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Approximation On Arcs and Dendrites Going to Infinity in $\mathbb{C}^n$

In memoriam: Herbert James Alexander 1940–1999

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Abstract. On a locally rectifiable arc going to infinity, each continuous function can be approximated by entire functions.

1 Introduction

A famous theorem of Torsten Carleman [4] asserts that for each continuous function f on the real line $\mathbb{R}$ and for each positive continuous function ϵ on $\mathbb{R}$, there exists an entire function g on $\mathbb{C}$ such that

$$|f(x) - g(x)| < \epsilon(x), \quad \text{for all } x \in \mathbb{R}.$$

Carleman's theorem was extended to $\mathbb{C}^n$ by Herbert Alexander [2] who replaced the line $\mathbb{R}$ by a piecewise smooth arc going to infinity in $\mathbb{C}^n$ and by Stephen Scheinberg [9] who replaced the real line $\mathbb{R}$ by the real part $\mathbb{R}^n$ of $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. In the present work, we extend Alexander's theorem to locally rectifiable closed connected subsets $\Gamma \subset \mathbb{C}^n$ which contain no closed curves.

Let X be a subset of $\mathbb{C}^n$. X is a continuum if it is a compact connected set. The length and area of X are the Hausdorff 1-measure and 2-measure of X respectively. The set X is said to be of *finite length* at a point $x \in X$ if this point has a neighbourhood in X of finite length, and X is said to be of *locally finite length* if X is of finite length at each of its points. Notice that if X is a set of locally finite length, then each compact subset of X has finite length (though X itself need not be of finite length). We denote the polynomial hull of a compact set X by $\hat{X}$. The algebra of continuous functions defined on X is denoted by $\mathcal{C}(X)$. Finally, the definition and some properties of the first Čech cohomology group with integer coefficients $\check{H}^1(X)$ are presented in [7] and [12].

2 The Alexander-Stolzenberg Theorem

John Wermer laid the foundations of approximation on curves in $\mathbb{C}^n$ and prepared the way for a fundamental result of Gabriel Stolzenberg [10] concerning hulls and

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smooth curves (for history see [11]). In [3], Alexander comments that Stolzenberg's theorem can be improved to consider *continua of finite length* instead of *smooth curves*. We shall refer to the following version as the Alexander-Stolzenberg Theorem.

Theorem 1 **(Alexander-Stolzenberg)** *Let X and Y be two compact subsets of $\mathbb{C}^n$, with X polynomially convex and $Y \setminus X$ of zero area. Then,*

A: *Every continuous function on $X \cup Y$ which is uniformly approximable on X by polynomials is uniformly approximable on $X \cup Y$ by rational functions.*

Suppose, moreover, there exists a continuum $\Upsilon \subset \mathbb{C}^n$ such that $\Upsilon \setminus X$ has locally finite length and $Y \subset (X \cup \Upsilon)$. Then:

B: $\widehat{X \cup Y} \setminus (X \cup Y)$ *is (if non-empty) a pure one-dimensional analytic subset of $\mathbb{C}^n \setminus (X \cup Y)$.*

C: *If the map $\check{H}^1(X \cup Y) \to \check{H}^1(X)$ induced by $X \subset X \cup Y$ is injective, then $X \cup Y$ is polynomially convex.*

The proof of this theorem is implicitly contained in the papers of Stolzenberg [11] and Alexander [3]. A sketch of the proof is available from the authors and will be published elsewhere. Note also, that in this Alexander-Stolzenberg Theorem, locally finite length is required only for parts B and C.

3 Approximation On Unbounded Sets

We now pass from approximation on compacta to approximation on closed sets. A closed set Y of $\mathbb{C}^n$, without interior, is called a *set of tangential approximation by entire functions* if, for each $f \in \mathcal{C}(Y)$ and each positive $\epsilon \in \mathcal{C}(Y)$, there is an entire function g such that $|f - g| < \epsilon$ on Y.

Carleman's theorem, stated in the introduction, asserts that the real line $\mathbb{R}$ in $\mathbb{C}$ is a set of tangential approximation by entire functions. This was originally proved employing inductive processes. Alexander [2] extended Carleman's theorem to piecewise smooth arcs Γ going to infinity in $\mathbb{C}^n$. That is, Γ is the the image of the real axis under a proper continuous embedding (a curve without self-intersections, *going to infinity in both directions*).

As a consequence of the Alexander-Stolzenberg Theorem, we can also approximate by entire functions on unbounded sets which are more general than arcs, but first, we need to introduce the polynomially convex hull of non-compact sets:

Definition Given an arbitrary subset Y of $\mathbb{C}^n$, its polynomially convex hull is defined by $\hat{Y} = \bigcup\{\hat{K} : K \subset Y \text{ is compact}\}$.

Proposition 1 *Let Γ be a closed set in $\mathbb{C}^n$ of zero area such that $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded for every compact set $D \subset \mathbb{C}^n$. Let B_1 be an open ball with center in the origin which contains the closure of $\hat{\Gamma} \setminus \Gamma$. That is, the set $B_1 \cup \Gamma$ contains the hull $\hat{K}$ of every compact set $K \subset \Gamma$.*

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Then, given two continuous functions $f, \epsilon \in \mathcal{C}(\Gamma)$ *such that* ϵ *is real positive and* f *can be uniformly approximated by polynomials on* $\Gamma \cap \bar{B}_1$, *there exists an entire function* F *such that* $|F(z) - f(z)| < \epsilon(z)$ *for* $z \in \Gamma$.

Proof Let B_0 be the empty set, B_1 as in the hypotheses and B_k open balls with center in the origin such that each B_k contains the closure of $\widehat{\Gamma \cup \bar{B}_{k-1}} \setminus \Gamma$. That is, the set $B_k \cup \Gamma$ contains the hull $\hat{K}$ of every compact set $K \subset (\Gamma \cup \bar{B}_{k-1})$. Define X_k to be the polynomially convex hull of $\bar{B}_{k+1} \cap (\Gamma \cup \bar{B}_{k-1})$, so $X_k \subset (B_k \cup \Gamma)$. The compact sets X_k and $X_k \cap \bar{B}_k$ are both polynomially convex.

The given hypotheses automatically imply that there exists a polynomial F_1 such that $|F_1(z) - f(z)| < (\frac{2}{3} - 2^{-1})\epsilon(z)$ on $\Gamma \cap \bar{B}_1$. Proceeding by induction, we shall construct a sequence of polynomials F_k which converges uniformly on compact sets to an entire function with the desired properties.

Given a polynomial F_k such that $|F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ on $\Gamma \cap \bar{B}_k$, let h_k be a continuous function equal to F_k on $\bar{B}_k$ and such that $|h_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ for $z \in \Gamma \cap \bar{B}_{k+1}$ as well. Fix a real number $0 < \lambda_k < 1$ strictly less than $\epsilon(z)$ for every $z \in \Gamma \cap \bar{B}_{k+1}$.

Notice that $X_k = (X_k \cap \bar{B}_k) \cup (\Gamma \cap \bar{B}_{k+1})$. Hence, by Theorem 1.A, the function h_k can be approximated by rational functions on X_k because $X_k \cap \bar{B}_k$ is polynomially convex and Γ has zero area. Moreover, the functions h_k can be approximated by polynomials by the Oka-Weil theorem. Thus, there exists a polynomial F_{k+1} such that $|F_{k+1}(z) - h_k(z)| < 2^{-1-k}\lambda_k$ for $z \in X_k$, and so $|F_{k+1}(z) - f(z)| < (\frac{2}{3} - 2^{-1-k})\epsilon(z)$ on $\Gamma \cap \bar{B}_{k+1}$.

Finally, the inequality $|F_{k+1}(z) - F_k(z)| < 2^{-1-k}$ holds for $z \in \bar{B}_{k-1}$, so the sequence F_k converges to an entire function with the desired properties. ■

On the other hand, if the equality $\hat{\Gamma} = \Gamma$ holds as well in the last proposition, we can choose the empty set instead of the open ball B_1 (because the proof is an inductive process); and so Γ becomes a set of tangential approximation by entire functions. There are many closed sets Γ which satisfy the hypotheses of the last proposition. For example, we have the following.

Theorem 2 *Let* Γ *be closed connected set of locally finite length in* $\mathbb{C}^n$ *whose first cohomology group* $\check{H}^1(\Gamma)$ *vanishes* (Γ *contains no simple closed curves*). *Then,* Γ *is a set of tangential approximation by entire functions.*

In particular, this theorem allows tangential approximation on locally rectifiable arcs tending to infinity. This was conjectured by Aupetit in [1] and announced by Alexander in [2].

Proof The proof strongly uses the topology of Γ. We show that each point of Γ has finite order, that is, has a basis of neighbourhoods in Γ having finite boundaries. Given a point $z \in \Gamma$, let B_r be the open ball in $\mathbb{C}^n$ of radius r and center z. Since Γ is locally of finite length, the intersection of Γ with the closed ball $\bar{B}_r$ has finite length, so the intersection of Γ with the boundary of B_s must be a finite set for almost all radii $0 < s < r$. Whence, each sub-continuum of Γ is locally connected [8,

p. 283]. On the other hand, there are no simple closed curves contained in Γ because $\check{H}^1(\Gamma) = 0$, so each sub-continuum of Γ is a dendrite, that is, a locally connected continuum containing no simple closed curves. In particular, if Γ is compact, then it is a dendrite.

Notice the following lemma.

Lemma 1 *Each compact subset $K \subset \Gamma$ is contained in a sub-continuum (dendrite) of Γ.*

Proof Since Γ is locally connected, the set K is contained in a finite union of sub-continua of Γ. The lemma now follows since Γ is arcwise connected (see Theorem 3.17 of [7]). ■

Let D be a compact set in $\mathbb{C}^n$. Notice that $D \cup \Gamma$ may contain simple closed curves Υ with $D \cap \Upsilon \neq \varnothing$ but $\Upsilon \not\subset D$. We shall call such a simple closed curve $\Upsilon \subset (D \cup \Gamma)$ a *loop*. We show there exists a ball which contains all of these loops. Henceforth, let B_r be open balls of radii r and center in the origin, and choose a radius $s > 0$ such that $D \subset B_s$. Recall that $\Gamma \cap \bar{B}_{s+1}$ has finite length, so there exists a ball B_t with $s < t < s + 1$ such that Γ meets the boundary of B_t only in a finite number of points $Q = \{q_1, \dots, q_m\}$. Let $\{\Upsilon_j\}$ be the possible loops which meet the complement of B_t. The set $\bigcup\{\Upsilon_j\} \setminus B_t$ is contained in Γ and can be expressed as the union of compact arcs (not necessarily disjoint) which lie outside of $\bar{B}_t$ except for their two end points which lie in Q. Since Γ cannot contain simple closed curves, two different arcs cannot share the same end points, and there can only be finitely many such arcs. Hence, there exists a ball B_δ which contains all the loops Υ, and $D \subset B_\delta$.

We shall show that $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded. Without loss of generality, we may suppose that D is a closed ball. Since Γ is connected, the hull $\widehat{D \cup \Gamma}$ is equal to $\bigcup_{r \geq \delta} \hat{K}_r$, where K_r is the connected component of $\bar{B}_r \cap (D \cup \Gamma)$ which contains D. We can prove that $\hat{K}_r = \hat{K}_\delta \cup K_r$, for every $r \geq \delta$, using Alexander's original argument. The following lemma is a literal translation of Lemma 1(a) of [2], to our context.

Lemma 2 *For every $r \geq \delta$, $\hat{K}_r = \hat{K}_\delta \cup \tau_r$ where $\tau_r = \overline{K_r \setminus K_\delta}$.*

Since the notation is quite complicated and different from Alexander's, and we need to invoke Theorem 1.B, we shall include the proof of Lemma 2, but first we conclude the proof of the theorem.

By Lemma 2, the set $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded because $\hat{K}_r = \hat{K}_\delta \cup \tau_r = \hat{K}_\delta \cup K_r$ and $\widehat{D \cup \Gamma} = (\hat{K}_\delta \cup \Gamma)$. Moreover, the equality $\hat{\Gamma} = \Gamma$ holds as well because each compact subset of Γ is contained in a dendrite of finite length and is polynomially convex (see Lemma 1 and Alexander's work [3]), so we can deduce from Proposition 1 that Γ is a set of tangential approximation. ■

Proof of Lemma 2 Let $T_r = \hat{K}_\delta \cup \tau_r$ be the set on the right hand side of the asserted equality. Clearly, we have $T_r \subset \hat{K}_r \subset \hat{T}_r$ (the second inclusion is in fact equality). Thus it suffices to show that T_r is polynomially convex. Arguing by contradiction, we

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suppose otherwise. By Theorem 1.B, $\hat{T}_r \setminus T_r$ is a 1-dimensional analytic subvariety of $\mathbb{C}^n \setminus T_r$.

Let V be a non-empty irreducible analytic component of $\hat{T}_r \setminus T_r$. We claim that $\bar{V} \setminus K_r$ is an analytic subvariety of $\mathbb{C}^n \setminus K_r$. Since $T_r = \hat{K}_\delta \cup \tau_r$, it suffices to verify this locally at a point $x \in \bar{V} \cap Q$ where

$$Q = \hat{K}_\delta \setminus K_\delta.$$

By Theorem 1.B, both $\hat{K}_r$ and Q are analytic near x, where *near* x refers to the intersection of sets with *small enough* neighbourhoods of x, here and below. Furthermore, near x, $\bar{V} \subset \hat{K}_r$, $V \subset \hat{K}_r \setminus Q$ and $Q \subset \hat{K}_r$. Thus, near x, Q is a union of some analytic components of $\hat{K}_r$. It follows that near x, $\bar{V}$ is just a union of some of the other local analytic components of $\hat{K}_r$ at x; in fact, near x, $\bar{V} = V \cup \{x\}$. Put

$$W = \bar{V} \setminus K_r.$$

Then W is an irreducible analytic subset of $\mathbb{C}^n \setminus K_r$ and moreover,

$$\bar{W} \setminus W \subset K_\delta \cup \tau_r = K_r.$$

Thus $\bar{W} \subset \hat{K}_r$ by the maximum principle.

Fix a point $p \in V \subset W$. Since $p \notin T_r$, we have $p \notin \hat{K}_\delta$ and therefore there exists a polynomial h such that $h(p) = 0$ and $\Re h < 0$ on $\hat{K}_\delta$. By the open mapping theorem, either $h(W)$ is an open neighbourhood of 0 or $h \equiv 0$ on W. In the latter case, $h \equiv 0$ on $\bar{W}$ and so $\bar{W} \setminus W$ is disjoint from K_δ. This implies that $\bar{W} \setminus W \subset \hat{\tau}_r$ so $W \subset \hat{\tau}_r$. We have a contradiction because τ_r is contained in a dendrite of finite length and is polynomially convex (see Lemma 1 and Alexander's work [3]), and moreover, a dendrite cannot contain a 1-dimensional analytic set. Hence, the former case holds. As $h(\tau_r)$ is nowhere dense in the plane (recall that it is of finite length), there is a small complex number $\alpha \in h(W)$ such that $\alpha \notin h(\tau_r)$. Now put $g = h - \alpha$. If α is sufficiently small, we conclude that (i) $\Re g < 0$ on $\hat{K}_\delta$, (ii) $g(q) = 0$ for some $q \in W$ and (iii) $0 \notin g(\tau_r)$.

Now (i) implies that the polynomial g has a continuous logarithm on $\hat{K}_\delta$ and so, by restriction, on K_δ. We can extend this logarithm of g on K_δ to a continuous logarithm of g on K_r because of (iii), since the ball B_δ was chosen such that every simple closed curve (loop) $\Upsilon \subset K_r$ is contained in B_δ and hence in K_δ. But K_r contains $\bar{W} \setminus W$. Applying the argument principle [10, p. 271] to g on the analytic set W gives a contradiction to (ii). ■

We remark that the condition of having zero area is essential in Proposition 1, as the following example (inspired by [5]) shows.

Example 1 Let $\mathfrak{I}$ be the closed unit interval $[0, 1]$ of the real line and $K \subset \mathfrak{I}$ the compact set $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. It is easy to see that the $(2 + \epsilon)$-dimensional Hausdorff measure of the closed connected set $Y = (\mathfrak{I} \times \{0\}) \cup (K \times \mathbb{C})$ in $\mathbb{C}^2$ is equal to zero for every $\epsilon > 0$, moreover, the equality $\hat{Y} = Y$ holds. However,

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the following continuous function $f \in \mathcal{C}(Y)$ cannot be uniformly approximated by holomorphic functions in $\mathcal{O}(Y)$:

$$f(w,z) = \begin{cases} z \text{ if } w = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Suppose there exists a real number $\epsilon > 0$ and a holomorphic function $g \in \mathcal{O}(Y)$ such that $|f - g| < \epsilon$ on Y. We automatically have that $g(w,z)$ is bounded, holomorphic and constant on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \dots$. Hence, the holomorphic function $\frac{\partial g}{\partial z}$ vanishes on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \dots$ as well. Since the zero set of $\frac{\partial g}{\partial z}$ is an analytic set, this derivative must be zero in a neighbourhood of $\{0\} \times \mathbb{C}$ and hence on the connected set Y. The last statement is a contradiction to the fact that $|g(1,z) - z| < \epsilon$ for every $z \in \mathbb{C}$.

On the other hand, to see that $\hat{Y} = Y$, notice that $Y = \bigcup_{r>0} Y_r$, where $Y_r = (\mathcal{I} \times \{0\}) \cup (K \times \Delta_r)$ and $\Delta_r \subset \mathbb{C}$ are closed discs of radius r. The set $K \times \Delta_r$ is polynomially convex because it is the Cartesian product of two polynomially convex sets in $\mathbb{C}$; and so Y_r is polynomially convex because of Theorem 1.

Although connectivity, as we have emphasized, plays a crucial role in this paper, similar results can be obtained for sets whose connected components form a locally finite family. Finally, we remark that, on a Stein manifold, analogous results also hold by simply embedding the Stein manifold into some $\mathbb{C}^n$.

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