Canad. Math. Bull. Vol. **51** (3), 2008 pp. 378–385

Cyclic Vectors in Some Weighted L^p Spaces of Entire Functions

Kou Hei Izuchi

Abstract. In this paper, we generalize a result recently obtained by the author. We characterize the cyclic vectors in $L^p_a(\mathbb{C}, \phi)$. Let $f \in L^p_a(\mathbb{C}, \phi)$ and $f\mathcal{C}$ be contained in the space. We show that f is non-vanishing if and only if f is cyclic.

1 Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the unit circle. We denote the polynomial ring by $\mathcal{C}$, and the space of all entire functions by $\mathrm{Hol}(\mathbb{C})$. Let X be a complete semi-normed space of holomorphic functions on a domain Ω in $\mathbb{C}$. For a subset E of X, let $\overline{E}$ be the closure of E in X. A function f is said to be cyclic in X if $f\mathcal{C} \subset X$ and $\overline{f\mathcal{C}} = X$. In the Hardy spaces $H^p(\mathbb{D})$, it is well known that a function is cyclic if and only if it is $H^p(\mathbb{D})$-outer [Gar]. Also in the Bergman spaces $L^p_a(\mathbb{D})$ $(0 < p < \infty)$, it is known that a function is cyclic if and only if it is $L^p_a(\mathbb{D})$-outer [HKZ]. The classical Fock space $L^2_a(\mathbb{C})$ is the space of all μ-square integrable entire functions on $\mathbb{C}$, where

$$d\mu(z) = e^{-\frac{|z|^2}{2}} dA(z)/2\pi$$

is the Gaussian measure on $\mathbb{C}$ and dA is the ordinary Lebesgue measure. See [CG] for the study of the Fock space $L^2_a(\mathbb{C})$. We have previously proved the following [Izu].

Theorem *Let $h \in \mathrm{Hol}(\mathbb{C})$. Then the following are equivalent:*

(i) *f is a nonvanishing function in $L^2_a(\mathbb{C})$.*
(ii) *$f = e^h$ where $h = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| < \frac{1}{4}$.*
(iii) *f is cyclic in $L^2_a(\mathbb{C})$.*

It is known that there are non-vanishing functions in $H^p(\mathbb{D})$ and $L^p_a(\mathbb{D})$ which are not cyclic in the respective spaces [Gar, HKZ]. On the other hand, the situation is quite different in $L^2_a(\mathbb{C})$. Brown and Shields [BS] posed the following question.

Question Let Ω be a bounded region in $\mathbb{C}$. Does there exist a polynomially dense Banach space of analytic functions in which a function f is cyclic if and only if $f(z) \neq 0$ for all $z \in \Omega$?

Received by the editors February 22, 2006.
AMS subject classification: Primary: 47A16; secondary: 46J15, 46H25.
Keywords: weighted L^p spaces of entire functions, cyclic vectors.

The above theorem is not the direct answer of this question, but it says that if $\Omega = \mathbb{C}$, then there exists a Banach space in which every non-vanishing function f is cyclic.

In this paper, we consider the cyclic vectors in more generalized spaces, some weighted L^p spaces of entire functions.

Let $0 < p < \infty$, $s > 0$ and $\alpha > 0$. Throughout this paper, we put $\phi(z) = \frac{\alpha}{p}|z|^s$. The space $L^p_a(\mathbb{C}, \phi)$ consists of those entire functions whose semi-norm

$$\|f\|_{L^p_a(\mathbb{C},\phi)} = \left\{ \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dA(z) \right\}^{1/p}$$

is finite. We study the cyclic vectors in $L^p_a(\mathbb{C}, \phi)$. The following is our main result.

Theorem 1.1 *Let f be a function in $L^p_a(\mathbb{C}, \phi)$ satisfying $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$. Then the following are equivalent:*

(i) *f is a non-vanishing function.*

(ii) *$f = e^h$ where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ for $a_k \in \mathbb{C}$, and in addition $|a_s| < \frac{\alpha}{p}$ if s is an integer.*

(iii) *f is cyclic in $L^p_a(\mathbb{C}, \phi)$.*

We know that every non-vanishing function in the classical Fock space $L^2_a(\mathbb{C})$ is cyclic. In our case, we notice that it is not valid for some positive numbers, that is, if s is not an integer or $s = 1, 2, 3, 4$, then $L^p_a(\mathbb{C}, \phi)$ has the same property as the one in $L^2_a(\mathbb{C})$, but if $s = 5, 6, 7, \cdots$, the situation is different. For example, although $f = e^{\frac{\alpha}{p}z^s}$ is a non-vanishing function in $L^p_a(\mathbb{C}, \phi)$, the function f does not satisfy $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$. Clearly, this function f is not cyclic. But if we consider the non-vanishing functions just satisfying $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$, then the situation is similar.

We prove the theorem in Section 2.

2 Proof of the Main Theorem

Guo and Zheng [GZ] proved that if $f \in L^p_a(\mathbb{C}, \phi)$ satisfies $fL^p_a(\mathbb{C}, \phi) \subset L^p_a(\mathbb{C}, \phi)$, then f is a constant. So first we consider which non-vanishing function $f \in L^p_a(\mathbb{C}, \phi)$ satisfies $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$. To do this, we deal with another space $\mathcal{F}^p_\phi$ which is studied in [MMO]. The space is

$$\mathcal{F}^p_\phi = \left\{ f \in \mathrm{Hol}(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p_\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} \rho^{-1} \Delta\phi \, dA(z) < \infty \right\},$$

where $\Delta\phi$ is the Laplacian of ϕ and $\rho^{-1}\Delta\phi$ is a regular version of $\Delta\phi$. If $p = 2$, then $\mathcal{F}^2_\phi$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2\phi(z)} \rho^{-1} \Delta\phi \, dA(z).$$

We denote the reproducing kernel of $\mathcal{F}^2_\phi$ by K_λ. The following lemma was proved by Marco, Massaneda and Ortega-Cerdà [MMO, Lemma 21].

https://doi.org/10.4153/CMB-2008-038-9 Published online by Cambridge University Press

Lemma 2.1 *There exists a positive number C such that for any $\lambda \in \mathbb{C}$*

$$C^{-1}e^{2\phi(\lambda)} \leq \|K_\lambda\|^2_{\mathcal{F}^2_\phi} \leq Ce^{2\phi(\lambda)}.$$

Chen, Guo and Hou [CGH] proved the following.

Lemma 2.2

$$\lim_{|\lambda|\to\infty} \frac{\langle f, K_\lambda\rangle_{\mathcal{F}^2_\phi}}{\|K_\lambda\|_{\mathcal{F}^2_\phi}} = 0$$

for any $f \in \mathcal{F}^2_\phi$.

By Lemma 2.1 and Lemma 2.2, we get the following.

Lemma 2.3 *The following are equivalent:*

(i) *$f \in L^p_a(\mathbb{C}, \phi)$ is a non-vanishing function satisfying $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$.*

(ii) *$f(z) = e^{h(z)}$, where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ and in addition $|a_s| < \frac{\alpha}{p}$ if s is an integer.*

Proof (i) $\Rightarrow$ (ii): Suppose s is an integer. First, we consider when $p = 2$. Let $e^h \in \mathcal{F}^2_\phi$. By Lemma 2.1,

$$\frac{\left|\langle e^h, K_\lambda\rangle_{\mathcal{F}^2_\phi}\right|}{\|K_\lambda\|_{\mathcal{F}^2_\phi}} \geq \frac{1}{\sqrt{C}} e^{-\frac{\alpha}{2}|\lambda|^s}|e^{h(\lambda)}|.$$

By Lemma 2.2, we get

$$\lim_{|\lambda|\to\infty} e^{-\frac{\alpha}{2}|\lambda|^s} e^{h(\lambda)} = 0.$$

Since $|e^{h(\lambda)}| \leq e^{\frac{\alpha}{2}|\lambda|^s}$ for sufficient large $|\lambda|$, we have $\Re h(\lambda) \leq \frac{\alpha}{2}|\lambda|^s$ for large $|\lambda|$, so that the order σ of e^h is

$$\sigma = \limsup_{r\to\infty} \frac{\log(\max_{|\lambda|=r} \Re h(\lambda))}{\log r} \leq \limsup_{r\to\infty} \frac{\log \frac{\alpha}{2} r^s}{\log r} = s.$$

By [BG, Corollary 4.5.11], we get $h \in \mathcal{C}$ and $\deg h \leq s$. Now we show that $e^h \notin \mathcal{F}^2_\phi$ for $h(z) = \sum_{n=0}^{s} a_n z^n$ with $|a_s| \geq \frac{\alpha}{2}$. By Lemma 2.2, we have that $\lim_{|\lambda|\to\infty} e^{-\frac{\alpha}{2}|\lambda|^s} f(\lambda) = 0$ for $f \in \mathcal{F}^2_\phi$. Then it is easy to show that $e^h \notin \mathcal{F}^2_\phi$.

If $p \neq 2$, then since

$$\|e^h\|^2_{\mathcal{F}^2_\phi} = \|e^{\frac{2}{p}h}\|^p_{\mathcal{F}^p_\phi},$$

we get $e^h \notin \mathcal{F}^p_\phi$ for $h(z) = \sum_{n=0}^{s} a_n z^n$ with $|a_s| \geq \frac{\alpha}{p}$. By the norms for two spaces, $\mathcal{F}^p_\phi$ and $L^p_a(\mathbb{C}, \phi)$, (i) $\Rightarrow$ (ii) is obvious.

If s is not an integer, we can also prove it similarly to the above.

(ii) $\Rightarrow$ (i): If s is an integer, it is enough to show that $e^h\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$ for $h(z) = \sum_{n=0}^{s} a_n z^n$ with $|a_s| < \frac{\alpha}{p}$. Let $q \in \mathcal{C}$ and ε be a number satisfying

$$0 < \varepsilon < \frac{1}{2}\left(\frac{\alpha}{p} - |a_s|\right). \tag{2.1}$$

For large $|z|$, we have $|q(z)| \le e^{\varepsilon |z|^s}$ and $|\sum_{n=0}^{s-1} a_n z^n| \le \varepsilon |z|^s$. Thus

$$\left| q(z)e^{h(z)} \right|^p = |q(z)|^p e^{p\Re h(z)} \le e^{p\varepsilon|z|^s} e^{p(|a_s|+\varepsilon)|z|^s} = e^{p(2\varepsilon+|a_s|)|z|^s}.$$

Then

$$\left| q(z)e^{h(z)} \right|^p e^{-\alpha|z|^s} \le e^{-\delta|z|^s}$$

for large $|z|$, where $\delta = \alpha - p(2\varepsilon + |a_s|)$. By (2.1), we have $\delta > 0$. Therefore there exists a positive constant C such that

$$\int_{\mathbb{C}} \left| q(z)e^{h(z)} \right|^p e^{-\alpha|z|^s}\, dA(z) \le C \int_{\mathbb{C}} e^{-\delta|z|^s}\, dA(z).$$

Since the last integral is finite, we get $e^h \mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$.

If s is not an integer, since $h(z) = \sum_{n=0}^{[s]} a_n z^n$ and $[s] < s$, the conclusion is trivial. ■

By Lemma (2.3), we have proved (i) $\Leftrightarrow$ (ii) in Theorem 1.1.

Now we shall prove that the polynomial ring $\mathcal{C}$ is dense in $L^p_a(\mathbb{C}, \phi)$. The following two lemmas are the generalizations of the results proved by Garling and Wojtaszczyk [GW, Lemma 1, Proposition 5].

Lemma 2.4 *Let $f \in L^p_a(\mathbb{C}, \phi)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then we have the following:*

(i) *There exists a constant $C_1 > 0$, which depends on f, satisfying*

$$|c_n| \le C_1 e^{\frac{2-s}{ps}} \left(\frac{s\alpha e}{pn + 2 - s} \right)^{\frac{n}{s}} \|f\|_{L^p_a(\mathbb{C},\phi)}.$$

(ii) *For large n,*

$$\begin{aligned} \|z^n\|^p_{L^p_a(\mathbb{C},\phi)} &= \frac{\alpha^{-\frac{pn+2}{s}}}{s} \Gamma\left(\frac{pn+2}{s} \right) \\ &\sim \frac{1}{s\alpha} \left(2\pi \frac{pn+2-s}{s} \right)^{\frac{1}{2}} \left(\frac{pn+2-s}{s\alpha e} \right)^{\frac{pn+2-s}{s}}, \end{aligned}$$

where Γ denotes the gamma function.

(iii) *There is a constant $C_2 > 0$, which depends on f, satisfying*

$$\|c_n z^n\|_{L^p_a(\mathbb{C},\phi)} \le C_2 \left(\frac{pn+2-s}{s} \right)^{\frac{1}{2p}} \left(\frac{pn+2-s}{s\alpha} \right)^{\frac{2-s}{ps}} \|f\|_{L^p_a(\mathbb{C},\phi)}.$$

Proof (i) We know that for any $R > 0$

$$c_n = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(Re^{i\theta})}{(Re^{i\theta})^n} d\theta.$$

Since

$$|f(Rz)| \le C_1 e^{\frac{\alpha}{p}|Rz|^s} \|f\|_{L^p_a(\mathbb{C},\phi)}$$

https://doi.org/10.4153/CMB-2008-038-9 Published online by Cambridge University Press

for some $C_1 > 0$ (see [OS, Lemma 1]), if we set $R^s = \frac{pn+2-s}{s\alpha}$, we get

$$|c_n| \le C_1 \frac{e^{\frac{\alpha}{p}R^s}}{R^n} \|f\|_{L^p_a(\mathbb{C},\phi)} \le C_1 \frac{e^{\frac{pn+2-s}{ps}}}{\left(\frac{pn+2-s}{s\alpha}\right)^{\frac{n}{s}}} \|f\|_{L^p_a(\mathbb{C},\phi)}$$
$$= C_1 e^{\frac{2-s}{ps}} \left(\frac{s\alpha e}{pn+2-s}\right)^{\frac{n}{s}} \|f\|_{L^p_a(\mathbb{C},\phi)}.$$

(ii) We have

$$\|z^n\|^p_{L^p_a(\mathbb{C},\phi)} = \frac{1}{2\pi}\int_{\mathbb{C}} |z^n|^p e^{-\alpha|z|^s}\, dA(z) = \int_0^\infty r^{pn+1} e^{-\alpha r^s}\, dr$$
$$= \frac{\alpha^{-\frac{pn+2}{s}}}{s}\int_0^\infty t^{\frac{pn+2}{s}-1} e^{-t}\, dt = \frac{\alpha^{-\frac{pn+2}{s}}}{s}\Gamma\left(\frac{pn+2}{s}\right).$$

By Stirling's formula $\Gamma(x+1) \sim \sqrt{2\pi x}(\frac{x}{e})^x$,

$$\frac{\alpha^{-\frac{pn+2}{s}}}{s}\Gamma\left(\frac{pn+2}{s}\right) \sim \frac{\alpha^{-\frac{pn+2}{s}}}{s}\left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2}}\left(\frac{pn+2-s}{se}\right)^{\frac{pn+2-s}{s}}$$
$$= \frac{1}{s\alpha}\left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2}}\left(\frac{pn+2-s}{s\alpha e}\right)^{\frac{pn+2-s}{s}}$$

(iii) By (i) and (ii),

$$\|c_n z^n\|_{L^p_a(\mathbb{C},\phi)} \le C_1 e^{\frac{2-s}{ps}}\left(\frac{s\alpha e}{pn+2-s}\right)^{\frac{n}{s}}\left(\frac{1}{s\alpha}\right)^{\frac{1}{p}}$$
$$\times\left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2p}}\left(\frac{pn+2-s}{s\alpha e}\right)^{\frac{pn+2-s}{ps}}\|f\|_{L^p_a(\mathbb{C},\phi)}$$
$$= C_1\left(\frac{1}{s\alpha}\right)^{\frac{1}{p}}\left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2p}}\left(\frac{pn+2-s}{s\alpha}\right)^{\frac{2-s}{ps}}\|f\|_{L^p_a(\mathbb{C},\phi)}$$

for some $C_1 > 0$. ■

Lemma 2.5 *The polynomial ring $\mathcal{C}$ is dense in $L^p_a(\mathbb{C},\phi)$.*

Proof It is clear that $\mathcal{C} \subset L^p_a(\mathbb{C},\phi)$. Let $\varepsilon > 0$ and $f \in L^p_a(\mathbb{C},\phi)$ with $f(z) = \sum_{n=0}^\infty c_n z^n$. Write $\gamma = \max\{1, 2^{p-1}\}$. Then there exists a constant $R > 0$ satisfying

$$\frac{\gamma}{2\pi}\int_{|z|>R} |f(z)|^p e^{-\alpha|z|^s}\, dA(z) < \frac{\varepsilon}{10}.$$

Put $f_r(z) = f(rz)$ for any r with $\frac{1}{2} < r < 1$. Then

$$\frac{\gamma}{2\pi}\int_{|z|>2R} |f_r(z)|^p e^{-\alpha|z|^s} dA(z) = \frac{\gamma}{2\pi r^2}\int_{|z|>2Rr} |f(z)|^p e^{-\frac{\alpha}{r^s}|z|^s}\, dA(z) < \frac{2\varepsilon}{5}.$$

https://doi.org/10.4153/CMB-2008-038-9 Published online by Cambridge University Press

There is a constant δ with $\frac{1}{2} < \delta < 1$ such that

$$|f_r(z) - f(z)| < \frac{s\alpha^{\frac{2}{s}}}{2\Gamma(\frac{2}{s})}\varepsilon$$

on $|z| \le 2R$ for any r with $\delta < r < 1$. By the proof of Lemma 2.4(ii),

$$\frac{1}{2\pi}\int_{\mathbb{C}} e^{-\alpha|z|^s}\, dA(z) = \frac{\Gamma\left(\frac{2}{s}\right)}{s\alpha^{\frac{2}{s}}}.$$

Hence

$$\begin{aligned}
\|f_r - f\|^p_{L^p_a(\mathbb{C},\phi)} &= \frac{1}{2\pi}\int_{|z|\le 2R} |f_r(z) - f(z)|^p e^{-\alpha|z|^s}\, dA(z)\\
&\quad + \frac{1}{2\pi}\int_{|z|>2R} |f_r(z) - f(z)|^p e^{-\alpha|z|^s} dA(z)\\
&\le \frac{\varepsilon}{2\pi}\frac{s\alpha^{\frac{2}{s}}}{2\Gamma(\frac{2}{s})}\int_{|z|\le 2R} e^{-\alpha|z|^s}\, dA(z)\\
&\quad + \frac{\gamma}{2\pi}\int_{|z|>2R} |f_r(z)|^p e^{-\alpha|z|^s}\, dA(z)\\
&\quad + \frac{\gamma}{2\pi}\int_{|z|>2R} |f(z)|^p e^{-\alpha|z|^s}\, dA(z)\\
&< \varepsilon
\end{aligned}$$

for any r with $\delta < r < 1$. Therefore it is enough to show that for each r with $\delta < r < 1$, f_r can be approximated by polynomials in the semi-norm. Let l be a non-negative integer. Put

$$q_l(z) = \sum_{n=0}^{l} r^n c_n z^n.$$

Suppose $p < 1$. By Lemma 2.4(iii),

$$\begin{aligned}
\|f_r - q_l\|^p_{L^p_a(\mathbb{C},\phi)} &= \frac{1}{2\pi}\int_{\mathbb{C}}\Big|\sum_{n=l+1}^{\infty} r^n c_n z^n\Big|^p e^{-\alpha|z|^s}\, dA(z)\\
&\le \frac{1}{2\pi}\int_{\mathbb{C}}\sum_{n=l+1}^{\infty} |r^n c_n z^n|^p e^{-\alpha|z|^s}\, dA(z)\\
&= \sum_{n=l+1}^{\infty} r^{pn}\|c_n z^n\|^p_{L^p_a(\mathbb{C},\phi)}\\
&\le \sum_{n=l+1}^{\infty} r^{pn} C_2^p \Big(\frac{pn+2-s}{s}\Big)^{\frac{1}{2}}\Big(\frac{pn+2-s}{s\alpha}\Big)^{\frac{2-s}{s}}\|f\|^p_{L^p_a(\mathbb{C},\phi)}\\
&< \infty.
\end{aligned}$$

https://doi.org/10.4153/CMB-2008-038-9 Published online by Cambridge University Press

Suppose $p \geq 1$. Also by Lemma 2.4(iii),

$$\begin{aligned}\|f_r - q_l\|_{L^p_a(\mathbb{C},\phi)} &= \Big\| \sum_{n=l+1}^{\infty} r^n c_n z^n \Big\|_{L^p_a(\mathbb{C},\phi)} \leq \sum_{n=l+1}^{\infty} r^n \|c_n z^n\|_{L^p_a(\mathbb{C},\phi)} \\ &\leq \sum_{n=l+1}^{\infty} C_2 r^n \Big(\frac{pn+2-s}{s}\Big)^{\frac{1}{2p}} \Big(\frac{pn+2-s}{s\alpha}\Big)^{\frac{2-s}{ps}} \|f\|_{L^p_a(\mathbb{C},\phi)} \\ &< \infty.\end{aligned}$$

Thus $\|f_r - q_l\| \to 0$ as $l \to \infty$. ■

Finally we show (ii) $\Leftrightarrow$ (iii) in Theorem 1.1. Since every cyclic vector is non-vanishing, (iii) $\Rightarrow$ (ii) is trivial.

Proof that (ii) $\Rightarrow$ (iii) in Theorem 1.1 We show that $\mathcal{C} \subset \overline{e^h \mathcal{C}}$. Suppose s is an integer. Let N be a positive integer satisfying

$$\Big(1 + \frac{1}{N}\Big)|a_s| < \frac{\alpha}{p}. \tag{2.2}$$

Put

$$q_n(z) = \sum_{k=0}^{n} \frac{\{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m\}^k}{k!}.$$

Let l be a non-negative integer. Since

$$|q_n(z)| \leq \sum_{k=0}^{\infty} \frac{\{\frac{1}{N}\sum_{m=0}^{s} |a_m||z|^m\}^k}{k!} = e^{\frac{1}{N}\sum_{m=0}^{s} |a_m||z|^m},$$

we have

$$\begin{aligned}&\Big| z^l q_n(z) e^{\sum_{m=0}^{s} a_m z^m} - z^l e^{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m} e^{\sum_{m=0}^{s} a_m z^m}\Big| \\ &\qquad \leq |z|^l \Big(e^{(1+\frac{1}{N})\sum_{m=0}^{s} |a_m||z|^m} + e^{(1-\frac{1}{N})\sum_{m=0}^{s} |a_m||z|^m}\Big) \\ &\qquad \leq 2|z|^l e^{(1+\frac{1}{N})\sum_{m=0}^{s} |a_m||z|^m}.\end{aligned}$$

Hence

$$\begin{aligned}&\Big| z^l q_n(z) e^{\sum_{m=0}^{s} a_m z^m} - z^l e^{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m} e^{\sum_{m=0}^{s} a_m z^m}\Big|^p e^{-\alpha|z|^s} \\ &\qquad \leq 2^p |z|^{pl} e^{\{p(1+\frac{1}{N})|a_s| - \alpha\}|z|^s + p(1+\frac{1}{N})\sum_{m=0}^{s-1} |a_m||z|^m}.\end{aligned}$$

By (2.2), we have $p(1+\frac{1}{N})|a_s| - \alpha < 0$, so that the last function is integrable with respect to dA. Since

$$q_n(z) \to e^{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m}$$

https://doi.org/10.4153/CMB-2008-038-9 Published online by Cambridge University Press

pointwise on $\mathbb{C}$ as $n \to \infty$, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{C}} \Big| z^l q_n(z) e^{\sum_{m=0}^{s} a_m z^m} - z^l e^{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m} e^{\sum_{m=0}^{s} a_m z^m} \Big|^p e^{-\alpha|z|^s} dA(z) \to 0 \quad \text{as } n \to \infty.$$

Therefore

$$e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m} \mathcal{C} \subset \overline{e^{\sum_{m=0}^{s} a_m z^m} \mathcal{C}}.$$

Applying this way again for $e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m}$, we have

$$\int_{\mathbb{C}} \Big| z^l q_n(z) e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m} - z^l e^{-\frac{1}{N}\sum_{m=0}^{s} a_m z^m} e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m} \Big|^p e^{-\alpha|z|^s} dA(z) \to 0 \quad \text{as } n \to \infty.$$

Thus $e^{(1-\frac{2}{N})\sum_{m=0}^{s} a_m z^m} \mathcal{C} \subset \overline{e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m} \mathcal{C}}$. Repeating this argument, we get

$$\mathcal{C} \subset \overline{e^{\frac{1}{N}\sum_{m=0}^{s} a_m z^m} \mathcal{C}} \subset \cdots \subset \overline{e^{(1-\frac{1}{N})\sum_{m=0}^{s} a_m z^m} \mathcal{C}} \subset \overline{e^{\sum_{m=0}^{s} a_m z^m} \mathcal{C}},$$

If s is not an integer, then since $[s] < s$, we can choose $N = 1$ in the above proof. In a similar way, we get the desired result. ∎

This completes the proof of Theorem 1.1.

References

[BG] C. Berenstein and R. Gay, *Complex Variables.* Graduate Texts in Mathematics 125, Springer-Verlag, New York, 1991.

[BS] L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space.* Trans. Amer. Math. Soc. **285**(1984), no. 1, 269–303.

[CG] X. Chen and K. Guo, *Analytic Hilbert Modules.* CRC Research Notes in Mathematics 433, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[CGH] X. Chen, K. Guo, and S. Hou, *Analytic Hilbert spaces over the complex plane.* J. Math. Anal. Appl. **268**(2002), no. 2, 684–700.

[Gar] J. B. Garnett, *Bounded Analytic Functions.* Pure and Applied Mathematics 96, Academic Press, New York, 1981.

[GW] D. J. H. Garling and P. Wojtaszczyk, *Some Bargmann spaces of analytic functions.* In: Function Spaces. Lecture Notes in Pure and Appl. Math. 172, Dekker, New York, 1995, pp. 123–138.

[GZ] K. Guo and D. Zheng, *Invariant subspaces, quasi-invariant subspaces, and Hankel operators.* J. Funct. Anal. **187**(2001), no. 2, 308–342.

[HKZ] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces.* Graduate Texts in Mathematics 199, Springer-Verlag, New York, 2000.

[Izu] K. H. Izuchi, *Cyclic vectors in the Fock space over the complex plane.* Proc. Amer. Math. Soc. **133**(2005), no. 12, 3627–3630.

[MMO] N. Marco, X. Massaneda, and J. Ortega-Cerdà, *Interpolating and sampling sequences for entire functions.* Geom. Funct. Anal. **13**(2003), no. 4, 862–914.

[OS] J. Ortega-Cerdà and K. Seip, *Beurling-type density theorems for weighted L^p spaces of entire functions.* J. Anal. Math. **75**(1998), 247–266.

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo, JAPAN 060-0810
e-mail: khizuchi@math.sci.hokudai.ac.jp