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CYCLIC MAPS FROM SUSPENSIONS TO SUSPENSIONS

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1. Introduction. In [**7**] Varadarajan defined the notion of a cyclic map $f : A \to X$. The collection of all homotopy classes of such cyclic maps forms the Gottlieb subset $G(A, X)$ of $[A, X]$. If $A = S^1$, this reduces to the group $G(X, x_0)$ of Gottlieb [**5**]. We show that a cyclic map f maps ΩA into the centre of ΩX in the sense of Ganea [**4**]. If A and X are both suspensions, we then show that if $f : A \to X$ maps ΩA into the centre of ΩX, then f is cyclic. Thus for maps from suspensions to suspensions, Varadarajan's cyclic maps are just those maps considered by Ganea. We also define $G(\Sigma A, \Sigma X)$ in terms of the generalized Whitehead product [**1**]. This gives the computations for $G(S^{n+k}, S^n)$ in terms of Whitehead products in $\pi_{2n+k}(S^n)$.

We work in the category of spaces with base points and having the homotopy type of countable CW-complexes. All maps and homotopies are with respect to base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.

Given spaces X and Y we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. For any space X, we denote by $e : \Sigma\Omega X \to X$ the map whose adjoint is the identity map of ΩX and by $e' : X \to \Omega\Sigma X$ the map which is the adjoint of the identity map of ΩX, where Ω and Σ are the loop and suspension functors respectively.

2. We first state some definitions and results we shall need to prove our results. Let $f : A \to X$ be a map. We say that f is *cyclic* [**7**] if we can find a map $F : X \times A \to X$ such that $Fj = \nabla(1 \vee f)$ where $j : X \vee A \to X \times A$ is the inclusion of the wedge product into the cartesian product, and $\nabla : X \vee X \to X$ is the folding map. The set of all homotopy classes of such cyclic maps is the subset $G(A, X)$ of $[A, X]$.

If $f : A \to X$ is a map, then for every space Z, we have a homomorphism $(\Omega f)_\# : [Z, \Omega A] \to [Z, \Omega X]$. Let $X \flat A$ be the flat product, that is, the fibre of the inclusion $j : X \vee A \to X \times A$. Then in [**4**], Ganea proved the following result.

THEOREM 1. *The following are equivalent:*
(i) $(\Omega f)_\#$ *maps* $[Z, \Omega A]$ *into the centre of the group* $[Z, \Omega X]$.
(ii) $\nabla(1 \vee f)\, i \simeq *$.

Any such map satisfying either of these conditions is referred to by Ganea as mapping ΩA into the centre of ΩX.

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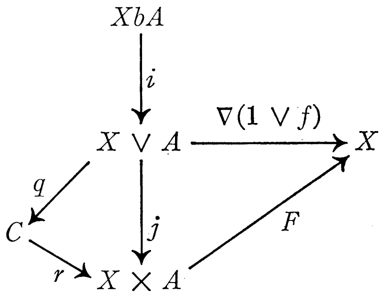
THEOREM 2. *Let $f : A \to X$ be a cyclic map. Then f maps ΩA into the centre of ΩX.*

Proof. Let $q : X \vee A \to C$ be the cofibre of the inclusion $i : XbA \to X \vee A$. Then we have a map $r : C \to X \times A$ such that $rq = j : X \vee A \to X \times A$. Since f is cyclic, we have a map $F : X \times A \to X$ giving the following diagram with commutative triangles:



Clearly $\nabla(1 \vee f)\, i \simeq *$. Hence f maps ΩA into the centre of ΩX.

THEOREM 3. *Let $f : \Sigma A \to \Sigma X$ map $\Omega\Sigma A$ into the centre of $\Omega\Sigma X$. Then f is cyclic.*

Proof. We have $\nabla(1 \vee f)\, i \simeq *$. By Lemma 5.1 of [**2**], it follows that $[e, fe] = 0$, where $e : \Sigma\Omega\, \Sigma X \to \Sigma X$, $fe : \Sigma\Omega\, \Sigma A \to \Sigma X$. Now in [**3**], Ganea showed that there is a map $\gamma : \Omega\Sigma X \wedge \Omega\Sigma X \to \Omega\Sigma X$ such that the composite $\gamma q : \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X \wedge \Omega\Sigma X \to \Omega\Sigma X$ is the commutator of the two projections $\Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X$ in the group $[\Omega\Sigma X \times \Omega\Sigma X, \Omega\Sigma X]$. Then by Lemma 2.1 of [**2**], it follows that $[e, fe] = 0$ if and only if $\gamma\{\Omega e \wedge \Omega(fe)\} \simeq *$. If we use the same notation $e' : \Omega X \to \Omega\Sigma\, \Omega X$, $e' : \Omega A \to \Omega\Sigma\, \Omega A$ for the obvious embeddings, then we have $(\Omega e)e' \simeq 1_X$. Hence we have

$$\gamma(1_{\Omega\Sigma X} \wedge \Omega f) \simeq *.$$

Hence by the same Lemma 2.1 of [**2**], we have $[1_{\Sigma X}, f] = 0$. Now let $i_1 : \Sigma X \to \Sigma X \vee \Sigma A$, $i_2 : \Sigma A \to \Sigma X \vee \Sigma A$ be the usual inclusions. Then $\nabla(1_{\Sigma X} \vee f)[i_1, i_2] = [1_{\Sigma X}, f] = 0$. Since the cofibre of $[i_1, i_2] : \Sigma(X \wedge A) \to \Sigma X \vee \Sigma A$ is $(\Sigma X \vee \Sigma A) \cup C\Sigma(X \wedge A) \simeq \Sigma X \times \Sigma A$, it follows that we can find a map $F : \Sigma X \times \Sigma A \to \Sigma X$ such that $Fj = \nabla(1 \vee f)$ where $j : \Sigma X \vee \Sigma A \to \Sigma X \times \Sigma A$ is the inclusion. Hence f is cyclic.

Remark 1. In the course of the proof, we have shown that if $f : \Sigma A \to \Sigma X$ maps $\Omega\Sigma A$ into the centre of $\Omega\Sigma X$, then $[1_{\Sigma X}, f] = 0$. Conversely, it is obvious that if $[1_{\Sigma X}, f] = 0$, then f is cyclic. Thus we have the following corollary.

COROLLARY 3. *Let $f : \Sigma A \to \Sigma X$. Then the following are equivalent.*

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(i) *f is cyclic.*
(ii) *f maps* $\Omega\Sigma A$ *into the centre of* $\Omega\Sigma X$.
(iii) $[1_{\Sigma X}, f] = 0$.

Remark 2. We can apply this result to spheres. Then we see that the computations of Varadarajan [**7**, Theorem 4.1] on $G(S^k, S^k)$ are just the well known results on the Whitehead product $[\iota, \iota]$. Further, the corollary could be applied to compute $G(S^{n+k}, S^n)$, for various k. The result depends on the computation of the Whitehead product on spheres. These have been extensively computed by Mahowald [**6**] and others.

We conclude by stating another result. We recall the following definition from [7], $P(\Sigma A, X) = \{\alpha \in [\Sigma A, X] \mid [\alpha, \beta] = 0$ for all $\beta \in [\Sigma^k A, X]$ and all $k \geqq 1\}$. Varadarajan proves that for all $k \geqq 1$, $G(S^k, S^k) = P(S^k, S^k)$. An obvious corollary of our results above is the following generalization,

THEOREM 4. $G(\Sigma X, \Sigma X) = P(\Sigma X, \Sigma X)$.

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