SPACES OF CLOSED SUBGROUPS OF A CONNECTED LIE GROUP

by N. OLER†

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In a sequence of two papers which appeared in 1968 and 1969 Herbert Abels [**1**, **2**] has developed, from a method originated by Gerstenhaber [**6**], a means for extending the study of properly discontinuous groups of transformations to that of proper transformation groups in general. We recall that, if G is a Hausdorff locally compact group of transformations of a locally compact space X, then the action of G is proper when, for any two compact subsets K and L, the subset $G(K, L) = \{g \in G : gL \cap K \neq 0\}$ of G is compact (see [**3**], p. 55). In what follows all groups and spaces will be Hausdorff and locally compact. If H is a closed subgroup of G, then it is clear that the property just defined is possessed by the action of H as a group of left translations of G.

Let $\Sigma(G)$ be the space of closed subgroups of G with the following topology. A neighbourhood $Nd(H; K, U)$ of a subgroup H is determined by each compact subset K of G and open neighbourhood U of the identity of G by the definition

$$Nd(H; K, U) = \{H' : H' \cap K \subset HU \text{ and } H \cap K \subset H'U\}.$$

We refer to this as the Chabauty topology, it having been defined first by Chabauty [**5**] for discrete subgroups. It is easy to verify (c.f. Bourbaki [**4**]) that the neighbourhoods defined above constitute a basis for a Hausdorff topology and that the same topology is generated if, in the above definition of $Nd(H; K, U)$, HU and $H'U$ are replaced respectively by UH and UH'.

The object of this note is to apply the method of Abels to obtain a proof of the following

THEOREM. *Let G be a connected Lie group, $\Sigma(G)$ the Chabauty space of closed subgroups of G and $\Sigma_0(G)$ the subspace of $\Sigma(G)$ consisting of subgroups H of G with the property that the quotient space G/H is compact. Then $\Sigma_0(G)$ is open in $\Sigma(G)$.*

We begin by describing the basic notion of a *fundamental system* as defined in [**1**]. (The application we wish to make does not require the more specialized " uniform fundamental system " introduced in [**2**].) Let G be a continuous group of homeomorphisms of a space X. For any two subsets K and L of X, we let $G(K, L) = \{g \in G : gL \cap K \neq 0\}$ and $H(K, L) = G(K, L) \cap H$ for any subset H of G. A subset F of X, is called a fundamental set for G acting on X when (i) F is a G-covering: $GF = X$ and (ii) $G(K, F)$ is relatively compact for each compact subset K of X. A fundamental system is a " localization " of the notion of a fundamental set obtained by restricting these requirements to suitable subsets of X and G respectively. More precisely, a triple (F, Q, E) in which F is a closed subset of X, Q is an open neighbourhood of F and E is an open neighbourhood of the identity of G is called a *fundamental system* (for

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G acting on X) when the following conditions are satisfied:

(1) $gF \cap Q \neq 0$, $\forall g \in E$, i.e., $E(Q, F) = E$.
(2) If K is a nonempty compact subset of Q, then $E(K, F)$ is nonempty and compact.
(3) For each $x \in Q$ and $g \in E(x, F)$,

$$E(g^{-1}x, F) = g^{-1}E(x, F).$$

(4) E generates G.

Condition (2) says in particular that $EF \supset Q$. Indeed, if we replace E by G and Q by X, then conditions (1), (3) and (4) hold for any nonempty subset F and condition (2) says precisely that F is a fundamental set. The problem is in fact just this: Given a fundamental system (F, Q, E), to determine conditions under which F is then necessarily a fundamental set. It is shown in [**1**] how to construct a space X' on which G acts as a continuous group of transformations and a local homeomorphism π of X' into X with the following properties. (i) π commutes with the actions of G on X and X', i.e., the diagram

$$\begin{array}{ccc} G \times X' & \longrightarrow & X' \\ {\scriptstyle id \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ G \times X & \longrightarrow & X \end{array}$$

is commutative. (ii) Furthermore, X' contains an open subset Q' which is homeomorphic under π to Q and $F' = \pi|_{Q'}^{-1}(F)$ is a closed fundamental set with respect to the action of G on X'. The principal result concerning the " associated pair " (X', π) which we shall employ is

THEOREM (H. Abels). *Let (F, Q, E) be a fundamental system with respect to the continuous group of transformations G acting on X and let (X', π) be the " associated pair ". If*

(1) *X is locally connected and connected,*
(2) *Q is connected,*
(3) *there exists a G-invariant uniform structure which induces the topology of X,*
(4) *F is compact,*

then (X', π) is a covering space of X.

We proceed now to the proof of our theorem. G being a connected Lie group, it is the product of a maximal compact subgroup and a euclidean space. Its fundamental group is therefore finitely generated. Furthermore, G admits a simply connected covering space. It follows (see [**1**, Proposition 4]) that G has the following property. There exists a compact subset K of G such that, for any covering space (X', p) of X, if X' contains a subset L which is homeomorphic to K under $p|_L$, then (X', p) is a trivial covering space.

Let H be in $\Sigma_0(G)$. Then there exists a compact subset F of G such that $HF = G$; we may further suppose that $F \supset K$ and $e \in F$. Since G is locally compact, locally connected and connected, any two points of G are contained in a relatively compact connected subset, from which it follows that F, being compact, can be covered by a finite number of relatively compact connected sets having a common point. Hence F has a relatively compact connected open neighbourhood. So let Q be such a neighbourhood of F and let $E = H(Q, F)$. Then E

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is a relatively compact subset containing the identity which is open in H. We choose W, a compact neighbourhood of the identity, such that $WF \subset Q$ and consider the neighbourhood $Nd(H, \bar{E}, W)$ of H in $\Sigma(G)$. Let $H' \in Nd(H, \bar{E}, W)$; then $H' \cap \bar{E} \subset HW$ and $H \cap \bar{E} \subset H'W$. If $q \in Q$, then $q = gf$ for some $g \in E$ and $f \in F$. Also, $g \in E$ implies that $g = g'w$ for some $g' \in H'$ and $w \in W$. Thus $q = g'wf$ and $Q \subset H'WF$. Set $E' = H'(Q, WF)$; then clearly $E' = E'(Q, WF)$ and $Q \subset E'WF$. Let K be any nonempty compact subset of Q. Then $E'(K, WF)$ is nonempty and furthermore $E'(K, WF) = KF^{-1}W^{-1} \cap H'$, a closed subset of $KF^{-1}W^{-1}$ which is compact; hence $E'(K, WF)$ is compact. Let $x \in Q$ and $g \in E'(x, WF)$. Since H' is a subgroup, it is clear that $g^{-1}xF^{-1}W^{-1} \cap H' = g^{-1}[xF^{-1}W^{-1} \cap H']$ and hence $E'(g^{-1}x, WF) = g^{-1}E'(x, WF)$. Now let H'' be the subgroup of H' generated by E'. What we have shown is that (WF, Q, E') is a fundamental system for H'' acting on G. The conditions of Abels's theorem are satisfied, enabling us to conclude therefore that (G', π), the associated space of (WF, Q, E'), is a covering space of G. We have moreover that G' contains a subset F' which is homeomorphic to WF under $\pi|_{F'}$. It follows that π must therefore be a homeomorphism. Since, furthermore, F' is a fundamental set and π commutes with the actions of H'' on G' and G, we find that F is a fundamental set for H'' acting on G, so that, in particular, $H''WF = G$. Now, for $g \in H'$, $g = g_1 f$, where $g_1 \in H''$ and $f \in WF$. This implies that $g^{-1}g_1 \in H'(e, WF) \subset H'(Q, WF) = E'$; hence $g^{-1} \in g_1^{-1}E' \subset H''$. Thus $H' = H''$ and the proof is complete.

We conclude by remarking that, if $S(G)$ and $S_0(G)$ denote the subspaces of $\Sigma(G)$ and $\Sigma_0(G)$, respectively, obtained by restricting to discrete subgroups, then A. M. Macbeath [7] has shown that $S_0(G)$ is the union of open subsets of $S(G)$ consisting of isomorphic subgroups.

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UNIVERSITY OF PENNSYLVANIA
and
UNIVERSITY OF BIRMINGHAM

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