NIL SUBRINGS OF GOLDIE RINGS ARE NILPOTENT

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Herstein and Small have shown (**1**) that nil rings which satisfy certain chain conditions are nilpotent. In particular, this is true for nil (left) Goldie rings. The result obtained here is a generalization of their result to the case of any nil subring of a Goldie ring.

Definition. L is a *left annihilator* in the ring R if there exists a subset $S \subset R$ with $L = \{x \in R \mid xS = 0\}$. In this case we write $L = l(S)$. A *right annihilator* $K = r(S)$ is defined similarly.

Definition. A ring R satisfies the *ascending chain condition* on left annihilators if any ascending chain of left annihilators terminates at some point. We recall the well-known fact that this condition is inherited by subrings.

Definition. R is a *Goldie ring* if R has no infinite direct sum of left ideals and has the ascending chain condition on left annihilators.

LEMMA 1 (**1**, Lemma 2). *Let N be a non-zero nil ring satisfying the ascending chain condition on left annihilators. Then $r(N) \neq (0)$.*

LEMMA 2. *Let R satisfy the ascending chain condition on left annihilators, and suppose that $r(S)$ is a two-sided ideal of R. Then $R/r(S)$ has the ascending chain condition on left annihilators.*

Proof. It is trivial that the inverse image of a right annihilator in $R/r(S)$ is a right annihilator in R. Now proceed as in (**1**, Lemma 3).

LEMMA 3. *Let R be a Goldie ring and S a nil subring of R. Then there exists a positive integer k with $r(R)S^k = (0)$.*

Proof. We need modify only slightly the proof of (**1**, Theorem 3). Let $A = r(R)$. A is a two-sided ideal of R and any additive subgroup of A is a left ideal of R. Let T be the torsion part of A and consider, for a prime p, the p-primary component T_p. Let $V_1 = \{x \in T_p \mid px = 0\}$. Then V_1 is a vector space over P, the field of P elements. Since R is a Goldie ring, V_1 is finite-dimensional over P. Let $V_i = \{x \in T_p \mid p^i x = 0\}$. Then it can easily be shown that V_i/V_{i-1} is a finite-dimensional vector space over P. As each V_i is a right ideal of R, V_i/V_{i-1} is a right R module, hence a right S module. S induces a finite nil ring of linear transformations on V_i/V_{i-1}. Therefore, this ring of transformations is nilpotent, and hence $V_i S^{n_i} \subset V_{i-1}$. It follows that

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$V_iS^{m_i} = (0)$, where $m_i = n_1 + n_2 + \ldots + n_i$. By the ascending chain condition on left annihilators in R, there exists an integer k such that if $xS^m = (0)$, then $xS^k = (0)$. Thus, we have $V_iS^k = (0)$ for each i. Since every element of T_p is in some V_i, we have $T_pS^k = (0)$. Clearly, $TS^k = (0)$.

Let $\bar{R} = R/T$ and $\bar{A} = A/T$. $\bar{A}$ is torsion free. Let $\bar{V} = \bar{A} \otimes_Z Q$, where Z is the ring of integers and Q is the field of rational numbers. As R is a Goldie ring, $\bar{V}$ is a finite-dimensional vector space over Q. Now $\bar{S} = (S + T)/T$ induces linear transformations on $\bar{V}$ via $(\bar{a} \otimes q)t_{\bar{s}} = \bar{a}\bar{s} \otimes q$. The homomorphism $\bar{s} \rightarrow t_{\bar{s}}$ has kernel exactly $r(\bar{A}) \cap \bar{S}$. Thus, $\bar{S}/r(\bar{A}) \cap \bar{S}$ is a nil ring of $n \times n$ matrices over Q, where $n = \dim_Q \bar{V}$. It follows that $\bar{S}/r(\bar{A}) \cap \bar{S}$ is nilpotent. Hence $\bar{S}^m \subset r(\bar{A}) \cap \bar{S} \subset r(\bar{A})$ for some m, thus $\bar{A}(\bar{S})^m = (\bar{0})$ and $AS^m \subset T$. Since $TS^k = (0)$, we obtain $AS^mS^k = (0)$ which yields $AS^{m+k} = (0)$. By the choice of k, we obtain $AS^k = (0)$.

Consider the ring S generated by elements $x_1, x_2, \ldots, x_n, \ldots$ with the relations $x_ix_j = 0$ for $i \geqq j$. It is easy to show that S is a nil ring with the ascending chain condition on left annihilators, and that S is not nilpotent.

LEMMA 4. *Let R be a Goldie ring. Then R has no subring isomorphic to the ring S defined above.*

Proof. Assume that R has S as a subring. Let $s_n = \prod_{i=1}^n x_i$. Then $s_n \neq 0$ and $l(s_1) \subset l(s_2) \subset \ldots$. As R is a Goldie ring, there is an integer k with $l(s_k) = l(s_m)$ for all $m \geqq k$. Furthermore, $Rs_n \neq (0)$ for any n. For if $Rs_n = (0)$, then by Lemma 3, $s_nS^p = 0$ for some p. But then $s_{n+p} = 0$, contradicting $s_i \neq 0$ for all i. Now consider $Rs_k + Rs_{k+1} + \ldots$. This sum is not direct, as R is a Goldie ring. Hence

$$\sum_{i=1}^{n} r_{j_i}s_{k+j_i} = 0,$$

where $r_{j_i}s_{k+j_i} \neq 0$ for all i, and $0 \leqq j_i < j_m$ for $i < m$. It follows that

$$\sum_{i=1}^{n} r_{j_i}s_{k+j_i}x_{k+j_1+1} = 0.$$

This implies that $r_{j_1}s_{k+j_1+1} = 0$. But then $r_{j_1} \in l(s_{k+j_1+1}) = l(s_{k+j_1})$, which yields $r_{j_1}s_{k+j_1} = 0$, contradicting our assumption. Hence R has no subring isomorphic to S.

In all that follows, let K be a nil ring with the ascending chain condition on left annihilators, and further assume that for any j, $l(K^j) = (0)$.

LEMMA 5. *Let A be a non-zero two-sided ideal of K. Then $K/r(A)$ is a non-zero nil ring with the ascending chain condition on left annihilators.*

Proof. Use Lemmas 1 and 2.

LEMMA 6. *There exist ideals $A_1, A_2, \ldots, A_n, \ldots$ in K with the properties:*
(1) $A_1A_2 \ldots A_k \neq (0)$ *for all k,*
(2) $A_1A_2 \ldots A_kA_n = (0)$ *if $n \leqq k$.*

Proof. We construct the A_i by induction. Let $A_1 = r(K)$. Assume that we have $A_1, A_2, \ldots, A_k$, where $A_i = r(A_1A_2 \ldots A_{i-1}K)$ if $i > 1$, and $A_1A_2 \ldots A_k \neq (0)$. By Lemma 5, $K/r(A_1A_2 \ldots A_k) = \bar{K}$ is a non-zero, nil ring and satisfies the ascending chain condition on left annihilators. Hence, by Lemma 1 there exists an ideal $\bar{T} \neq (\bar{0})$ in $\bar{K}$ with $\overline{KT} = (\bar{0})$. Let T be the inverse image of $\bar{T}$ in K. Then $A_1A_2 \ldots A_kT \neq (0)$, but $A_1A_2 \ldots A_kKT = (0)$. Hence, $A_1A_2 \ldots A_kr(A_1A_2 \ldots A_kK) \neq (0)$. Let $r(A_1A_2 \ldots A_kK) = A_{k+1}$. Now as $A_1 = r(K)$, $A_1A_2 \ldots A_kA_1 = (0)$. For $1 < n \leqq k$, $A_1A_2 \ldots A_k \subset A_1A_2 \ldots A_{n-1}K$. But then $A_1A_2 \ldots A_kA_n \subset A_1A_2 \ldots A_{n-1}KA_n = (0)$ by definition of A_n.

LEMMA 7. *There exist elements $x_1, x_2, \ldots, x_n, \ldots$ in K such that*
(1) $\prod_{j=1}^{k} x_j = s_k \neq 0$,
(2) $s_kx_n = 0$ *for* $n \leqq k$.

Proof. Let $\{A_i\}$ be the ideals of Lemma 6. For any A_j we have

$$l(A_j) \subset l(A_jA_{j+1}) \subset l(A_jA_{j+1}A_{j+2}) \subset \ldots .$$

As K has the ascending chain condition on left annihilators, there must exist an integer p with $l(A_jA_{j+1} \ldots A_p)$ maximal in the above ascending chain. Suppose that we have elements $x_i \in A_i$ for $i \leqq k$ such that

$$x_1x_2 \ldots x_k = s_k \notin l(A_{k+1}A_{k+2} \ldots A_{k+n})$$

for any n. If for each $y \in A_{k+1}$ we have $s_ky \in l(A_{k+2}A_{k+3} \ldots A_{k+n(y)})$, then $s_ky \in l(A_{k+2}A_{k+3} \ldots A_p)$ for some p independent of y. Hence,

$$s_kA_{k+1} \subset l(A_{k+2}A_{k+3} \ldots A_p),$$

and therefore $s_k \in l(A_{k+1}A_{k+2} \ldots A_p)$, contradicting $s_k \notin l(A_{k+1}A_{k+2} \ldots A_{k+n})$ for any n. Thus, there exists $x_{k+1} \in A_{k+1}$ with

$$s_kx_{k+1} = s_{k+1} \notin l(A_{k+2}A_{k+3} \ldots A_{k+n})$$

for any n. As $s_{k+1} \notin l(A_{k+2})$ for all k, we have $s_k \neq 0$. This proves the first part of the lemma. The second part follows immediately from our construction and the second part of Lemma 6.

Let us return to the situation where N is a nil subring of a Goldie ring R. If it happens that $l(N^j) = (0)$ for all j, then using Lemmas 7 and 4 we would arrive at a contradiction. Hence, $l(N^j) \neq (0)$ for some j. Now as R is a Goldie ring, the ascending sequence $l(N) \subset l(N^2) \subset l(N^3) \subset \ldots$ must terminate at some point, say at k_0. Consider $\bar{N} = N/l(N^{k_0})$. If N is not nilpotent, $\bar{N}$ is a non-zero nil ring with ascending chain condition on left annihilators, by Lemma 2. Further, $\bar{N}$ has the property that $l(\bar{N}^j) = (\bar{0})$ for all j. For suppose that $l(\bar{N}^j) \neq (\bar{0})$. Let S be the inverse image of $l(\bar{N}^j)$ in N. Then $SN^j \subset l(N^{k_0})$, which implies that $SN^jN^{k_0} = SN^{j+k_0} = (0)$. However, $l(N^{k_0}) = l(N^{j+k_0})$. Thus $SN^{k_0} = (0)$ and $\bar{S} = l(\bar{N}^j) = (\bar{0})$.

Therefore we can apply Lemma 7 to $\bar{N}$ to obtain elements $\bar{x}_i \in \bar{N}$ with $\bar{x}_1\bar{x}_2 \ldots \bar{x}_k \neq \bar{0}$ but $\bar{x}_1\bar{x}_2 \ldots \bar{x}_k\bar{x}_n = \bar{0}$ for $n \leqq k$. Let x_i be an inverse image of $\bar{x}_i$. Then we have $x_1x_2 \ldots x_k \neq 0$, and for $n \leqq k$, $x_1x_2 \ldots x_kx_n \in l(N^{k_0})$. We have proved the following lemma.

LEMMA 8. *Let N be a nil ring with the ascending chain condition on left annihilators which is not nilpotent. Let k_0 be the integer at which $l(N) \subset l(N^2) \subset l(N^3) \subset \ldots$ terminates. Then there exist elements $x_1, x_2, \ldots, x_k, \ldots$ in N such that*

(1) $x_1x_2 \ldots x_k \neq 0$ *for all* k,

(2) $x_1x_2 \ldots x_kx_nx_{n+1} \ldots x_{n+k_0} = 0$ *if* $n \leqq k$.

THEOREM 1. *Let R be a Goldie ring and N a proper nil subring. Then N is nilpotent.*

Proof. Assume that N is not nilpotent. Let $x_1, x_2, \ldots, x_n, \ldots$ be the elements of Lemma 8. The proof now follows that of Lemma 4. If $s_k = x_1x_2 \ldots x_k$ we again have $Rs_k \neq (0)$ and an integer k with $l(s_k) = l(s_m)$ for $k \leqq m$. As before we obtain

$$\sum_{i=1}^{n} r_{j_i}s_{k+j_i} = 0.$$

Now multiply this sum by $x_{k+j_1+1}x_{k+j_1+2} \ldots x_{k+j_1+k_0+1}$ on the right and proceed as in the lemma. The contradiction which arises forces us to conclude that N is nilpotent.

REFERENCE

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