

A NOTE ON CAUCHY'S ROOT TEST

by I. S. MURPHY

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Throughout this note we deal with a series $\sum a_n$ of positive terms. The following tests for the convergence of this series are well-known.

TEST 1. (Ratio test). Let $\lim_{n \rightarrow \infty} a_{n+1}/a_n = K$. Then, if $K < 1$, $\sum a_n$ converges, while if $K > 1$, $\sum a_n$ diverges.

TEST 2. (Root test). Let $\lim_{n \rightarrow \infty} a_n^{1/n} = K$. Then, if $K < 1$, $\sum a_n$ converges, while if $K > 1$, $\sum a_n$ diverges.

It is known that Test 2 is more powerful than Test 1. Theorem 5.3 of Phillips [3] states that if $\lim_{n \rightarrow \infty} a_{n+1}/a_n = K$, then $\lim_{n \rightarrow \infty} a_n^{1/n} = K$. This shows that Test 2 is at least as powerful as Test 1. In the wake of this, consideration of a series like

$$a + a + a^2 + a^2 + a^3 + a^3 + \dots \quad (0 < a < 1),$$

to which Test 2 applies while Test 1 does not, shows that Test 2 is strictly more powerful than Test 1.

A third test is Gauss's test. Like the Ratio test, it scans the *ratio* of successive terms. It is however much more incisive than the Ratio test. The form of Gauss's test given as Test 3 is Ex.34.1 of Rankin [4]. Other forms occur in Bromwich [1] and Knopp [2].

TEST 3. (Gauss's test). Suppose that there exists X such that for all $n \geq X$,

$$a_{n+1}/a_n = 1 - \alpha/n + \varepsilon_n.$$

- (i) If $\alpha > 1$, and $\varepsilon_n = o(1/n)$, then $\sum a_n$ converges.
- (ii) If $\alpha < 1$, and $\varepsilon_n = o(1/n)$, then $\sum a_n$ diverges.
- (iii) If $\alpha = 1$, and $\varepsilon_n = o(1/(n \log n))$, then $\sum a_n$ diverges.

The purpose of this note is to provide a fourth test. The relationship between Test 4 and Gauss's test is analogous to the relationship between the Root test and the Ratio test.

TEST 4. Suppose there exists X such that for all $n \geq X$,

$$a_n^{1/(n \log n)} = 1 - \alpha/n + \varepsilon_n.$$

- (i) If $\alpha > 1$, and $\varepsilon_n = o(1/n)$, then $\sum a_n$ converges.
- (ii) If $\alpha < 1$, and $\varepsilon_n = o(1/n)$, then $\sum a_n$ diverges.
- (iii) If $\alpha = 1$, and $\varepsilon_n = o(\log \log n / (n \log n))$, then $\sum a_n$ diverges.

Proof. Parts (i) and (ii) are similar. We prove part (ii). Suppose that $\alpha < 1$ and $\varepsilon_n = o(1/n)$. Then for all $\varepsilon > 0$, there exists an integer N such that

$$a_n^{1/(n \log n)} \geq 1 - (\alpha + \varepsilon)/n \quad \text{for all } n \geq N.$$

Choosing $\varepsilon = \frac{1}{2}(1 - \alpha)$, we deduce that, for all sufficiently large n ,

$$\begin{aligned} a_n &\geq [1 - (\alpha + 1)/2n]^{n \log n} \\ &= \exp \{n \log n \log [1 - (\alpha + 1)/2n]\} \\ &= \exp [-\frac{1}{2}(\alpha + 1) \log n + O(\log n/n)] \\ &\geq \frac{1}{2}n^{-\frac{1}{2}(\alpha + 1)}. \end{aligned}$$

So $\sum a_n$ diverges by comparison with $\sum n^{-\frac{1}{2}(\alpha + 1)}$.

To prove part (iii), suppose that $\alpha = 1$ and that $\varepsilon_n = o(\log \log n / (n \log n))$. Then, for all $\varepsilon > 0$, we deduce that, for all sufficiently large n ,

$$\begin{aligned} a_n &\geq (1 - 1/n - \varepsilon \log \log n / n \log n)^{n \log n} \\ &= \exp [n \log n \log (1 - 1/n - \varepsilon \log \log n / n \log n)] \\ &= \exp [-\log n - \varepsilon \log \log n + O(\log n/n)] \\ &\geq \frac{1}{2}n^{-1}(\log n)^{-\varepsilon}. \end{aligned}$$

Taking $\varepsilon = 1$, we can deduce that $\sum a_n$ diverges by comparison.

The following result shows that Test 4 is at least as powerful as Gauss's test.

THEOREM. *Consider the following two conditions.*

(A) *There exists X such that for all $n \geq X$,*

$$a_{n+1}/a_n = 1 - \alpha/n + \varepsilon_n.$$

(B) *There exists Y such that for all $n \geq Y$,*

$$a_n^{1/(n \log n)} = 1 - \alpha/n + \delta_n.$$

Then the following results hold.

(i) *If $\alpha \neq 1$ and (A) holds with $\varepsilon_n = o(1/n)$, then (B) holds with $\delta_n = o(1/n)$.*

(ii) *If $\alpha = 1$ and (A) holds with $\varepsilon_n = o(1/(n \log n))$, then (B) holds with $\delta_n = o(\log \log n / (n \log n))$.*

Proof. (i) Suppose (A) holds with $\alpha \neq 1$ and $\varepsilon_n = o(1/n)$. Then for all $\varepsilon > 0$, there exists N such that

$$1 - \frac{\alpha + \varepsilon}{n} \leq \frac{a_{n+1}}{a_n} \leq 1 - \frac{\alpha - \varepsilon}{n} \quad \text{for all } n \geq N.$$

So for all $n \geq N$, we have

$$\log \left(1 - \frac{\alpha + \varepsilon}{n} \right) \leq \log \frac{a_{n+1}}{a_n} \leq \log \left(1 - \frac{\alpha - \varepsilon}{n} \right).$$

By increasing N , if necessary, and by demanding that ε is sufficiently small, we can ensure

that, for all $n \geq N$,

$$-\left(\frac{\alpha + 2\varepsilon}{n}\right) \leq \log \frac{a_{n+1}}{a_n} \leq -\left(\frac{\alpha - \varepsilon}{n}\right).$$

So, on summation, we find that

$$-(\alpha + 2\varepsilon) \sum_{r=N}^{n-1} \frac{1}{r} \leq \sum_{r=N}^{n-1} \log \frac{a_{r+1}}{a_r} \leq -(\alpha - \varepsilon) \sum_{r=N}^{n-1} \frac{1}{r}.$$

Recalling that

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \rightarrow \gamma \text{ as } n \rightarrow \infty,$$

we can write

$$K_1 - (\alpha + 2\varepsilon) \log(n - 1) \leq \log \frac{a_n}{a_N} \leq K_2 - (\alpha - \varepsilon) \log(n - 1),$$

where K_1 and K_2 are constants. Hence

$$K_3 - (\alpha + 2\varepsilon) \log(n - 1) \leq \log a_n \leq K_4 - (\alpha - \varepsilon) \log(n - 1),$$

where K_3 and K_4 are constants. Dividing by $n \log n$ and noting that $\log(n - 1)/\log n = 1 + O(1/(n \log n))$, we deduce that there exist constants K_5 and K_6 such that for all sufficiently large n

$$\frac{K_5}{n \log n} - \frac{\alpha + 2\varepsilon}{n} \leq \log a_n^{1/(n \log n)} \leq \frac{K_6}{n \log n} - \frac{\alpha - \varepsilon}{n}.$$

So, for all sufficiently large n ,

$$\log a_n^{1/(n \log n)} = -\alpha/n + o(1/n),$$

and therefore

$$a_n^{1/(n \log n)} = 1 - \alpha/n + o(1/n).$$

(ii) If we start from

$$1 - 1/n - \varepsilon/(n \log n) \leq \frac{a_{n+1}}{a_n} \leq 1 - 1/n + \varepsilon/(n \log n)$$

for all $n \geq N$, the proof is similar to the proof of (i) except that in this case we are forced to consider not only $\sum_{r=N}^{n-1} 1/r$ but also $\sum_{r=N}^{n-1} 1/(r \log r)$. The latter can be dealt with by using the fact that

$$\frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \dots + \frac{1}{n \log n} - \log \log n$$

tends to a finite limit as $n \rightarrow \infty$.

COROLLARY. *When Gauss's test works, so also does Test 4.*

REMARKS. 1. That Test 4 is strictly more powerful than Gauss's test can be seen from the series $\sum a_n$, where $a_{2n-1} = a_{2n} = [1 - (3/2n)]^{2n \log 2n}$ for all $n \geq 1$. Here Gauss's test is inapplicable since $a_{2n}/a_{2n-1} = 1$, for all $n \geq 1$. However, $a_n^{1/(n \log n)} = 1 - (3/n) + o(1/n)$ for large values of n , so that the series converges by Test 4.

2. In practice, it is found that the Ratio test is more useful than the Root test, despite being less powerful. So also is it with Gauss's test and Test 4.

REFERENCES

1. T. J. I'A. Bromwich, *Infinite series* (Macmillan, 1908).
2. K. Knopp, *Theory and application of infinite series* (Blackie, 1928).
3. E. G. Phillips, *A course of analysis* (Cambridge, 1962).
4. R. A. Rankin, *An introduction to mathematical analysis* (Pergamon, 1963).

UNIVERSITY OF GLASGOW
SCOTLAND

CORRIGENDUM

to the paper

ABELIAN ERGODIC THEOREMS FOR VECTOR-VALUED FUNCTIONS†

by P. E. KOPP

The definition of $\Omega_{f,a}$ on p. 57 should read

$$\Omega_{f,a} = \bigcup_{0 < \rho < 1} \{w \in \Omega : (1 - \rho) \|(R_\rho f)(w)\|_X > a\}.$$

† Vol. 16 (1975), 57–60.