Canad. Math. Bull. Vol. 22 (1), 1979

A SECOND NOTE ON INGHAM'S SUMMATION METHOD

BY
S. L. SEGAL

A series $\sum a_n$ is said to be summable (I) to the limit A if

$$(*)\qquad \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\sum_{d\mid n} da_d = A.$$

Clearly the limit is the same whether $x\to\infty$ through all real values or only positive integer values, and the expression whose limit is being taken can also be expressed in the two equivalent forms

$$\frac{1}{x}\sum_{d\le x} da_d\left[\frac{x}{d}\right] \quad\text{and}\quad \frac{1}{x}\sum_{d\le x}\sum_{m\le x/d} ma_m$$

where $[x]$ is the greatest integer $\le x$. The method called (I) was introduced by Ingham [4] in connection with a novel proof of the prime number theorem and independently by Wintner [8]. The denomination (I) for it was coined by Hardy [2, Appendix IV]. Despite the fact that the method (I) is not regular, it has a certain number-theoretic interest as the definition (*) would suggest. Such number theoretic connections are discussed in [4], [2, Appendix IV], [6]. Two limitation theorems are known for (I). If $\sum_{a_n}$ is summable (I), then

(i) $a_n = o(\log\log n)$ as $n\to\infty$ [2, Appendix IV]

and

(ii) $\sum_{n\le x} a_n = o(\log x)$ as $x\to\infty$ [7]

Clearly neither (i) nor (ii) includes the other and it had been an open question for sometime whether these were best possible. In [1], the author and P. Erdös show that (i) is best possible by actual construction of an appropriate series. I have recently realized that (ii) can also be shown to be best possible, but the proof is non-constructive. The purpose of this brief note is to give that proof. Throughout, all variables other than x indicate positive integers, $\mu(n)$ is the Möbius function, $N(x)=\sum_{n\le x}(\mu(n)/(n)$, and $[x]$ is the greatest integer $\le x$.

Received by the editors April 25, 1978 and, in revised form, June 13, 1978.

THEOREM. *Let $\varepsilon(x)$ be any positive function decreasing to 0 monotonically but arbitrarily slowly as $x\to\infty$. Then there exists a series $\sum a_n$ which is (I)-summable and such that*

$$\sum_{n\le x} a_n \neq o(\varepsilon(x)\log x) \quad \text{as} \quad x\to\infty.$$

Proof. Let $K(n)$ be any sequence indexed by the positive integers tending to 0 as $n\to\infty$. (Define, for convenience, $K(0)=0$.) Define b_n by

$$(1) \qquad b_n = \frac{1}{n}\sum_{d\mid n}\mu\left(\frac{n}{d}\right)(dK(d)-(d-1)K(d-1)).$$

Then for positive integers t,

$$\frac{1}{t}\sum_{n\le t}\sum_{d\mid n} db_d = K(t),$$

And so $\sum b_n$ is (I)-summable to 0.

Furthermore

$$(2) \qquad \begin{aligned}\sum_{n\le r} b_n &= \sum_{d\le r}\frac{dK(d)-(d-1)K(d-1)}{d}\sum_{m\le r/d}\frac{\mu(m)}{m}\\ &= \sum_{d\le r}(K(d)-K(d-1))\,N\left(\frac{r}{d}\right)+\sum_{d\le r}\frac{K(d-1)}{d}N\left(\frac{r}{d}\right).\end{aligned}$$

For the second sum on the right, we have, since $K(d)$ is bounded,

$$(3) \qquad \left|\sum_{d\le r}\frac{K(d-1)}{d}N\left(\frac{r}{d}\right)\right| = 0\left(\sum_{d\le r}\frac{1}{d}\left|N\left(\frac{r}{d}\right)\right|\right) = 0(1) \quad \text{as} \quad r\to\infty$$

by a remark of Rubel [5, Correction].

For the first sum on the right,

$$(4) \qquad \sum_{d\le r}(K(d)-K(d-1))\,N\left(\frac{r}{d}\right) = \sum_{d\le r}K(d)\left(N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right)$$

since $N(X)=0$ for $X<1$.

Substituting (4) and (3) into (2) gives

$$(5) \qquad \sum_{d\le r}K(d)\left(N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right) = \sum_{n\le r} b_n + 0(1) \quad \text{as} \quad r\to\infty.$$

Suppose now the theorem were false; that is suppose there is a positive function $\varepsilon(x)\to 0$ as $x\to\infty$, such that for every I-summable series $\sum a_n$, $\sum_{n\le x} a_n = o(\varepsilon(x)\log x)$ as $x\to\infty$. Then by the above construction, (5) says that for every sequence $K(n)\to 0$ as $n\to\infty$,

$$(6) \qquad \sum_{d\le r}K(d)\left(N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right) = o(\varepsilon(r)\log r).$$

Define $C_{r,d}$ by

$$C_{r,d}=\frac{1}{\varepsilon(r)\log r}\left(N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right)$$

Then (6) says that the matrix $\|C_{r,d}\|$ transforms all sequences tending to 0 into sequences tending to 0. The conditions for an infinite matrix to have this property are well-known (e.g. [2, p. 49]) and so, in particular, we must have

$$\text{(7)}\qquad \frac{1}{\varepsilon(r)\log r}\sum_{d\le r}\left|N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right|<C$$

for all r, where C is a constant independent of r.

On the other hand,

$$\text{(8)}\qquad \begin{aligned}\sum_{d\le r}\left|N(r/d)-N(r/d+1)\right|&=\sum_{d\le r}\left|\sum_{(r/d+1)<m\le r/d}\frac{\mu(m)}{m}\right|\\ &\ge\sum_{r^{1/2}<d\le r}\left|\sum_{(r/d+1)<m\le r/d}\frac{\mu(m)}{m}\right|.\end{aligned}$$

Now for $r^{1/2}<d\le r$, $r/d(d+1)<1$, and hence the inner sum contains at most one term. As is well-known [e.g. [3], Theorem 343 and partial summation],

$$\sum_{m\le y}\frac{|\mu(m)|}{m}=\frac{6}{\pi 2}\log y+0(1)\quad\text{as}\quad y\to\infty.$$

Hence, we have from (8),

$$\text{(9)}\qquad \begin{aligned}\sum_{d\le r}\left|N\left(\frac{r}{d}\right)-N\left(\frac{r}{d+1}\right)\right|&\ge\sum_{r^{1/2}<d\le r}\left|\sum_{(r/d+1)<m\le r/d}\frac{\mu(m)}{m}\right|\\ &=\sum_{r^{1/2}<d\le r}\ \sum_{r/d+1<m\le r/d}\frac{|\mu(m)|}{m}\\ &=\sum_{1\le d<(r/[r^{1/2}]+1)}\frac{|\mu(m)|}{m}\\ &=(3/\pi^2)\log r+0(1)\quad\text{as}\quad r\to\infty.\end{aligned}$$

But (9) contradicts (7) since $\varepsilon(r)\to 0$ as $r\to\infty$, which proves the theorem.

ACKNOWLEDGEMENT. I wish to thank Prof. K. Ramachandra for a remark which indicated that (ii) was in fact best possible.

REFERENCES

1. Erdös, P. and S. L. Segal, *A note on Ingham's Summation Method*, Jour. Number Theory, **10**, (1978), 95–98.

2. Hardy, G. H. *Divergent Series*, Oxford, 1949.

3. Hardy, G. H. and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford (**4th** edition, 1965).

https://doi.org/10.4153/CMB-1979-019-2 Published online by Cambridge University Press

4. Ingham, A. E., *Some Tauberian Theorems connected with the Prime Number Theorem*, J. London Math. Soc. **20,** (1945), 171–180.

5. Rubel, L. A., *An Abelian Theorem for Number-Theoretic Sums*, Acta Arithmetica **6,** (1960), 175–177, Correction Acta Arithmetica **6,** (1961), 523.

6. Segal, S. L., *Ingham's Summation Method and the Riemann Hypothesis*, Proc. Lond. Math. Soc. (3) **30,** (1975), 129–142, Carrijendum **34** (1977), 438.

7. Segal, S. L., *On Ingham's Summation Method*, Canadian Journal Math., **18,** (1966), 97–105.

8. Wintner, A., *Eratosthenian Averages*, Baltimore, 1943.

Department of Mathematics
The University of Rochester
Rochester, New York 14627

https://doi.org/10.4153/CMB-1979-019-2 Published online by Cambridge University Press