|\Bigl(\frac{p-2}{4p}\Bigr) - \frac{p-1}{2p}s. \end{aligned}$$

Since G is not a 2-group, so we have either $\sum_{a=0}^{a_0} r'_p(a) > 0$ or $s > 0$. If some $r'_p(a) > 0$, then

$$m > \frac{n}{4} + \frac{p-2}{4} - \frac{p-1}{2^{a+1}p}.$$

But we note that since $p \geqslant 5$, for each $a \geqslant 0$,

$$\frac{p-2}{4} - \frac{p-1}{2^{a+1}p} \geqslant \frac{p-2}{4} - \frac{p-1}{2p} > 0.$$

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Hence,

$$m > \frac{n}{4} + \frac{p-2}{4} - \frac{p-1}{2p} = \frac{n}{4} + \frac{p^2-4p+2}{4p}.$$

On the other hand if $s > 0$, then $|\Delta| \geqslant (p+1)s \geqslant p+1$. Thus,

$$\begin{aligned} m &> \frac{n}{4} + |\Delta|\Big(\frac{p-2}{4p}\Big) - \frac{p-1}{2p}s \\ &\geqslant \frac{n}{4} + s\Big(\frac{(p+1)(p-2)}{4p} - \frac{p-1}{2p}\Big) \\ &\geqslant \frac{n}{4} + \frac{p^2-3p}{4p}. \end{aligned}$$

So in either case we must have,

$$m > \frac{n}{4} + min\Big\{\frac{p^2-4p+2}{4p}, \frac{p^2-3p}{4p}\Big\} = \frac{n}{4} + \frac{p^2-4p+2}{4p}.$$

Hence,

$$n < 4m - \frac{p^2-4p+2}{4p} = 4m - p + 4 - \frac{2}{p}.$$

That is, $n < 4m - p + 4$ and so $n \leqslant 4m - p + 3$. Hence the proof of Theorem 1.1 is complete. ∎

4. The proof of Theorem 1.2

In this section with the notation defined in Section 3 we refine the upper bound $4m - p + 3$ for n and obtain an upper bound that is attained for an infinite family of groups. For this purpose suppose that G is not a 2-group on any orbit, that is $u = 0$. Then we have the following lemma.

Lemma 4.1. *With the above notation,*

$$n < \frac{2p^2m}{(p-1)^2} - \Big[\frac{p}{p-1}\sum_a \frac{2^a-1}{2^a}r'_p(a) + \frac{1}{p-1}|\Delta| - \frac{p}{p-1}s\Big].$$

Proof: Since G is not a 2-group on any orbit, $u = 0$ and so $|\Phi| = 0$ the inequality of Lemma 3.2, yields

$$m > \Big(\sum_{a=0}^{a_0} \frac{p-1}{2}.\Big(1 - \frac{1}{2^a p}\Big)r'_p(a) + \frac{p-1}{2p}(|\Delta| - s)\Big).$$

Since $n = pr_p + pr'_p + |\Delta|$, we have

$$\begin{aligned} \frac{2p^2m}{(p-1)^2} &> pr_p + \frac{p}{p-1}\sum_a \frac{2^a p - 1}{2^a}r'_p(a) + \frac{p}{p-1}(|\Delta| - s) \\ &= pr_p + pr'_p + \frac{p}{p-1}\sum_a \frac{2^a-1}{2^a}r'_p(a) + |\Delta| + \frac{1}{p-1}|\Delta| - \frac{p}{p-1}s. \end{aligned}$$

Hence,

$$\frac{2p^2m}{(p-1)^2} > n + \left[\frac{p}{p-1}\sum_a \frac{2^a-1}{2^a} r'_p(a) + \frac{1}{p-1}|\Delta| - \frac{p}{p-1}s\right],$$

and we have the stated inequality.

Now define $\eta := 2p^2m - (p-1)^2n$. Clearly η is an integer and from Lemma 3.3 we have that

$$0 \leqslant p(p-1)\sum_a \frac{2^a-1}{2^a} r'_p(a) + (p-1)|\Delta| - (p-1)p.s < \eta.$$

If $\eta > p(p-1)$, then part 1 of Theorem 1.2 follows. To prove the theorem we suppose that $\eta \leqslant p(p-1)$ and seek to discover what configurations may occur.

From the inequality

$$0 \leqslant p(p-1)\sum_a \frac{2^a-1}{2^a} r'_p(a) + (p-1)|\Delta| - (p-1)p.s < \eta \leqslant p(p-1)$$

we obtain,

$$0 \leqslant p\sum_a \frac{2^a-1}{2^a} r'_p(a) + \big(|\Delta| - (p-1)p.s\big) < p.$$

First we show that $s = 0$. Since if $s \neq 0$, then by [**4**, Lemma 2]

$$p.t < n \leqslant t + \frac{2mp}{p-1} - 1.$$

Hence,

$$t < \frac{1}{p-1}\Big(\frac{2mp}{p-1} - 1\Big),$$

and then we have

$$n \leqslant t + \frac{2mp}{p-1} - 1 < \frac{p}{p-1}\Big(\frac{2mp}{p-1} - 1\Big),$$

which contradicts our assumption. Hence $s = 0$, and so $|\Delta| = 0$ and

$$0 \leqslant p\sum_a \frac{2^a-1}{2^a} r'_p(a) < \eta/(p-1) \leqslant p.$$

This inequality implies that $r'_p \leqslant 1$. If $r_p = 0$ and $r'_p \neq 0$, then $n = p = 2mp/(p-1)$. Following [**2**] and Lemma 2.3, the last equality holds if and only if $m = (p-1)/2$ and $G = Z_{2^a} \ltimes Z_p$, where $a \geqslant 1$, and $2^a \mid (p-1)$.

Moreover, suppose that $r'_p = 0$. Then $n = p.r_p$ and by Example 2.2 G is an elementary Abelian p-group.

Now we consider the case that $r_p > 0$, and $r'_p > 0$, that is $r'_p = 1$. Define $\Sigma_1 := \bigcup_{i=1}^{r_p} \Omega_i$, the union of the orbits of length p on which G acts as Z_p, and Σ_2, the unique of the orbit of length p on which G acts as $G = Z_{2^a} \ltimes Z_p$, where $a \geqslant 1$ and $2^a \mid (p-1)$.

Let $m_1 := \mathrm{move}(G^{\Sigma_1})$, and $m_2 := \mathrm{move}(G^{\Sigma_2}) = (p-1)/2$. By Example 2.2 each non-identity element of G^{Σ_1} have the same movement m_1, so by considering of the each p-element of G^{Σ_2} it follows easily that $m = m_1 + m_2$. Defining $n_1 := |\Sigma_1|$, and $n_2 := |\Sigma_2| = p = 2mp/(p-1)$. We have from Example 2.2 that

$$n_1 \leqslant \frac{p}{p-1}\left(\left\lfloor \frac{2m_1p}{p-1}\right\rfloor - 1\right),$$

and so

$$n = n_1 + n_2 \leqslant \frac{p}{p-1}\left(\left\lfloor \frac{2m_1p}{p-1}\right\rfloor - 1\right) + \left\lfloor \frac{2m_2p}{p-1}\right\rfloor < \frac{p}{p-1}\left(\left\lfloor \frac{2m_1p}{p-1}\right\rfloor - 1\right) + \frac{p}{p-1}\left\lfloor \frac{2m_2p}{p-1}\right\rfloor.$$

Hence,

$$n < \frac{p}{p-1}\left(\left\lfloor \frac{2mp}{p-1}\right\rfloor - 1\right).$$

What this has shown is that, if $\eta := 2p^2m - (p-1)^2n$, that is, $n \geqslant \Big(\lfloor (2mp/p-1)\rfloor - 1\Big)/(p-1)$, then either $r_p = 0$ or $r'_p = 0$ and we have the situation posed in above. In this cases

$$n = \left\lfloor \frac{2mp}{p-1}\right\rfloor = \frac{p}{p-1}\left(\left\lfloor \frac{2mp}{p-1}\right\rfloor - 1\right).$$

Thus the result now follows and the proof of Theorem 1.2 is complete. ∎

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