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PATH PROPERTIES OF THE PRIMITIVES OF A BROWNIAN MOTION

ZHENGYAN LIN

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Abstract

Let $\{W(t), t \geq 0\}$ be a standard Brownian motion. For a positive integer m, define a Gaussian process

$$X_m(t) = \frac{1}{m!}\int_0^t (t-s)^m \, dW(s).$$

Watanabe and Lachal gave some asymptotic properties of the process $X_m(\cdot)$, $m \geq 1$. In this paper, we study the bounds of its moduli of continuity and large increments by establishing large deviation results.

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1. Introduction

Let $\{W(t), t \geq 0\}$ be a standard Brownian motion. For a positive integer m, define a Gaussian process

$$X_m(t) = \frac{1}{m!}\int_0^t (t-s)^m \, dW(s), \tag{1.1}$$

which was first mentioned by Shepp [4]. This class of processes arises in several domains of applied mathematics. For instance, the process $X_1(\cdot)$, which has been studied at length, is the solution of Langevin's equation under certain physical conditions. Wahba [5,6] used $X_n(\cdot)$ to derive a correspondence between smoothing by splines and Bayesian estimation in certain stochastic models.

Watanabe [7] established a law of the iterated logarithm for $X_1(\cdot)$ (in fact, his result concerns a larger class of Gaussian processes). Lachal [2,3] studied the law of the

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iterated logarithm and regular points for $X_m(\cdot)$, $m \geq 1$. Moreover, Lachal [2] obtained some integral tests that precisely characterize the upper functions for X_m, which is an important result in the asymptotic study of X_m.

In this paper we study path behaviour of the process $X_m(\cdot)$. By establishing results on large deviations, we investigate the moduli of continuity and large increment properties for $X_m(\cdot)$, $m \geq 1$, and give their upper and lower bounds. Note that increments of $X_m(\cdot)$ are neither independent nor stationary, moreover $X_m(\cdot)$ is also not a stationary process. Usually, stationarity of increments is required for investigating the moduli of continuity and large increments of a process.

First of all, we give some moment results. We have

$$EX_m^2(t) = \frac{1}{(m!)^2}\int_0^t (t-s)^{2m}\,ds =: b_m t^{2m+1}, \tag{1.2}$$

where $b_m = (m!)^{-2}(2m+1)^{-1}$, and for any $h > 0$

$$\begin{aligned} &E(X_m(t+h) - X_m(t))^2 \\ &= \frac{1}{(m!)^2} E\left(\int_0^{t+h} (t+h-s)^m\,dW(s) - \int_0^t (t-s)^m\,dW(s)\right)^2 \\ &= \frac{1}{(m!)^2}\left\{E\left(\int_0^t \left(\sum_{j=1}^m \binom{m}{j}(t-s)^{m-j}h^j\right)dW(s)\right)^2\right. \\ &\quad \left. +E\left(\int_t^{t+h}(t+h-s)^m\,dW(s)\right)^2\right\} \\ &=: \sum_{j=2}^{2m+1} b_{mj}h^j t^{2m+1-j} \end{aligned} \tag{1.3}$$

for some positive b_{mj}, $j = 2, \ldots, 2m+1$, where $b_{m2} = ((m-1)!)^{-2}(2m-1)^{-1}$. Equality (1.3) implies

$$E(X_m(t+h) - X_m(t))^2 = (1+\delta(h/t))b_{m2}h^2t^{2m-1}, \tag{1.4}$$

where $0 < \delta(x) \to 0$ as $x \to 0$. Hence

$$\begin{aligned} E(X_m(t+h)X_m(t)) &= \frac{1}{2}E\left\{X_m^2(t+h) + X_m^2(t) - (X_m(t+h) - X_m(t))^2\right\} \\ &= \frac{1}{2}b_m\left((t+h)^{2m+1} + t^{2m+1}\right) - \frac{1}{2}\sum_{j=2}^{2m+1} b_{mj}h^j t^{2m+1-j}. \end{aligned} \tag{1.5}$$

Put $Y_m(t) = X_m(t)/t^{m-1/2}$. By (1.2)

$$EY_m^2(t) = b_m t^2. \tag{1.6}$$

Using (1.2), (1.3) and (1.5) we have

(1.7)
$$
\begin{aligned}
&E(Y_m(t+h)-Y_m(t))^2\\
&= E\left\{\frac{X_m(t+h)-X_m(t)}{(t+h)^{m-1/2}}-\left(\frac{1}{t^{m-1/2}}-\frac{1}{(t+h)^{m-1/2}}\right)X_m(t)\right\}^2\\
&= \frac{\sum_{j=2}^{2m+1} b_{mj}h^j t^{2m+1-j}}{(t+h)^{2m-1}}+\frac{((t+h)^{m-1/2}-t^{m-1/2})^2}{t^{2m-1}(t+h)^{2m-1}}b_m t^{2m+1}\\
&\quad -\frac{2((t+h)^{m-1/2}-t^{m-1/2})}{t^{m-1/2}(t+h)^{2m-1}}\left\{\frac{1}{2}b_m\left((t+h)^{2m+1}-t^{2m+1}\right)\right.\\
&\quad \left.-\frac{1}{2}\sum_{j=2}^{2m+1} b_{mj}h^j t^{2m+1-j}\right\}\\
&=: B_m h^2 + g_m(h,t),
\end{aligned}
$$

where

$$
\begin{aligned}
B_m &= b_{m2}+b_m\left\{\left(m-\frac{1}{2}\right)^2-\left(m-\frac{1}{2}\right)(2m+1)\right\}\\
&= b_{m2}-b_m\left(m-\frac{1}{2}\right)\left(m+\frac{3}{2}\right),
\end{aligned}
$$
$$
g_m(h,t)=O(h^3 t) \quad \text{as } ht\to 0,
$$

which implies that

(1.8)
$$
E(Y_m(t+h)-Y_m(t))^2=(1+o(1))B_m h^2 \quad \text{as } ht\to 0.
$$

2. Large deviations

First we quote a well-known lemma.

LEMMA 2.1 (Fernique). *Let $G(t)$ be a Gaussian process on $[0,1]$ with $EG^2(t)\le A^2$ and $E(G(t)-G(s))^2\le\sigma^2(|t-s|)$, where $\sigma(\cdot)$ is a continuous nondecreasing function satisfying*

$$
\int_1^\infty \sigma(e^{-x^2})\,dx<\infty.
$$

Then, for $x\ge 2$, we have

$$
P\left\{\sup_{0\le t\le 1}|G(t)|\ge x\left(A+\int_1^\infty\sigma(e^{-y^2})\,dy\right)\right\}\le ce^{-x^2/2},
$$

where c is an absolute constant.

The following is a large deviation result for small time increments.

PROPOSITION 2.1. *For any* $\varepsilon > 0$, *there exist positive numbers* h_0, x_0, c_1 *and* C_1 *such that for any* $0 < h \le h_0$ *and* $x \ge x_0$

$$P\left\{\sup_{0<t\le 1-h}\sup_{0\le s\le h}\frac{|X_m(t+s)-X_m(t)|}{(t\vee h)^{m-1/2}} \ge (1+\varepsilon)b_{m2}^{1/2}hx\right\} \le C_1(e^{-c_1x^2}+h^{-1}e^{-x^2/2}).$$

PROOF. For any $t > 0$ and integer $r > 0$, let $t_r = [t2^r/h]/(2^r/h)$, and write, for $rh < 1-h$,

(2.1)
$$\begin{aligned}&\sup_{0<t\le 1-h}\sup_{0\le s\le h}\frac{|X_m(t+s)-X_m(t)|}{(t\vee h)^{m-1/2}}\\&= \sup_{0<t\le rh}\sup_{0\le s\le h}\frac{|X_m(t+s)-X_m(t)|}{(t\vee h)^{m-1/2}} \vee \sup_{rh\le t\le 1-h}\sup_{0\le s\le h}\frac{|X_m(t+s)-X_m(t)|}{t^{m-1/2}}\\&=: I_1\vee I_2.\end{aligned}$$

Noting $t\vee h \ge (t+h)/2$, we have

$$I_1 \le 2^{m-1/2}\sup_{0<t\le rh}\sup_{0\le s\le h}\frac{|X_m(t+s)-X_m(t)|}{(t+h)^{m-1/2}} \le 2^{m+1/2}\sup_{0<t\le(1+r)h}|Y_m(t)|.$$

Let $Z_m(t) = Y_m((1+r)ht)$, $0 < t \le 1$. We will use Lemma 2.1 with $A = b_m^{1/2}(1+r)h$ and $\sigma(s) = (2B_m)^{1/2}(1+r)hs$. Put $D = (1+r)(b_m^{1/2} + (2B_m)^{1/2}\int_1^\infty e^{-y^2}\,dy)$. For any given $\varepsilon > 0$, take $r = r(\varepsilon)$ to be specified later on. By Lemma 2.1, we have

$$P\{I_1 \ge b_{m2}^{1/2}hx\} \le P\left\{\sup_{0<t\le 1}|Z_m(t)| \ge (b_{m2}^{1/2}2^{-(m+1/2)}D^{-1})Dhx\right\} \le Ce^{-c_1x^2} \tag{2.2}$$

for $x \ge x_0 := b_{m2}^{-1/2}2^{m+1/2}D$, where $c_1 = b_{m2}2^{-(2m+1)}D^{-2}/2$.

Consider I_2 now. We shall use a method similar to that in [1]. For $rh < t \le 1-h$, $0 \le s \le h$, which implies that

$$\frac{1}{t^{m-1/2}} \le \left(1+\frac{1}{r}\right)^{m-1/2}\frac{1}{(t+s)^{m-1/2}} \le \left(1+\frac{1}{r}\right)^{m-1/2}\frac{1}{(t+s)_{r+j+1}^{m-1/2}}$$

for any $j \ge 0$, we have

(2.3)
$$\begin{aligned}&\frac{|X_m(t+s)-X_m(t)|}{t^{m-1/2}}\\&\le \frac{|X_m((t+s)_r)-X_m(t_r)|}{t^{m-1/2}} + \frac{|X_m((t+s)_r)-X_m(t+s)|}{t^{m-1/2}}\end{aligned}$$

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$$
\begin{aligned}
&+\frac{|X_m(t_r)-X_m(t)|}{t^{m-1/2}}\\
&\le \frac{|X_m((t+s)_r)-X_m(t_r)|}{t^{m-1/2}}+\sum_{j=0}^{\infty}\frac{|X_m((t+s)_{r+j+1})-X_m((t+s)_{r+j})|}{t^{m-1/2}}\\
&+\sum_{j=0}^{\infty}\frac{|X_m(t_{r+j+1})-X_m(t_{r+j})|}{t^{m-1/2}}\\
&\le \frac{|X_m((t+s)_r)-X_m(t_r)|}{t_r^{m-1/2}}\\
&+\left(1+\frac{1}{r}\right)^{m-1/2}\sum_{j=0}^{\infty}\frac{|X_m((t+s)_{r+j+1})-X_m((t+s)_{r+j})|}{(t+s)_{r+j+1}^{m-1/2}}\\
&+\sum_{j=0}^{\infty}\frac{|X_m(t_{r+j+1})-X_m(t_{r+j})|}{t_{r+j+1}^{m-1/2}}.
\end{aligned}
$$

For the first term of the right hand side of (2.3), by (1.4) we have

$$
E\left(\frac{X_m((t+s)_r)-X_m(t_r)}{t_r^{m-1/2}}\right)^2 \le \left(1+\frac{\varepsilon}{4}\right)^2 b_{m2}\left(1+2^{-r}\right)^2 h^2 \le \left(1+\frac{\varepsilon}{3}\right)^2 b_{m2}h^2,
$$

provided $r = r(\varepsilon)$ is large enough. Hence, noting that the number of points lying within the grid $[0, h] \times [rh, 1]$ with step $h/2^r$ is less than $2^{2r}/h$, we obtain

$$
\begin{aligned}
(2.4)\quad & P\left\{\sup_{rh<t\le 1-h}\sup_{0\le s\le h}\frac{|X_m((t+s)_r)-X_m(t_r)|}{t_r^{m-1/2}} \ge \left(1+\frac{\varepsilon}{3}\right)b_{m2}^{1/2}hx\right\}\\
&\le \frac{2^{2r}}{h}\sup_{rh<t\le 1-h}\sup_{0\le s\le h} P\left\{\frac{|X_m((t+s)_r)-X_m(t_r)|}{t_r^{m-1/2}} \ge \left(1+\frac{\varepsilon}{3}\right)b_{m2}^{1/2}hx\right\}\\
&\le \frac{2^{2r}}{h}e^{-x^2/2}
\end{aligned}
$$

by recalling the well-known inequality $1-\Phi(x) \le (1/\sqrt{2\pi}x)e^{-x^2/2}$. (Without loss of generality, assume that $x_0 \ge 1/\sqrt{2\pi}$.)

Consider the second term of the right hand side of (2.3). Note the following inequality:

$$
\begin{aligned}
P\left\{\sup_{i\in I}\sum_{j=0}^{\infty}X_{ij} \ge \sum_{j=0}^{\infty}x_j\right\} &\le \sum_{i\in I}P\left\{\sum_{j=0}^{\infty}X_{ij} \ge \sum_{j=0}^{\infty}x_j\right\}\\
&\le \#(I)\sup_{i\in I}P\{\exists j\ge 0 : X_{ij}\ge x_j\}\\
&\le \#(I)\sup_{i\in I}\sum_{j=0}^{\infty}P\{X_{ij}\ge x_j\},
\end{aligned}
$$

where X_{ij}, $i \in I$, $j = 0, 1, \ldots$, are random variables and x_j, $j = 0, 1, \ldots$, are real numbers. Moreover, by (1.4) again, we have

$$E\left(\frac{X_m(t_{r+j+1}) - X_m(t_{r+j})}{t_{r+j+1}^{m-1/2}}\right)^2 \le 2b_{m2}h^2/2^{2(r+j+1)}$$

for any $0 < t \le 1$, provided r is large enough. Furthermore, we may demand

$$\sqrt{2}\sum_{j=0}^{\infty} 2^{-(r+j+1)/2} \le \left(1+\frac{1}{r}\right)^{-m+1/2}\frac{\varepsilon}{3}.$$

Then we have

(2.5)

$$\begin{aligned}
P&\left\{\sup_{rh<t\le 1-h}\sup_{0\le s\le h}\sum_{j=0}^{\infty}\frac{|X_m((t+s)_{r+j+1})-X_m((t+s)_{r+j})|}{(t+s)_{r+j+1}^{m-1/2}} \ge \left(1+\frac{1}{r}\right)^{-m+1/2}\frac{\varepsilon}{3}b_{m2}^{1/2}hx\right\}\\
&\le P\left\{\sup_{rh<t\le 1}\sum_{j=0}^{\infty}\frac{|X_m(t_{r+j+1}) - X_m(t_{r+j})|}{t_{r+j+1}^{m-1/2}} \ge \sum_{j=0}^{\infty}\sqrt{2}b_{m2}^{1/2}\left(\frac{h}{2^{(r+j+1)/2}}\right)x\right\}\\
&\le \frac{2^r}{h}\sum_{j=0}^{\infty} e^{-2^{r+j+1}x^2/2} \le \frac{2^r}{h}e^{-x^2/2}
\end{aligned}$$

for large r. Similarly, for the third term of the right hand side of (2.3) we have

$$P\left\{\sup_{rh<t\le 1-h}\sup_{0\le s\le h}\sum_{j=0}^{\infty}\frac{|X_m(t_{r+j+1}) - X_m(t_{r+j})|}{t_{r+j+1}^{m-1/2}} \ge \frac{\varepsilon}{3}b_{m2}^{1/2}hx\right\} \le \frac{2^r}{h}e^{-x^2/2}. \tag{2.6}$$

Combining (2.3)–(2.6) we obtain

$$P\left\{I_2 \ge (1+\varepsilon)b_{m2}^{1/2}hx\right\} \le \left(2^{2r}+2^{r+1}\right)\frac{1}{h}e^{-x^2/2}. \tag{2.7}$$

(2.2) and (2.7) together imply the conclusion of Proposition 2.1. □

An analogue of Proposition 2.1 in the large increment case is the following.

PROPOSITION 2.2. *Let a_T be a function of T with $0 < a_T \le T$ and $a_T/T \to 0$ as $T \to \infty$. Then for any $\varepsilon > 0$, there exist positive numbers T_0, x_1, c_2 and C_2 such that for any $T \ge T_0$ and $x \ge x_1$,*

$$P\left\{\sup_{0<t\le T-a_T}\sup_{0\le s\le a_T}\frac{|X_m(t+s)-X_m(t)|}{(t\vee a_T)^{m-1/2}} \ge (1+\varepsilon)b_{m2}^{1/2}a_Tx\right\} \le C_2(e^{-c_2x^2}+Ta_T^{-1}e^{-x^2/2}).$$

The proof is similar to that of Proposition 2.1, and hence, is omitted.

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3. Moduli of continuity

We need another well-known lemma.

LEMMA 3.1 (Slepian). *Let $G(t)$ and $G^*(t)$ be Gaussian processes on $[0, T]$ for some $0 < T < \infty$, possessing continuous sample path functions with $EG(t) = EG^*(t) = 0$, $EG^2(t) = EG^{*2}(t) = 1$, and let $\rho(s,t)$ and $\rho^*(s,t)$ be their respective covariance functions. Suppose that we have $\rho(s,t) \geq \rho^*(s,t)$, $s, t \in [0, T]$. Then for any real u,*

$$P\left\{\sup_{0\leq t\leq T} G(t) \leq u\right\} \geq P\left\{\sup_{0\leq t\leq T} G^*(t) \leq u\right\}.$$

Put $\log x = \ln(e \vee x)$.

THEOREM 3.1.

$$\limsup_{h\to 0} \sup_{0<t\leq 1-h} \sup_{0\leq s\leq h} \frac{|X_m(t+s) - X_m(t)|}{b_{m2}^{1/2}(t\vee h)^{m-1/2} h (2\log h^{-1})^{1/2}} \leq 1 \quad \textit{almost surely,} \tag{3.1}$$

$$\liminf_{h\to 0} \sup_{0<t\leq 1-h} \frac{|X_m(t+h) - X_m(t)|}{b_{m2}^{1/2}(t\vee h)^{m-1/2} h (2\log\log h^{-1})^{1/2}} \geq 1 \quad \textit{almost surely.} \tag{3.2}$$

REMARK 3.1. It is interesting to find the exact factors such that equality signs in (3.1) and/or (3.2) hold. For Lévy's moduli of continuity of a Brownian motion $W(\cdot)$, the '$(\log h^{-1})^{1/2}$' makes the equality sign in (3.1) hold. For $X_m(\cdot)$, there are certain difficulties because its increments are neither independent nor stationary.

PROOF. First we prove (3.1). For any given $\varepsilon > 0$, by Propositon 2.1, there exist $c_1 = c_1(\varepsilon) > 0$ and $C_1 = C_1(\varepsilon) > 0$ such that

$$\begin{aligned} P&\left\{\sup_{0<t\leq 1-h} \sup_{0\leq s\leq h} \frac{|X_m(t+s) - X_m(t)|}{b_{m2}^{1/2}(t\vee h)^{m-1/2} h (2\log h^{-1})^{1/2}} \geq (1+\varepsilon)^2\right\} \\ &\leq C_1\left(\exp\left\{-2c_1(1+\varepsilon)^2 \log h^{-1}\right\} + h^{-1}\exp\left\{-(1+\varepsilon)^2\log h^{-1}\right\}\right) \\ &\leq C_1(h^{2c_1} + h^{2\varepsilon}). \end{aligned}$$

Taking $h_n = n^{-A}$ with $A > (2(\varepsilon \wedge c_1))^{-1}$, we obtain

$$\sum_{n=1}^{\infty} P\left\{\sup_{0<t\leq 1-h_n} \sup_{0\leq s\leq h_n} \frac{|X_m(t+s) - X_m(t)|}{b_{m2}^{1/2}(t\vee h_n)^{m-1/2} h_n (2\log h_n^{-1})^{1/2}} \geq (1+\varepsilon)^2\right\} < \infty,$$

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which, in combination with the Borel-Cantelli lemma, implies

$$\limsup_{n\to\infty} \sup_{0<t\le 1-h_n} \sup_{0\le s\le h_n} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee h_n)^{m-1/2}h_n(2\log h_n^{-1})^{1/2}} \le (1+\varepsilon)^2 \quad \text{a.s.} \tag{3.3}$$

The procedure from (3.3) to (3.1) is routine, and hence, is omitted.

Next we show (3.2). Let $h_n = n^{-A_n}$ with $A_n = n^{(\log\log n)^{-1}} \uparrow \infty$ as $n \to \infty$. Define

$$Y(i) = \frac{X_m((i+1)h_n) - X_m(ih_n)}{(ih_n)^{m-1/2}}, \qquad 0 < i \le n^{A_n} - 1.$$

By (1.4), $EY(i)^2 \ge b_{m2}h_n^2$. We have that, for $i \le j$,

$$\begin{aligned}
&E(Y(i)Y(j)) \\
&= \frac{1}{(m!)^2(ih_n)^{m-1/2}(jh_n)^{m-1/2}} \left\{ \int_0^{(i+1)h_n} ((i+1)h_n - s)^m((j+1)h_n - s)^m ds \right. \\
&\quad - \int_0^{(i+1)h_n} ((i+1)h_n - s)^m (jh_n - s)^m ds \\
&\quad \left. - \int_0^{ih_n} (ih_n - s)^m((j+1)h_n - s)^m ds + \int_0^{ih_n} (ih_n - s)^m (jh_n - s)^m ds \right\} \\
&= h_n^2 \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p}\binom{m}{q} \frac{1}{(m!)^2(2m-p-q+1)(ij)^{m-1/2}} \\
&\quad \times \left\{ (i+1)^{2m-q+1}(j+1)^q - (i+1)^{2m-q+1}j^q - i^{2m-q+1}(j+1)^q + i^{2m-q+1}j^q \right\} \\
&= h_n^2 \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p}\binom{m}{q} \frac{((i+1)^{2m-q+1} - i^{2m-q+1})((j+1)^q - j^q)}{(m!)^2(2m-p-q+1)(ij)^{m-1/2}} \\
&= h_n^2 \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p}\binom{m}{q} \frac{4(2m-q+1)q\,(i/j)^{m-q+1/2}}{(m!)^2(2m-p-q+1)}(1 + O(1/i)).
\end{aligned} \tag{3.4}$$

Let $n_1 = [A_n \log n]$, $Z(i) = Y(e^i)$, $i = 0, 1, \dots, n_1$,

$$c_m = \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p}\binom{m}{q} \frac{4(2m-q+1)q}{(m!)^2(2m-p-q+1)},$$

and $D_n = 3\log\log n$. (3.4) implies that for $i \ge n_1/3$ and $j - i \ge D_n$,

$$E(Z(i)Z(j)) \le h_n^2 c_m e^{-(j-i)/2}(1 + O(1/i)) \le c_m(\log n)^{-1}h_n^2, \tag{3.5}$$

provided n is large enough. Let $\{\xi_i, i \ge 0\}$ and ζ be independent normal random variables with means zero and $E\xi_i^2 = EZ(i)^2 - c_m(\log n)^{-1}h_n^2 = (1 + o(1))b_{m2}h_n^2$ as $n \to \infty$ (recalling (1.4)), $E\zeta^2 = c_m(\log n)^{-1}h_n^2$. Define $\gamma_i = \xi_i + \zeta$. Then

$E\gamma_i^2 = EZ(i)^2$ and $EZ(i)Z(j) \le E\gamma_i\gamma_j$. Let $I = \{i : n_1/3 \le i \le n_1 - 1, i \bmod D_n\}$, then $\#(I) \ge n_1/(2D_n)$ for large n. Hence by Slepian's lemma and using the well-known inequality

$$1 - \Phi(x) \ge \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)e^{-x^2/2},$$

we obtain that for large n

$$\begin{aligned}
(3.6)\quad & P\left\{\max_{\substack{n_1/3\le i\le n_1-1\\ i \bmod D_n}} Z(i) \le (1-\varepsilon)b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\} \\
&\le P\left\{\max_{\substack{n_1/3\le i\le n_1-1\\ i \bmod D_n}} \gamma_i \le (1-\varepsilon)b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\} \\
&\le P\left\{\max_{\substack{n_1/3\le i\le n_1-1\\ i \bmod D_n}} \xi_i \le \left(1-\frac{\varepsilon}{2}\right)b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\} \\
&\quad + P\left\{\zeta \ge \frac{\varepsilon}{2}b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\} \\
&\le \left(1 - P\left\{\xi_i > (1-\frac{\varepsilon}{2})b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\}\right)^{n_1/(2D_n)} \\
&\quad + \exp\left\{-\frac{\varepsilon^2 b_{m2}}{4c_m(\log n)^{-1}}\log\log h_n^{-1}\right\} \\
&\le \left(1 - \frac{1}{(8\pi\log\log h_n^{-1})^{1/2}}\exp\left\{-\left(1-\frac{\varepsilon}{2}\right)\log\log h_n^{-1}\right\}\right)^{n_1/(2D_n)} + n^{-2} \\
&= \left(1 - \frac{(\log h_n^{-1})^{(1-\varepsilon/2)}}{(8\pi\log\log h_n^{-1})^{1/2}}\right)^{n_1/(2D_n)} + n^{-2} \\
&\le \exp\left\{-\frac{(\log h_n^{-1})^{-(1-\varepsilon/2)}n_1}{2D_n(8\pi\log\log h_n^{-1})^{1/2}}\right\} + n^{-2} \le 2n^{-2}.
\end{aligned}$$

Inequality (3.6) implies

$$\sum_{n=1}^{\infty} P\left\{\max_{0\le i\le n_1-1} Z(i) \le (1-\varepsilon)b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}\right\} < \infty,$$

and by the Borel-Cantelli lemma it follows that

$$(3.7)\qquad \liminf_{n\to\infty}\max_{0\le i\le n_1-1}\frac{Z(i)}{b_{m2}^{1/2}h_n(2\log\log h_n^{-1})^{1/2}} \ge 1-\varepsilon \quad \text{a.s.}$$

And hence we conclude

$$(3.8)\qquad \liminf_{n\to\infty} \sup_{h_n\le t\le 1-h_n} \frac{X_m(t+h_n)-X_m(t)}{b_{m2}^{1/2} t^{m-1/2} h_n (2\log\log h_n^{-1})^{1/2}} \ge 1-\varepsilon \quad \text{a.s.}$$

Considering $h_{n+1} < h \le h_n$, we have

$$\begin{aligned}(3.9)\qquad &\sup_{0<t\le 1-h} \frac{|X_m(t+h)-X_m(t)|}{b_{m2}^{1/2}(t\vee h)^{m-1/2} h (2\log\log h^{-1})^{1/2}} \\ &\ge \sup_{h_n\le t\le 1-h} \frac{|X_m(t+h_n)-X_m(t)+X_m(t+h)-X_m(t+h_n)|}{b_{m2}^{1/2} t^{m-1/2} h (2\log\log h^{-1})^{1/2}} \\ &\ge \sup_{h_n\le t\le 1-h_n} \frac{|X_m(t+h_n)-X_m(t)|}{b_{m2}^{1/2} t^{m-1/2} h_n (2\log\log h_n^{-1})^{1/2}} \\ &\quad -2 \sup_{h_n<t\le 1-(h_n-h_{n+1})} \sup_{0\le s\le h_n-h_{n+1}} \frac{|X_m(t+h_{n+1}+s)-X_m(t+h_{n+1})|}{b_{m2}^{1/2}(t+h_{n+1})^{m-1/2}(h_n-h_{n+1})} \\ &\quad \times \frac{(t+h_{n+1})^{m-1/2}(h_n-h_{n+1})(\log(h_n-h_{n+1})^{-1})^{1/2}}{(2\log(h_n-h_{n+1})^{-1})^{1/2} t^{m-1/2} h_{n+1} (\log\log h_{n+1}^{-1})^{1/2}}.\end{aligned}$$

By the derivative calculus for the function $f(x) = x^{-A_x}$, we have

$$h_n - h_{n+1} = h_{n+1}\frac{A_n \log n}{n\log\log n}(1+o(1)).$$

Therefore,

$$\lim_{n\to\infty} \sup_{h_n<t\le 1-(h_n-h_{n+1})} \frac{(t+h_{n+1})^{m-1/2}(h_n-h_{n+1})(\log(h_n-h_{n+1})^{-1})^{1/2}}{t^{m-1/2} h_{n+1}(\log\log h_{n+1}^{-1})^{1/2}} = 0.$$

Consequently we conclude (3.2) by (3.8), (3.9) and (3.1). This completes the proof of Theorem 3.1. □

4. Large increments

THEOREM 4.1. *Let a_T be a continuous function of T with $0 < a_T \le T$ and suppose that*

$$(4.1)\qquad \lim_{n\to\infty} \frac{\sup_{n-1<t\le n} a_t}{\inf_{n-1<t\le n} a_t} = 1$$

and

$$(4.2)\qquad \lim_{T\to\infty} \log(T/a_T)/\log\log T = \infty.$$

Then

$$(4.3)\qquad \limsup_{T\to\infty} \sup_{0<t\le T-a_T} \sup_{0\le s\le a_T} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee a_T)^{m-1/2} a_T (2\log(T/a_T))^{1/2}} \le 1 \quad \textit{a.s.}$$

If, instead of (4.2), *for any* $\varepsilon > 0$ *there exists* $T_0 > 0$ *such that for* $T > T_0$

$$\left(\log \frac{T}{a_T}\right)^{(\log\log\log T)^{1/\varepsilon}} \geq \log T, \tag{4.4}$$

$$(\log a_T)^{2(1-\varepsilon)\log\log\log a_T} \geq \log T, \tag{4.5}$$

then

$$\liminf_{T\to\infty} \sup_{0<t\leq T-a_T} \frac{|X_m(t+a_T)-X_m(t)|}{b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log\log(T/a_T))^{1/2}} \geq 1 \quad a.s. \tag{4.6}$$

PROOF. First we prove (4.3). Let $\theta > 1$ and for integers k and j let

$$A_{kj} = \{T : \theta^{k-1} < T \leq \theta^k,\ \theta^{j-1} < a_T \leq \theta^j\}. \tag{4.7}$$

In the sequel, we always consider k and j such that A_{kj} is non-empty. For any $A > 0$, by condition (4.2), there exists k_0 such that for $k \geq k_0$

$$\log\theta^{k-j}/\log\log\theta^k \geq A,$$

that is,

$$j \leq k - [(A/\log\theta)\log k + \theta_1] =: k_1, \tag{4.8}$$

where $\theta_1 = A(\log\log\theta)/\log\theta$. Then, noting that $b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log(T/a_T))^{1/2}$ is an increasing function of both T and a_T, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \sup_{0<t\leq T-a_T} \sup_{0\leq s\leq a_T} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log(T/a_T))^{1/2}} \\
&\quad\leq \limsup_{k\to\infty} \sup_{-\infty<j\leq k_1} \sup_{0<t\leq\theta^k-\theta^{j-1}} \sup_{0\leq s\leq\theta^j} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee\theta^{j-1})^{m-1/2}\theta^{j-1}(2\log\theta^{k-j})^{1/2}} \\
&\quad\leq \limsup_{k\to\infty} \sup_{-\infty<j\leq k_1} \sup_{0<t\leq\theta^{k+1}-\theta^j} \sup_{0\leq s\leq\theta^j} \frac{\theta^{m+1/2}|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee\theta^j)^{m-1/2}\theta^j(2\log\theta^{k-j})^{1/2}}.
\end{aligned} \tag{4.9}$$

Using Proposition 2.2 and (4.8) we have

$$\begin{aligned}
&P\left\{\sup_{-\infty<j\leq k_1} \sup_{0<t\leq\theta^{k+1}-\theta^j} \sup_{0\leq s\leq\theta^j} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee\theta^j)^{m-1/2}\theta^j(2\log\theta^{k-j})^{1/2}} \geq (1+\varepsilon)^2\right\} \\
&\quad\leq C_2 \sum_{j=-\infty}^{k_1} \left(\exp\left\{-2c_2(1+\varepsilon)^2\log\theta^{k-j}\right\} + \theta^{k-j+1}\exp\left\{-(1+\varepsilon)^2\log\theta^{k-j}\right\}\right) \\
&\quad\leq C_2 \sum_{j=-\infty}^{k_1} \left(\theta^{-2c_2(1+\varepsilon)^2(k-j)} + \theta^{-2\varepsilon(k-j)+1}\right) \\
&\quad\leq c\left(\theta^{-2c_2(1+\varepsilon)^2[(A/\log\theta)\log k+\theta_1]} + \theta^{-2\varepsilon[(A/\log\theta)\log k+\theta_1]+1}\right) \leq ck^{-2}
\end{aligned}$$

for some $c > 0$ by taking $A = (\log\theta)/(c_2(1+\varepsilon)^2 \wedge \varepsilon)$. Hence, from the Borel-Cantelli lemma we obtain

$$\limsup_{k\to\infty} \sup_{-\infty<j\le k_1} \sup_{0\le t\le\theta^{k+1}-\theta^j} \sup_{0\le s\le\theta^j} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee\theta^j)^{m-1/2}\theta^j(2\log\theta^{k-j})^{1/2}} \le (1+\varepsilon)^2 \quad \text{a.s.}$$

which, in combination with (4.9), implies (4.3) by arbitrariness of $\theta > 1$.

Next we show (4.6). Let $A_j = j^{(\log\log j)^{-1}}$ again, and let $B_0 = 0$, $B_j = j^{A_j}$, $j = 1, 2, \ldots$, $C_{kj} = \{T : B_{k-1} < T \le B_k, B_{j-1} < a_T \le B_j\}$. By condition (4.4), for any $A > 0$, there exists an integer j_0 such that for $j \ge j_0$

$$\log(B_k/B_j) \ge (\log B_k)^{(\log\log k)^{-A}} \ge A_k^{(\log\log k)^{-A}}. \tag{4.10}$$

On the other hand, by the derivative calculus for the function $g(x) = \log B_x$, we have

$$\log B_k - \log B_j \le 2(k-j)\frac{A_k \log k}{k \log\log k},$$

which, in combination with (4.10), implies that

$$j \le k - \left[\frac{k\log\log k}{2\log k} A_k^{-1+(\log\log k)^{-A}}\right] =: k_2.$$

Noting that $b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log\log(T/a_T))^{1/2}$ is an increasing function of both T and a_T we can write

$$\begin{aligned}
(4.11)\quad & \liminf_{T\to\infty} \sup_{0<t\le T-a_T} \frac{|X_m(t+a_T)-X_m(t)|}{b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log\log(T/a_T))^{1/2}} \\
&\ge \liminf_{k\to\infty} \inf_{1\le j\le k_2} \inf_{T\in C_{kj}} \sup_{0<t\le T-a_T} \frac{|X_m(t+a_T)-X_m(t)|}{b_{m2}^{1/2}(t\vee a_T)^{m-1/2}a_T(2\log\log(T/a_T))^{1/2}} \\
&\ge \liminf_{k\to\infty} \inf_{1\le j\le k_2} \sup_{0<t\le B_{k-1}/2} \frac{|X_m(t+B_j)-X_m(t)|}{b_{m2}^{1/2}(t\vee B_j)^{m-1/2}B_j(2\log\log(B_k/B_j))^{1/2}} \\
&\quad - \limsup_{k\to\infty} \sup_{0<t\le B_k-(B_j-B_{j-1})} \sup_{0\le s\le B_j-B_{j-1}} \frac{|X_m(t+s)-X_m(t)|}{b_{m2}^{1/2}(t\vee(B_j-B_{j-1}))^{m-1/2}(B_j-B_{j-1})} \\
&\quad \times \frac{(t\vee(B_j-B_{j-1}))^{m-1/2}(B_j-B_{j-1})(\log(B_k/(B_j-B_{j-1})))^{1/2}}{(2\log(B_k/(B_j-B_{j-1})))^{1/2}(t\vee B_j)^{m-1/2}B_j(\log\log(B_k/B_j))^{1/2}} \\
&=: J_1 - J_2.
\end{aligned}$$

By the derivative calculus for the function $h(x) = B_x$, we have

$$\frac{B_j - B_{j-1}}{B_j} \le \frac{2A_j \log j}{j\log\log j}.$$

The last inequality and condition (4.5) imply that, as $k \to \infty$,

$$\log B_k \le (1 + o(1)) \log B_{k-1} \le 2(\log B_j)^{2(1-\varepsilon)\log\log\log B_j} \le 2(A_j \log j)^{2(1-\varepsilon)\log\log j}.$$

Hence

$$\begin{aligned}(4.12) \qquad & \frac{(t \vee (B_j - B_{j-1}))^{m-1/2}(B_j - B_{j-1})(\log(B_k/(B_j - B_{j-1})))^{1/2}}{(t \vee B_j)^{m-1/2} B_j (\log\log(B_k/B_j))^{1/2}} \\ & \le \frac{B_j - B_{j-1}}{B_j}(\log B_k)^{1/2} \le \frac{2\sqrt{2}A_j \log j}{j \log\log j} \cdot (A_j \log j)^{(1-\varepsilon)\log\log j} \\ & = \frac{2\sqrt{2}A_j (\log j)^{1+(1-\varepsilon)\log\log j}}{j^{\varepsilon} \log\log j} \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Then by (4.3) and (4.12) we obtain

$$(4.13) \qquad J_2 = 0 \quad \text{a.s.}$$

Consider J_1 and for fixed k, define

$$Y_j(i) = \frac{X_m((i+1)B_j) - X_m(iB_j)}{(iB_j)^{m-1/2}}, \quad 0 < i \le B_k/B_j - 1,\ j = 0, 1, \dots, k_2.$$

Furthermore, let $Z_j(i) = Y_j(e^i)$, $i = 0, 1, \dots, k_3 - 1$ with $k_3 = [\log(B_k/B_j)]$. Similarly to (3.5), we have

$$EZ_j(i_1)Z_j(i_2) \le c'_m (\log\log k)^{-A-2} B_j^2$$

for some $c'_m > 0$ and any $i_1 \ge k_3/3$, $i_2 - i_1 \ge D'_k := 3(A+2)\log\log\log k$. Let $\{\xi_{ij}, i \ge 0\}$ and ζ_j be independent normal random variables with means zero and $E\xi_{ij}^2 = EZ_j(i)^2 - c'_m(\log\log k)^{-A-2}B_j^2$, $E\zeta_j^2 = c'_m(\log\log k)^{-A-2}B_j^2$. Then, similarly to (3.6), using (4.10) with $A > 6/\varepsilon$ we obtain for all large k

$$\begin{aligned}(4.14) \qquad & P\left\{\inf_{0 \le j \le k_2} \max_{\substack{k_3/3 \le i \le k_3 - 1 \\ i \bmod D'_k}} Z_j(i) \le (1-\varepsilon) b_{m2}^{1/2} B_j (2\log\log(B_k/B_j))^{1/2}\right\} \\ & \le \sum_{j=0}^{k_2} \left(\exp\left\{ -\frac{(\log(B_k/B_j))^{-(1-\varepsilon/2)}k_3}{2D'_k(8\pi \log\log(B_k/B_j))^{1/2}} \right\} \right. \\ & \qquad \left. + \exp\left\{ -\frac{\varepsilon^2 b_{m2}}{4c'_m(\log\log k)^{-A-2}} \log\log(B_k/B_j) \right\} \right) \\ & \le c \sum_{j=0}^{k_2} \left(\exp\left\{ -\frac{(\log(B_k/B_j))^{\varepsilon/2}}{D'_k(8\pi \log\log(B_k/B_j))^{1/2}} \right\} \right.\end{aligned}$$

$$+\exp\left\{-\frac{\varepsilon^2 b_{m2}}{4c'_m}(\log\log k)\log k\right\}\Bigg).$$

It is easy to see that

$$D'_k = o(\log(B_k/B_j)), \quad \log\log(B_k/B_j) = o(\log(B_k/B_j)).$$

So for large k,

$$\exp\left\{-\frac{(\log(B_k/B_j))^{\varepsilon/2}}{D'_k(8\pi\log\log(B_k/B_j))^{1/2}}\right\} \le \exp\{-(\log(B_k/B_j))^{\varepsilon/3}\}.$$

Combining it with (4.14) implies

$$\sum_{k=1}^{\infty} P\left\{\inf_{0\le j\le k_2}\max_{0\le i\le k_3-1} Z_j(i) \le (1-\varepsilon)b_{m2}^{1/2}B_j(2\log\log(B_k/B_j))^{1/2}\right\} < \infty.$$

Hence

$$J_1 \ge 1-\varepsilon \quad \text{a.s.} \tag{4.15}$$

Combining (4.15) with (4.13) we conclude that (4.4) holds. This completes the proof of Theorem 4.1. □

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Department of Mathematics
Zhejiang University, Xixi Campus
Hangzhou
Zhejiang 310028
P. R. China
e-mail: zlin@mail.hz.zj.cn