Can. J. Math. Vol. 48 (4), 1996 pp. 849–870

FIBER COMPLETIONS, CONTACT SINGULARITIES AND SINGLE VALUED SOLUTIONS FOR C^∞-SECOND ORDER ODE

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ABSTRACT. An implicitly defined second order ODE is said to be singular if the second derivative cannot be smoothly written in terms of lower order variables. The standard existence and uniqueness theory cannot be applied to such ODE and the graphs of solutions may fail to be regular curves (*i.e.*, the solutions may have isolated C^0-points or may fail to be single valued). In this paper we describe a local analysis for a large class of implicit second order ODE whose singular points satisfy a regularity condition. Within this class of ODE there is a secondary notion of (contact) singularity which is analogous to rest points for regular ODE. Theorems 5, 6, 7 and 8 produce invariants for these singularities which control the existence, uniqueness and the level of regularity in solutions.

1. **Introduction.** An implicitly defined second order ODE is said to be *singular* if the second derivative cannot be smoothly written in terms of lower order variables. The standard existence and uniqueness theory cannot be applied to such ODE and the graphs of solutions may fail to be regular curves (*i.e.*, the solutions may have isolated C^0-points or may fail to be single valued). In this paper we describe a local analysis for a large class of implicit second order ODE whose singular points satisfy a regularity condition. Within this class of ODE there is a secondary notion of (contact) singularity which is analogous to rest points for regular ODE. Theorems 5, 6, 7, and 8 produce invariants for these singularities which control the existence, uniqueness and the level of regularity in solutions.

Given a smooth second order ODE (*i.e.*, a smooth relation $R(x,y,y',y'')=0$) we seek local single valued solutions $y=f(x)$ which fail to be smooth at an isolated point where, they are C^0, and $\big(f'(x)\big)^{-m}$ has a smooth zero of order n, for some $m, n \in \mathbf{Z}^+$. This is accomplished by observing that the 2-jet space $J^2(\mathbf{R},\mathbf{R}) \longrightarrow J^1(\mathbf{R},\mathbf{R})$ has a natural completion. This completion is a fibered 4-manifold, $G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$, with a canonical differential ideal $\mathcal{J}^2$ (Section 2). Smooth immersive integral curves for this differential system correspond to graphs of multivalued functions. In Section 3 we identify special points in $G^2(\mathbf{R}^2)$ with the property that a smooth immersive integral curve which contains such a point locally defines a single valued function of the type we seek. We then view an ODE as 3-variety in $J^2(\mathbf{R},\mathbf{R})$ and examine its completion to $G^2(\mathbf{R}^2)$. If this completion contains a special point of $G^2(\mathbf{R}^2)$ and satisfies C^∞-open regularity conditions,

Research partially supported by NSF Grant DMS-9123752.
Received by the editors June 5, 1995.
AMS subject classification: Primary: 35A30, 35B32, 35F20.

https://doi.org/10.4153/CJM-1996-043-9 Published online by Cambridge University Press

then there exists a local solution of the type we seek (Theorem 5 and 6). This establishes the existence of single valued solutions for initial data at singular points of an ODE (*i.e.*, points in the fold locus or points of contact singularity where the natural exterior differential system drops rank). In the process we identify classes of ODEs for which these single valued solutions are locally stable under perturbation of initial conditions.

We then examine several classes of ODEs which fail to satisfy all of the regularity conditions of Theorem 6 yet respond to a similar analysis (Section 5). First we define the notion of a *second order* ODE with singular solutions [C], [K1]. Theorem 7 then provides a normal form for such an ODE and the existences of local single valued solutions. In Theorem 9 we consider second order ODE with a nondegenerate Morse singularity and provide necessary conditions for the existence of local C^∞, or C^0, single valued solutions with initial data at the Morse singularity. Such Morse singularities can occur in Quasi-Linear ODEs.

The classical literature on complex second order ODEs approaches the topic of this paper from the opposite direction [A]. Namely, a global classification of complex quasi-linear second order ODEs which do not contain (or contain few of) the special points of $G^2(\mathbf{C} \times \mathbf{CP}^1)$. This classification was up to the prolonged action of analytic fiber preserving transformations $\mathbf{C} \times \mathbf{CP}^1 \longrightarrow \mathbf{C} \times \mathbf{CP}^1$ which are Möbius on the $\mathbf{CP}^1$-fibers. (A base completion of $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C} \times \mathbf{CP}^1$ was implemented in order to study the poles of solutions). The motive here was to avoid those complex solutions whose graphs ramify over the space of independent variables. Such solutions would be relevent to this paper provided that certain reality conditions were satisfied. However, the Möbius-reduction of the transformation group is inappropriate for the study of local solutions [KS].

2. **The Grassman bundles.** Let $\pi^1: G^1(\mathbf{R}^2) \longrightarrow (\mathbf{R}^2)$ denote the Grassman bundle whose fiber at a point of $\mathbf{R}^2$ is the set of unoriented lines in the tangent space of $\mathbf{R}^2$ (at the given point). This bundle is diffeomorphic to $S^1 \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ and carries a canonical differential ideal, $\mathcal{J}^1$. A tangent vector at a point e^1 of $G^1(\mathbf{R}^2)$ is annihilated by this ideal if and only if the π^1_*-projection of the vector lies in the line corresponding to the point e^1. This defines a smooth non-integrable 2-plane distribution Ann^1 on the 3-manifold $G^1(\mathbf{R}^2)$.

A choice of domain projection $\pi_d: \mathbf{R}^2 \longrightarrow \mathbf{R}_d$ defines a subundle IG^1 in $G^1(\mathbf{R}^2) \longrightarrow \mathbf{R}^2$ consisting of lines which are tangent to the π_d-fibers. Over each point of $\mathbf{R}^2$ lies exactly one point of IG^1. The notation is mnemonic so as to suggest "infinite first derivative", see (3). The complement $G^2(\mathbf{R}^2) - IG^1$ can be identified with the 1-jet bundle of sections for $\pi_d: \mathbf{R}^2 \longrightarrow \mathbf{R}_d$. With a choice of range projection $\pi_r: \mathbf{R}^2 \longrightarrow \mathbf{R}_r$, transverse to π_d, the above complement can be identified with the 1-jet bundle, $\pi^1: J^1(\mathbf{R}_d, \mathbf{R}_r) \longrightarrow \mathbf{R}_d \times \mathbf{R}_r$, of functions $f: \mathbf{R}_d \longrightarrow \mathbf{R}_r$, [GG]. The group of smooth diffeomorphisms of $G^1(\mathbf{R}^2)$ which preserve the ideal $\mathcal{J}^1$ is denoted $C - \mathrm{Diff}^\infty(G^1)$ and is large. In particular the group of smooth diffeomorphisms of $\mathbf{R}^2$ acts naturally on $G^1(\mathbf{R}^2)$ by (the projectivization of) their differential maps. This defines a subgroup $P - \mathrm{Diff}^\infty(\mathbf{R}^2) \subset C - \mathrm{Diff}^\infty(G^1)$ which preserves the π^1-fibration. If a domain projection $\pi_d: \mathbf{R}^2 \longrightarrow \mathbf{R}_d$, has also been chosen, then the group of smooth π_d-fiber preserving diffeomorphisms defines a subgroup

$\mathrm{FP}-\mathrm{Diff}^\infty(\mathbf{R}^2) \subset P-\mathrm{Diff}^\infty(\mathbf{R}^2)$ which preserves the projections: π^1, π_d. Finally if a range projection, transverse to π_d, has also been chosen then we have the subgroup of product diffeomorphisms $\mathrm{Diff}^\infty(\mathbf{R}_d \times \mathbf{R}_r) \subset \mathrm{FP}-\mathrm{Diff}^\infty(\mathbf{R}^2)$ which preserve the projections: π^1, π_d, π_r.

If we choose coordinates (x,y) on $\mathbf{R}^2$ so that π_d and π_r are represented by

$$\pi_d(x,y) = x, \qquad \pi_r(x,y) = y, \tag{1}$$

respectively, then every line in the tangent space $T_{(x,y)}\mathbf{R}^2$ which is transverse to $\mathrm{Ker}(\pi_{d*}) = \partial y$ can be uniquely represented by $\partial x + p\partial y, p \in \mathbf{R}$. Thus (x,y,p) are coordinates on $G^2(\mathbf{R}^2)$ in which we have representations:

$$\begin{aligned} \pi^1(x,y,p) &= (x,y); \\ \theta^1 = dy - pdx, &\quad \text{generates } \mathcal{J}^1; \\ \partial x + p\partial y, \partial p &\quad \text{frame Ann}^1 \text{ at } (x,y,p). \end{aligned} \tag{2}$$

Similarly, every line in the tangent space $T_{(x,y)}\mathbf{R}^2$ which is transverse to $\mathrm{Ker}(\pi_{r*})$ can be uniquely represented by $\bar{p}\partial x + \partial y, \bar{p} \in \mathbf{R}$. Thus $(x,y,\bar{p})$ are coordinates on $G^1(\mathbf{R}^2)$ in which we have representations:

$$\begin{aligned} \pi^1(x,y,\bar{p}) &= (x,y); \\ \theta^1 = dx - \bar{p}dy, &\quad \text{generates } \mathcal{J}^1; \\ \partial y + \bar{p}\partial x, \partial \bar{p} &\quad \text{frame Ann}^1 \text{ at } (x,y,\bar{p}); \\ IG^1 &= \{\bar{p} = 0\}. \end{aligned} \tag{3}$$

The change of coordinate map between (2) and (3) is given by

$$\bar{p} = \frac{1}{p}. \tag{4}$$

Now let $\pi^2 \colon G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$ denote the Grassman bundle whose fiber at a point e^1 of $G^1(\mathbf{R}^2)$ is the set of unoriented lines, in the tangent space $T_{e^1}G^1(\mathbf{R}^2)$, which lie in the 2-plane $\mathrm{Ann}^1(e^1)$ (*i.e.*, are annihilated by $\mathcal{J}^1$ at e^1). This bundle is diffeomorphic to $S^1 \times S^1 \times \mathbf{R}^2 \longrightarrow S^1 \times \mathbf{R}^2$ and carries a canonical differential ideal $\mathcal{J}^2 = \mathcal{K} \oplus \pi^{2*}(\mathcal{J}^1)$. We will abuse notation and write $\mathcal{J}^1$ for $\pi^{2*}(\mathcal{J}^1)$. A tangent vector at a point $e^2 \in G^2(\mathbf{R}^2)$ annihilates the ideal $\mathcal{K}$ if and only if π^2_*-projection of this vector lies in the line corresponding to e^2. This defines two smooth non-integrable 3-plane distributions K, and $\pi^{2*}\,\mathrm{Ann}^1$ on $G^2(\mathbf{R}^2)$. We will abuse notation and write Ann^1 for $\pi^{2*}\,\mathrm{Ann}^1$. The intersection $\mathrm{Ann}^2 = K \cap \mathrm{Ann}^1$ is a smooth 2-plane distribution on $G^2(\mathbf{R}^2)$. The group $C-\mathrm{Diff}(G^1)$ acts naturally on $G^2(\mathbf{R}^2)$ by projectivization on their differential maps $TG^1(\mathbf{R}^2) \longrightarrow TG^1(\mathbf{R}^2)$. This defines a *transitive subgroup* $C-\mathrm{Diff}(G^2)$ of the group of smooth diffeomorphisms of $G^2(\mathbf{R}^2)$. This subgroup preserves the π^2-fibration and the ideals $\mathcal{K}$, $\mathcal{J}^1$. By (2) and (3) the π^1-

fibers are tangent to Ann^1. Hence there is a natural subbundle IG^2 in $G^2(\mathbf{R}^2)$ consisting of lines which are tangent to π^1-fibers. The notation is mnemonic so as to suggest "infinite second derivative", see (6). Upon making the choices π_d, π_r as above, the complement $G^2(\mathbf{R}^2)-\{IG^2 \cup IG^1\}$ can be identified with the 2-jet bundle $\pi^2 : J^2(\mathbf{R}_d, \mathbf{R}_r) \longrightarrow J^1(\mathbf{R}_d, \mathbf{R}_r)$. Notice that there is a natural section $ig^1 : IG^1 \longrightarrow IG^1$. Namely, at every point of IG^1 the line tangent to IG^1 also lies in Ann^1. This tangent line defines a unique point in each π^2-fiber, see (8). Recall that IG^1 and ig^1 require a choice of domain projection. Hence they are well defined only up to the action of $\text{FP} - \text{Diff}(\mathbf{R}^2)$.

If we choose coordinates (2) on $G^1(\mathbf{R}^2)$ then every line in the tangent space $T_{(x,y,p)}G^1(\mathbf{R}^2)$ which is annihilated by $\mathcal{J}^1$ and is transverse to $\text{Ker}(\pi^1_*) = \partial p$ can be uniquely represented by $\partial x + p\partial y + s\partial p$, $s \in \mathbf{R}$. Thus (x, y, p, s) are natural coordinates on $G^2(\mathbf{R}^2)$ in which we have representations:

$$
\begin{aligned}
&\pi^2(x, y, p, s) = (x, y, p);\\
&\theta^1 = dy - pdx, \quad \text{generates } \mathcal{J}^1;\\
&\theta^2 = dp - sdx, \quad \text{generates } \mathcal{K};\\
&\partial x + p\partial y + s\partial p, \partial s \quad \text{frame Ann}^2 \text{ at } (x, y, p, s);\\
&(x, y, p, s) \longrightarrow (x, y + g, p + g_x, s + g_{xx}).
\end{aligned} \tag{5}
$$

The last line represents the prolongation of a π_d-fiber preserving map $(x, y) \longrightarrow \big(x, y + g(x, y)\big)$ which induces the identity map on $\mathbf{R}_d$. Similarly the lines which annihilate $\mathcal{J}^2$ and lie in $IG^2 - IG^1$ can be uniquely represented by $\partial p + \tilde{s}(\partial x + p\partial y)p$, $\tilde{s} \in \mathbf{R}$, $\tilde{s} = 0$. Thus $(x, y, p, \tilde{s})$ are coordinates on $G^2(\mathbf{R}^2)$ in which we have representations:

$$
\begin{aligned}
&\pi^2(x, y, p, \tilde{s}) = (x, y, p);\\
&\theta^1 = dy - pdx, \quad \text{generates } \mathcal{J}^1;\\
&\theta^2 = \tilde{s}dp - dx, \quad \text{generates } \mathcal{K};\\
&\partial p + \tilde{s}(\partial x + p\partial y), \partial\tilde{s} \quad \text{frame Ann}^2 \text{ at } (x, y, p, \tilde{s});\\
&IG^2 = \{\tilde{s} = 0\};\\
&(x, y, p, \tilde{s}) \longrightarrow \left(x, y + g(x, y), p + g_x, \frac{\tilde{s}}{1 + \tilde{s}g_{xx}}\right).
\end{aligned} \tag{6}
$$

The last line represents the prolongation of a π_d-fiber preserving map which induces the identity map on $\mathbf{R}_d$. The change of coordinate map between (5) and (6) is given by

$$\tilde{s} = \frac{1}{s}. \tag{7}$$

Furthermore, the lines which annihilate $\mathcal{J}^2$ and lie in $IG^1 - IG^2$ have unique representatives $\partial y + \bar{p}\partial x + \bar{s}\partial\bar{p}$ with $\bar{p} = 0$ and $\bar{s} \in \mathbf{R}$. Thus $(x, y, \bar{p}, \bar{s})$ are coordinates on $G^2(\mathbf{R}^2)$ in which we have the representation:

$$(8)\qquad \begin{aligned} &\pi^2(x,y,\bar{p},\bar{s}) = (x,y,\bar{p});\\ &\theta^1 = dx - \bar{p}dy, \quad \text{generates } \mathcal{J}^1;\\ &\theta^2 = d\bar{p} - \bar{s}dy, \quad \text{generates } \mathcal{K};\\ &\partial y + \bar{p}\partial x + \bar{s}\partial p, \partial\bar{s} \quad \text{frame Ann}^2 \text{ at } (x,y,\bar{p},\bar{s});\\ &IG^1 - IG^2 = \{\bar{p} = 0\}; \quad ig^1 \cong \{\bar{s} = 0 = \bar{p}\};\\ &(x,y,\bar{p},\bar{s}) \longrightarrow \left(x, y + g(x,y), \frac{\bar{p}}{1+\bar{p}g_x}, \frac{\bar{s} - \bar{p}^3 g_{xx}}{(1+\bar{p}g_x)^3}\right). \end{aligned}$$

The last line represents the prolongation of a π_d-fiber preserving map which induces the identity map on $\mathbf{R}_d$. The change of coordinate map between (5) and (8) is given by

$$(9)\qquad \bar{p} = \frac{1}{p}, \quad \bar{s} = \frac{-s}{p^3}.$$

Finally, the lines which annihilate $\mathcal{J}^2$ and lie in $IG^1 \cap IG^2$ have unique representatives $\bar{\bar{s}}(\partial y + \bar{p}\partial x) + \partial\bar{p}$ with $\bar{p} = 0 = \bar{\bar{s}}$. Thus $(x,y,\bar{p},\bar{\bar{s}})$ are coordinates on $G^2(\mathbf{R}^2)$ in which we have the representations:

$$(10)\qquad \begin{aligned} &\pi^2(x,y,\bar{p},\bar{\bar{s}}) = (x,y,\bar{p});\\ &\theta^1 = dx - \bar{p}dy, \quad \text{generates } \mathcal{J}^1;\\ &\theta^2 = dy - \bar{\bar{s}}d\bar{p}, \quad \text{generates } \mathcal{K};\\ &\partial\bar{p} + \bar{\bar{s}}(\partial y + p\partial x), \partial\bar{\bar{s}} \quad \text{frame Ann}^2 \text{ at } (x,y,\bar{p},\bar{\bar{s}});\\ &IG^1 = \{\bar{p} = 0\}; \quad IG^2 = \{\bar{\bar{s}} = 0\};\\ &(x,y,\bar{p},\bar{\bar{s}}) \longrightarrow \left(x, y + g(x,y), \frac{\bar{p}}{1+\bar{p}g_x}, \frac{\bar{\bar{s}}(1+\bar{p}g_x)^3}{1 - \bar{\bar{s}}\bar{p}^3 g_{xx}}\right). \end{aligned}$$

The last line represents the prolongation of a π_d-filter preserving map which induces the identity map on $\mathbf{R}_d$. The change of coordinate map between (8) and (10) is given by

$$(11)\qquad \bar{\bar{s}} = \frac{1}{\bar{s}}.$$

3. **Integral curves.** A smooth immersion $c\colon \mathbf{R} \longrightarrow G^2(\mathbf{R}^2)$ is called C^∞ *immersive integral curve* if it is everywhere tangent to the 2-plane distribution Ann^2 (*i.e.*, $c^*(\mathcal{J}^2) = 0$). The *i-rank* of c at a point $t_0 \in (\mathbf{R}^2)$ is: the rank of the differential $(\pi^2 \circ c)_*(t_0)$, if $i = 2$, and the rank of the differential $(\pi^1 \circ \pi^2 \circ c)_*(t_0)$, if $i = 1$. There are exactly three possible cases: CASE 1: 2-rank $= 1 =$ 1-rank; CASE 2: 2-rank $= 0 =$ 1-rank; CASE 3: 2-rank $= 1$, 1-rank $= 0$.

If the 1-rank of c at t_0 is zero then the curve $\pi^1 \circ \pi^2 \circ c$ will fail to immerse at t_0. However, by the very existence of $\pi^2 \circ c$, this curve will have a well defined limiting tangent line at t_0. Similarly if the 2-rank of c at t_0 is zero then $\pi^2 \circ c$ will fail to immerse at t_0, yet it will have a well defined limiting tangent line. From this point of view c is a resolution or blow up for the nonimmersive singularities in $\pi^1 \circ \pi^2 \circ c$. Thus, on the most primitive level, integral curves can be interpreted as special singular curves in $\mathbf{R}^2$ up to the action of Diff($\mathbf{R}^2$). In order to use such integral curves in the context of ODE we choose projections π_d, π_r, restrict to the action of Diff($\mathbf{R}^2_d$) $\times$ Diff($\mathbf{R}_r$), and interpret $\pi^1 \circ \pi^2 \circ c \colon \mathbf{R} \longrightarrow \mathbf{R}_d \times \mathbf{R}_r$ as the graph of a multivalued function.

From this viewpoint the most interesting integral curves are those for which $c_d = \pi_d \circ \pi^1 \circ \pi^2 \circ c \colon \mathbf{R} \longrightarrow \mathbf{R}_d$ is one-to-one. We will say a continuous function $f \colon \mathbf{R}_d \longrightarrow \mathbf{R}_r$, which fails to be differentiable at 0, admits a (smooth) *G^2-resolution* c at 0 if the 2-jet of f in the (5)-coordinates (*i.e.*, $x = t, y = f(t), p = f'(t), s = f''(t)$) extends to 0, so that its image agrees with that of a smooth immersed integral curve $c \colon \mathbf{R} \longrightarrow G^2(\mathbf{R}^2)$ with $c_d(0) = 0$. (In other words the first and second derivatives of f blow up in a coherent manner at 0). We will now present local models for integral curves c and indicate when they are the G^2-resolution of a single valued function.

CASE 1. If the 1-rank of an integral curve c is 1 at $0 \in \mathbf{R}$ then the 2-rank is also 1 at 0 and $c(0) \notin IG^2$.

CASE 1a). If $c(0) \notin IG^1$ then $\pi^1 \circ \pi^2 \circ c \colon \mathbf{R} \longrightarrow \mathbf{R}_d \times \mathbf{R}_r$ can be locally parameterized as the graph of a smooth function, $f \colon \mathbf{R}_d \longrightarrow \mathbf{R}_r$. Thus we can locally parameterize c in the (5)-coordinates as $c(t) = \big(t, f(t), p(t), s(t)\big)$ where $f(t)$, $p(t)$, $s(t)$ are smooth functions which satisfy the differential relations: $0 = c^*\theta^1 = df - p\,dx = (f' - p)dt$; $0 = c^*\theta^2 = dp - s\,dx = (p' - s)dt$. It follows that c can be parameterized as the 2-jet of a smooth function $f \colon \mathbf{R}_d \longrightarrow \mathbf{R}_r$ (*i.e.*, $x = t, y = f(t), p = f'(t), s = f''(t)$ in the (5) coordinates).

CASE 1b). Similarly, if $c(0) \in IG^1$ then c can be locally parameterized as the 2-jet of a smooth function $f \colon \mathbf{R}_r \longrightarrow \mathbf{R}_d$ (*i.e.*, $x = f(t), y = t, \bar{p} = f'(t), \bar{s} = f''(t)$ with $f'(0) = 0$, in the (8)-coordinates). Note the reversal of $\mathbf{R}_d$ and $\mathbf{R}_r$.

CASE 2. If the 2-rank of c at 0 is zero then the 1-rank of c is also zero at 0 and $\dot{c}(0)$ must be tangent to the π^2-fiber at $c(0)$. Hence both curves $\pi^2 \circ c$ and $\pi^1 \circ \pi^2 \circ c$ fail to immerse at 0.

CASE 2a). If $c(0) \notin IG^2 \cup IG^1$ then (by the implicit function theorem) we can locally parameterize c in the (5)-coordinates as $c(t) = \big(x(t), y(t), p(t), s(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), p_0 + t^n c(t), s_0 + t\big)$ where: $x_0, y_0, p_0, s_0 \in \mathbf{R}$; $2 \leq \ell, m, n \in \mathbf{Z}$ and a, b, c are smooth functions which do not vanish at $t = 0$. They must satisfy the differential relations $0 = c^*\theta^1 = dy - p\,dx = (y' - px')dt$, $0 = c^*\theta^2 = dp - s\,dx = (p' - sx')dt$. (The lower bound on the exponents ℓ, m, n is exactly the i-rank condition). This implies that the choice of $2 \leq \ell$, and $a(t)$ determines m, n, $b(t)$, $c(t)$.

CASE 2b). If $c(0) \in IG^2 - IG^1$ then we can locally parameterize c in the (6)-coordinates as $c(t) = \big(x(t), y(t), p(t), \tilde{s}(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), p_0 + t^n c(t), t\big)$ where: $x_0, y_0, p_0 \in \mathbf{R}$; $2 \leq \ell, m, n \in \mathbf{Z}$ and a, b, c are smooth functions which do not vanish at $t = 0$. They must satisfy the differential relations $0 = c^*\theta^1 = dy - p\,dx = (y' - px')dt$; $0 = c^*\theta^2 = \tilde{s}\,dp - dx = (\tilde{s}p' - x')dt$. This implies that the choice of $2 \leq n, p_0$ and $c(t)$ determines ℓ, m, $a(t)$, $b(t)$.

CASE 2c). If $c(0) \in IG^1 - IG^2$ then we can locally parameterize c in the (8)-coordinates as $c(t) = \big(x(t), y(t), \bar{p}(t), \bar{s}(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), t^n c(t), \bar{s}_0 + t\big)$ where: x_0, y_0, $\bar{s}_o \in \mathbf{R}$: $2 \le \ell$, m, n and a, b, c are smooth functions which do not vanish at $t = 0$. They must satisfy the differential relations $0 = c^*\theta^1 = dx - \bar{p}dy = (x' - \bar{p}y')dt$, $0 = c^*\theta^2 = d\bar{p} - \bar{s}dy = (\bar{p}' - \bar{s}y')dt$. This implies that the choice of $2 \le m$, $\bar{s}_0$, and $b(t)$ determines ℓ, m, $a(t)$, $c(t)$. In particular, if $\bar{s}_0 = 0$ then $\ell = 2m + 1$; hence $x(t)$ is locally one-to-one. In this case $p^{2m+1} = (dy/dx)^{2m+1} \equiv (t^{-m-2})^{2m+1}$, $x \equiv t^{2m+1}$ and we have the following.

OBSERVATION 1. Given a smooth immersed integral curve c with $c(0) \in ig^1 \subset IG^1 - IG^2$ and 2-rank $= 0 =$ 1-rank at 0, then there exists a unique locally defined function $f: U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ continuous at $c_d(0)$, C^∞ on $U - \{c_d(0)\}$ whose G^2-resolution at $c_d(0)$ is c. Furthermore, there exists $m \in \mathbf{Z}$, $2 \le m$, such that $\big(f'(x)\big)^{-2m-1}$ has a zero of order $m + 1$ at $c_d(0)$.

EXAMPLE. Consider the integral curve c in (8)-coordinates $\big(x(t), y(t), \bar{p}(t), \bar{s}(t)\big) = (t^5/15, t^2/2, t^3/3, t)$ with $\ell = 5$, $m = 2$, $n = 3$ then $f(x) = (15x)^{\frac{2}{5}}/2$ has c as G^2-resolution.

CASE 2d). If $c(0) \in IG^2 \cap IG^1$ then we can locally parameterize c in the (10)-coordinates as $c(t) = \big(x(t), y(t), \bar{p}(t), \tilde{\bar{s}}(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), t^n c(t), t\big)$ where $2 \le \ell, m, n \in \mathbf{Z}$ and a, b, c are smooth functions which do not vanish at $t = 0$. They must satisfy the differential relations $0 = c^*\theta^1 = dx - \bar{p}dy = (x' - \bar{p}y')dt$, $0 = c^*\theta^2 = dy - \tilde{\bar{s}}d\bar{p} = (y' - \tilde{\bar{s}}\bar{p}')dt$. This implies that the choice of $2 \le n$ and $c(t)$ determines $\ell = 2n + 1$, $m = n + 1$, $a(t)$, $b(t)$. We have the following.

OBSERVATION 2. Given a smooth, immersed integral curve c with $c(0) \in IG^2 \cap IG^1$ and 2-rank $= 0 =$ 1-rank at 0 then there exists a unique locally defined function $f: U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ continuous at $c_d(0)$, C^∞ on $U - \{c_d(0)\}$, whose G^2-resolution at $c_d(0)$ is c. Furthermore, there exists $n \in \mathbf{Z}$, $2 \le n$, such that $\big(f'(x)\big)^{-2n-1}$ has a zero of order n at $c_d(0)$.

EXAMPLE. Consider the integral curve c in (10)-coordinates $\big(x(t), y(t), \bar{p}(t), \tilde{\bar{s}}(t)\big) = (t^5/2, t^3/3, t^2/2, t)$ with $\ell = 5$, $m = 3$, $n = 2$, then $f(x) = (2x)^{\frac{3}{5}}/3$ has c as G^2-resolutions.

CASE 3. If the 2-rank of c at 0 is one, then $\pi^2 \circ c$ is an immersed curve in $G^1(\mathbf{R}^2)$ which is everywhere tangent to the 2-plane distribution Ann^1. In this case $\pi^1 \circ \pi^2 \circ c$ will have rank zero at 0 exactly when $\pi^2 \circ c$ is tangent to the π^1-fiber at $\pi^2 \circ c(0)$ (*i.e.*, 1-rank $= 0$ exactly when $c(0) \in IG^2$).

CASE 3a). If $c(0) \in IG^1 - IG^2$ then we can locally parameterize c in the (6)-coordinates so that $c(t) = \big(x(t), y(t), p(t), \bar{s}(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), p_0 + t, t^n c(t)\big)$ satisfies $1 \le n$, $2 \le \ell, m$ and the differential relations of Case 2b). This implies that the choice of $1 \le n$, p_0, and $c(t)$ determines $\ell = n + 1$, m, $a(t)$, $b(t)$.

CASE 3b). If $c(0) \in IG^2 \cap IG^1$ then we can locally parameterize c in the (10)-coordinates so that $c(t) = \big(x(t), y(t), \bar{p}(t), \tilde{\bar{s}}(t)\big) = \big(x_0 + t^\ell a(t), y_0 + t^m b(t), t, t^n c(t)\big)$ satisfies

$1 \leq n$, $2 \leq m$, ℓ and differential relations of Case 2d). This implies that the choice of $1 \leq n$, $c(t)$ determines $\ell = n+2$, $m = n+1$, $a(t)$, $b(t)$. In this case c is tangent to IG^2 at 0 if and only if $2 \leq n$ and we have the following.

OBSERVATION 3. Given a smooth immersed integral curve c with $c(0) \in IG^2 \cap IG^1$ which is transverse to IG^2 at $c(0)$, and satisfies 2-rank $= 1$, 1-rank $= 0$, then there exists a unique locally defined function $f\colon U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ continuous at $c_d(0)$, C^∞ on $U - \{c_d(0)\}$ whose G^2-resolution at $c_d(0)$ is c. Furthermore, $\big(f'(x)\big)^{-3}$ has a zero of order 1 at $c_d(0)$.

EXAMPLE. Consider the integral curve c in (10)-coordinates $\big(x(t), y(t), \bar{p}(t), \tilde{\bar{s}}(t)\big) = (t^3/3, t^2/2, t, t)$ with $\ell = 3$, $m = 2$, $n = 1$, then $f(x) = (3x)^{\frac{2}{3}}/2$ has c as G^2-resolution.

Notice that in Observations 1, 2, 3 the integral curves are required to intersect submanifolds if $G^2(\mathbf{R}^4)$ whose codimension is greater than 1. Thus a smooth perturbation of such an integral curve will typically fail to intersect the required submanifold. That is to say, these isolated singularities are unstable.

4. **Second Order ODE.** A second order ODE is usually defined as a smooth relation, $0 = R(x, y, y', y'')$. This can be viewed as a 3-variety $0 = R(x, y, p, s)$ in the (5) coordinates on $G^2(\mathbf{R}^2)$. Usually, this variety is assumed to be a manifold (*i.e.*, $dR \neq 0$ on $R = 0$) and, hence, defines a smooth imbedded hypersurface $\Sigma^3 \longrightarrow \mathrm{J}^2(\mathbf{R}_d, \mathbf{R}_r)$. From this viewpoint it is natural to ask if this hypersurface can be *completed* to $IG^2 \cup IG^1$. To verify this, one pulls back the relation $R(x, y, p, s) = 0$ over the composites of the coordinate change maps (7), (9), (11). If these pullback relations can be rewritten as smooth nonsingular relations then the ODE can be completed to the points in the relevant coordinate patch $G^2(\mathbf{R}^2)$. Naturally, it is possible that the ODE completes to points of $IG^1 - IG^2$, but not to points of IG^1 or vice versa. We will use the two descriptions $R = 0$, and $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2)$ interchangeably and (with the exception of Section 5.2) assume that this 3-variety is an embedded hypersurface.

A solution to $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2)$ is defined to be an integral curve $c\colon \mathbf{R} \longrightarrow G^2(\mathbf{R}^2)$ whose image lies in the hypersurface Σ^3. Our emphasis here will be on local properties of solutions viewed as single-valued functions $f\colon \mathbf{R}_d \longrightarrow \mathbf{R}_r$ or sections $s\colon \mathbf{R}_d \longrightarrow \mathbf{R}^2$. Thus the projections have been implicitly chosen and we restrict to the group $\mathrm{Diff}(\mathbf{R}_d) \times \mathrm{Diff}(\mathbf{R}_r)$ or $\mathrm{FP} - \mathrm{Diff}(\mathbf{R}^2)$.

We will say the ODE, $R = 0$, at e^2 is locally *C-equivalent* to the ODE, $R' = 0$, at $e^{2'}$ if there exists a locally defined $\Psi \in C - \mathrm{Diff}^\infty\big(G^2(\mathbf{R}^2)\big)$ such that: $\Psi(e^2) = e^{2'}$; the pull back $\Psi^*(R')$ and R generate the same ideal in the ring of germs of smooth functions on $G^2(\mathbf{R}^2)$. If in addition Ψ can be chosen to lie in the prolongation of $\mathrm{FP} - \mathrm{Diff}^\infty(\mathbf{R}^2)$ then we will say the ODEs are locally FP-*equivalent*. A C-equivalence of an ODE with itself is called a *C-symmetry*. The set of local C-symmetries of an ODE is a germ of a smooth group action.

We will say a point e^2 in the ODE Σ^2 is π^2*-regular* if the composite $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$ has maximal rank at $e^2 \simeq 0$. Near such a π^2-regular point the tangent space to Σ^3 intersects the 2-distribution Ann^2 in a smooth line field L. Any curve $c\colon \mathbf{R} \longrightarrow \Sigma^3$

which is everywhere tangent to L is a solution. Thus we have unique, local existence of smooth immersive solutions for initial data $e^2 \in \Sigma^3$ near a π^2-regular point. We do not distinguish such solutions c if they differ by a smooth change of parameterization.

THEOREM 4. *Given $e^2 \in \Sigma^3$ a π^2-regular point of an ODE.*

i) If $e^2 \in IG^2 \cap IG^1$ then the ODE is locally FP-equivalent to

$$\begin{aligned} 0 &= (y')^4 + y''\Big[y'xa\Big(x,y,\frac{1}{y'}\Big) + y'yb\Big(x,y,\frac{1}{y'}\Big) + c\Big(x,y,\frac{1}{y'}\Big)\Big] \\ &= \tilde{\bar{s}} - xa(x,y,\bar{p}) - yb(x,y,\bar{p}) + \bar{p}c(x,y,\bar{p}), \end{aligned}$$

where a, b, c are smooth functions of $(x,y,\bar{p})$. If $c(0,0,0) \neq 0$ then there exists a unique locally defined solution $c\colon \mathbf{R} \longrightarrow \Sigma^2$ such that $c(0) = e^2$, $\pi_d(0) = 0$, and c defines a local single valued $f\colon U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ as in Observation 3.

ii) If Σ^3 contains the set $IG^2 \cap IG^1$ then it is locally FP-equivalent to

$$\begin{aligned} 0 &= (y')^4 + y''y'c(x,y,1/y'), \\ &= \tilde{\bar{s}} - \bar{p}c(x,y,\bar{p}), \end{aligned}$$

where $c(x,y,\bar{p})$ is a smooth function. In this case the solutions of part i) are locally stable under perturbation of initial conditions.

PROOF. By the π^2-regularity condition we may apply the implicit function theorem to represent Σ^3 as a smooth graph $\tilde{\bar{s}} = F(x,y,\bar{p})$ in the (10) coordinates with e^2 represented by the origin. The ODE intersects $IG^2 \subset IG^1$ at e^2 if and only if $F(0,0,0) = 0$ and we have the normal form. At this point $\mathrm{Ann}^2 = \partial\bar{p} \oplus \partial\tilde{\bar{s}}$, hence the solution c has 2-rank $= 0 =$ 1-rank at 0. Since the line field L is transverse to $IG^2 = \{\tilde{s} = 0\}$ if and only if $c(0,0,0) \neq 0$ we need only apply Observation 3. Since $IG^2 \cap IG^1$ is codimension 1 in Σ^3 and L is not tangent to $IG^2 \cap IG^1$. The stability assertion follows.

EXAMPLE. Consider the ODE

$$0 = y'' + (y')^4$$

which contain the set $IG^2 \cap IG^1$. The solutions are given by $y = -\frac{1}{2}(-3x+\alpha)^{\frac{2}{3}} + \beta$, $\alpha, \beta \in \mathbf{R}$. The G^2-resolutions of such solution intersects $IG^2 \cap IG^1$ over the point $(x,y) = (\frac{\alpha}{3}, \beta)$. Clearly the singularity in a solution persists under perturbation of initial conditions.

If e^2 fails to be π^2-regular we will assume (with the exception of Section 5.2) that this composite is a *submersion with fold* at e^2 (see the examples below and [GG]). Such ODE form a C^∞ open set in the space of smooth germs of functions R on $G^2(\mathbf{R}^2)$. In this case the set of points where the composite has rank 2 is a smoothly embedded 2-manifold $\mathrm{FL}^2 \longrightarrow \Sigma^3$ which we will call the *fold locus*. The composite $\mathrm{FL}^2 \longrightarrow \Sigma^3 \longrightarrow G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$ is a smooth embedding. This fold locus is well defined under C-equivalence. The tangent space to Σ^3 at e^2, $T_{e^2}\Sigma^3 \subset T_{e^2}G^2(\mathbf{R}^2)$ intersects $\mathrm{Ann}^2(e^2)$ in either a line $L(e^2)$ or all of $\mathrm{Ann}^2(e^2)$. In the latter case we say the ODE has a *contact singularity* at e^2. This defines a smooth line field L, on the complement of the contact singularities $\mathrm{CS} \subset \Sigma^3$. This line field and its contact singularities are well defined under C-equivalence. Thus we have the unique local existence of smooth immersive solutions $c\colon \mathbf{R} \longrightarrow \Sigma^3$, if the initial data $c(0) = e^2 \in \Sigma^2$ is not a point of contact singularity. We will see that near a point of contact singularity the line field L can be generated by a vector field with zeros exactly at points of contact singularity. For a C^∞-open set of ODE germs at e^2 the set of contact singularities is an immersed curve which is not tangent to $\mathrm{Ann}^2(e^2)$ and the linear part of L at a contact singularity has exactly two nonzero eigenvalues whose real parts are also nonzero. (Technically the eigenvalues are well defined up to a common real multiple). Such germs of ODE with contact singularities will be called *generic*. We will now provide local coordinate models for such germs.

PROPOSITION 5. *i) If $e^2 \in \mathrm{FL} \subset \Sigma^3$ is a generic contact singularity such that $T_{e^2}\Sigma^3$ does not lie in the* 3*-plane annihilated by $\mathcal{J}^2$ at e^2 then the ODE is locally C-equivalent to*

$$\begin{aligned} 0 &= y' - c(x,y) + y''b(x,y) - (y'')^2 a(x,y,y''), \\ &= p - c(x,y) + sb(x,y) - s^2 a(x,y,s), \end{aligned}$$

where a, b, c are smooth functions $a(0,0,0) \neq 0, b(0,0) = 0 = c(0,0)$, $c_x(0,0) = 0 \neq c_{xx}(0,0)$.

ii) In addition:

α) If $e^2 \in \Sigma^3 - (IG^2 \cup IG^1)$ then the ODE is locally FP*-equivalent to*

$$\begin{aligned} 0 &= y' - c(x,y) + y''b(x,y) - (y'')^2 a(x,y,y'') \\ &= p - c(x,y) + sb(x,y) - s^2 a(x,y,s), \end{aligned}$$

where a, b, c are as above.

β) If $e^2 \in \Sigma^3 \cap (IG^1 - IG^2)$ then the ODE is locally FP*-equivalent to*

$$\begin{aligned} 0 &= y''[x - c(y,y')] + b(y,y') - \frac{1}{y''} a\left(y, y', \frac{1}{y''}\right), \\ &= x - c(y,p) + \tilde{s}b(y,p) - \tilde{s}^2 a(x,y,\tilde{s}), \end{aligned}$$

where a, b, c are smooth functions of $(x,y,\tilde{s})$, with $a(0,0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$, $c_p(0,0) = 0 \neq c_{pp}(0,0)$.

γ) If $e^2 \in \Sigma^3 \cap ig^1$ then ODE is locally FP*-equivalent to*

$$\begin{aligned} 0 &= (y')^3[1 - y'c(x,y')] + y'y''b(x,y') - (y'')^2 a\left(x, y, \frac{1}{y''}\right) \\ &= \bar{p} - c(x,y) + \bar{s}b(x,y) - \bar{s}^2 a(x,y,\bar{s}), \end{aligned}$$

where a, b, c are smooth functions of $(x,y,\bar{s})$, with $a(0,0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$, $c_y(0,0) = 0 \neq c_{yy}(0,0)$.

δ) If $e^2 \in \Sigma^3 \cap (IG^2 \cap IG^1)$ then the ODE is locally FP*-equivalent to*

$$\begin{aligned} 0 &= (y'')^2\left[y - c\left(x, \frac{1}{y'}\right)\right] - y''(y')^3 b\left(x, \frac{1}{y'}\right) - (y')^6 a\left(x, \frac{1}{y}', \frac{(y')^3}{y''}\right). \\ &= y - c(x,\bar{p}) + \tilde{\bar{s}}b(x,\bar{p}) - (\tilde{\bar{s}})^2 a(x,\bar{p},\tilde{\bar{s}}), \end{aligned}$$

where a, b, c are smooth functions of $(x, \bar{p}, \bar{\bar{s}})$, *with* $a(0,0,0) \neq 0$, $b(0,0) = c(0,0)$, $c_{\bar{p}}(0,0) = 0 \neq c(0,0)$.

PROOF. Since the group $C - \mathrm{Diff}^{\infty}\big(G^2(\mathbf{R}^2)\big)$ is transitive and preserves the π^2-fiber we may assume Σ^3 is represented by $R(x,y,p,s) = 0$. Since the 3-plane annihilated by $\mathcal{J}^1$ at the origin is $\partial x \oplus \partial p \oplus \partial s$ and $R_s(0) = 0 = R_x(0)$ we must have $R_p(0) \neq 0$. Thus we may apply the implicit function theorem to locally write Σ^3 as a graph $p = F(x,y,s)$ and we have part 1. The remaining parts follow in a similar manner.

THEOREM 6. *a) Given a point* $e^2 \in \mathrm{FL} - \mathrm{CS}$ *on a smooth, regular ODE with* $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$ *a submersion with fold, let* $c\colon \mathbf{R} \longrightarrow \Sigma^3$ *be the unique local smooth immersive solution with* $c(0) = e^2$, *then the* 2*-rank and* 1*-rank of c at* 0 *are both zero.*

i) If $e^2 \in ig^1$ *then there exists a unique single valued local solution* $f\colon U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ *as in Observation 1 whose* G^2*-resolution is c.*

ii) If $e^2 \in IG^2 \cap IG^1$ *then there exists a unique single valued local solution* $f\colon U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ *as in Observation 2 whose* G^2*-resolution is c.*

b) Given a generic point of contact singularity $e^2 \in \mathrm{CS}$ *on a smooth ODE,* $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2)$, *then there exists a smooth immersive solution* $c\colon \mathbf{R} \longrightarrow \Sigma^3$, $c(0) = e^2$ *if and only if the eigenvalues of the linearization line field L at* e^2 *have zero imaginary part. Such a solution has* 2*-rank* $= 1$ *and* 1*-rank* $= 1$ *at* 0 *if and only if* $e^2 \in \Sigma^3 - IG^2$.

i) If the eigenvalues have opposite signs then there exist exactly two smooth immersive solutions at e^2 *and exactly four single valued solutions which are* C^0 *at* $c(0) = e^2$.

ii) If the eigenvalues have the same sign then there exist at least two smooth immersive solutions at e^2. *If in addition* $T_{e^2}\Sigma^3$ *does not lie in the* 3*-plane annihilated by* $\mathcal{J}^1$ *at* e^2 *we have:*

α) If $e^2 \in \Sigma^3 - (IG^2 \cup IG^2)$ *then each of the smooth immersive solutions defines local smooth single valued solutions as in Case 1a).*

β) If $e^2 \in \Sigma^3 \cap (IG^2 - IG^1)$ *then each of the smooth immersive solutions fails to locally define a single valued solution.*

γ) If $e^2 \in \Sigma^3 \cap IG^2 \cap IG^1$ *then each of the smooth immersive solutions defines local single valued solutions* $f\colon U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$, *as in Observation 3.*

c) In either case bi) or bii) there exists $e^{2'} \in \Sigma^3 - \mathrm{CS}$ *arbitrarily near* $e^2 \in \mathrm{CS}$ *such that the associated solution c with* $c(0) = e^{2'}$ *intersects the* $\mathrm{FL} - \mathrm{CS}$ *nontrivially.*

See the Appendix for illustrations of the above cases.

PROOF. For part a) we need only observe that the line field L is tangent to the π^2-filter at any point of $\mathrm{FL} - \mathrm{CS}$ and apply the observations of Section 2.

For part b i) and b ii) we apply the invariant manifold Theorem of Hirsch, Pugh and Shub [A]. If the eigenvalues have opposite signs then there exists unique smoothly immersed stable and unstable L-invariant curves which intersect transversally at e^2. These are the integral curves we seek. Because of the submersion with fold condition neither curve is tangent to the π^2-fiber at e^2. Hence the 2-rank of both integral curves is 1. The remainder of part i) follows from the observations of Section 2.

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If the eigenvalues have same sign then there exists an L-invariant smoothly immersed 2-disk $D^2 \longrightarrow \Sigma^3$ which is not tangent to both the curve of contact singularities at e^2, and the π^2-fiber at e^2. Hence the 2-rank of both integral curves is 1. The remainder of part i) follows from the observations of Section 2. We may rescale the vector field which spans the canonical line field L and assume the eigenvalues are 1 and λ. If $\lambda = 1$ or if λ is not an integer then we may apply Sternberg's Theorem [S], [A] to represent the vector field on the 2-disk by the linear part of L at e^2, *i.e.*, $u\partial u + \lambda v\partial v$. This provides the two smoothly immersed integral curves $u = 0$, $v = 0$. If $\lambda > 1$ is an integer then we may blow up the line field $L \cong a(u,v)\partial u + b(u,v)\partial u$ at $e^2 \cong (u,v) = (0,0)$ to construct the desired integral curves. (That is to say, use u, v, q as type (2)-coordinates on $G^1(D^2)$ and $(u,v,\bar{q})$ as type (3) coordinates on $G^1(D^2)$ then the condition for an integral curve $c\colon \mathbf{R} \longrightarrow D^2$ can be rewritten as a line field on $G^1(D^2)$ which annihilates the differential systems: $a(u,v)q - b(u,v)$, $dv - qdu$ in the (2)-coordinates; $a(u,v) - \bar{q}b(u,v)$, $du - \bar{q}dv$ in the (3)-coordinates. There is a L-invariant Möbius strip in $G^1(\mathbf{D}^2)$ which contains two zero points: $(u,v,q) = (0,0,0)$; $(u,v,\bar{q}) = (0,0,0)$. These zero points correspond to the two eigenvectors of the linear part of L at $(0,0) = e^2$. The eigenvalues of the linear part of the blow up line field on the Möbius strip are: 1 and $\lambda - 1$; λ and $1 - \lambda$. So if $\lambda = 2$ then by Sternberg's Theorem there exists an uncountable number of local smooth immersive integral curves (for the blow up line field) which contain one of the zero points on the Möbius strip. Composing with the projection $\pi^1\colon G^1(\mathbf{D}^2) \longrightarrow D^2$ yields the desired immersive solutions. If the integer λ is greater than 2 then we iterate the blow up construction until we can apply Sternberg's Theorem.

For part c) we need only recall that the eigenvectors with nonzero eigenvalue cannot be tangent to FL at $e^2 \in$ CS. Choose $e^{2'} \in \mathrm{FL} - DS$ near e^2 and apply part a).

REMARK. Illustrations of Theorem 6 can be found in the Appendix. It should be clear from the proof of part ii) that if a ratio of eigenvalues is rational $\neq 1$ then there exists an uncountable number of C^1-immersive solutions at e^2 which have the same order of differentiability as the curves $\alpha u^p = \beta u^q$ in the (u,v)-plane. If this ratio is one then there are an uncountable number of immersed solutions. A local analysis as in Section 2 would determine when these C^r-immersive solutions determine single valued solutions. Such functions would be smooth on the complement of a point where they would be C^0 not C^1 and $\big(f'(x)\big)^{-m}$ would have a zero with a finite order of differentiability. Also notice that in part i) one could construct C^0-solutions by concatenating pieces of the two smooth immersive solutions.

5. Special second order ODE.

Here we will discuss several classes of ODE which do not satisfy all of the regularity conditions of Theorem 5 yet respond to a similar analysis.

5.1 *Special fold loci.*

a) Second order ODE with singular solutions. Another way to identify a point of contact singularity in $\mathrm{FL} \subset \Sigma^3$ is by way of the following condition: the canonical ideals $\mathcal{K}$

https://doi.org/10.4153/CJM-1996-043-9 Published online by Cambridge University Press

and $\mathcal{J}^1$ when pulled back to FL become linearly dependent at a point of contact singularity. We define a *second order ODE with singular solutions* to be an ODE which satisfies the submersion with fold condition of Section 3 and the condition: $\mathcal{K}$ and $\mathcal{J}^1$ are linearly dependent when pulled back to FL but $\mathcal{J}^1$ is nonzero. This implies that every point of the fold locus is also a contact singularity and that the fold locus is foliated by smooth immersive integral curves. Since these integral curves lie in Σ^3 we will refer to them as *singular solutions* (compare [C]).

THEOREM 7. *i) A second order ODE with singular solutions is locally C-Equivalent to*

$$\begin{aligned} 0 &= y' - (y'')^2 a(x,y,y'') \\ &= p - s^2 a(x,y,s), \end{aligned}$$

where $a(x,y,s)$ is a smooth function in the (5)-coordinates with $a(0,0,0) \neq 0$. Furthermore, for all $e^2 \in$ FL there exist exactly two smooth immersed solutions with $c(0) = e^2$. The tangent vectors to these two solutions are not colinear at e^2.

ii) If an ODE with singular solutions satisfies $\mathrm{FL} \subset G^2(\mathbf{R}^2) - (IG^2 \cup IG^1)$*, and* FL *is not tangent to the π^1-fiber at e^2 then Σ^3 is locally* FP*-equivalent to*

$$\begin{aligned} 0 &= y' - (y'')^2 a(x,y,y'') \\ &= p - s^2 a(x,y,s) \end{aligned}$$

where $a(x,y,s)$ is as in part i). The two smooth immersive solutions of part i) define smooth single valued solutions $f(x)$, $g(x)$ as in Case 1a). Their difference $f(x) - g(x)$ vanishes to exactly second order at $\pi_d \circ \pi^1 \circ \pi^2(e^2)$.

iii) If an ODE with singular solutions satisfies $\mathrm{FL} \subset ig^1$ *then Σ^3 is locally FP-equivalent to*

$$\begin{aligned} 0 &= (y')^5 - (y'')^2 a\Big(x,y,\frac{-y''}{(y')^3}\Big) \\ &= \bar{p} - \bar{s}^2 a(x,y,\bar{s}) \end{aligned}$$

where $a(x,\bar{p},\bar{s})$ is a smooth function in the (8)-coordinates with $a(0,0,0) \neq 0$. One of the two smooth immersive solutions of part i) is the π_d-fiber through $c_d(0)$. The other defines a local single valued solution $f\colon U \subset \mathbf{R}_d \to \mathbf{R}_r$ which fails to be C^∞ exactly at $\pi_d \circ \pi^1 \circ \pi^2(e^2) c_d(0)$ where $\big(f'(x)\big)^{-3}$ has a zero of order 2. The C^0-solutions fail to define local single valued solutions.

PROOF. We may assume that e^2 is the origin in the (5)-coordinates. View the image $\pi^2(\mathrm{FL}) \subset G^1(\mathbf{R}^2)$ as a first order ODE. Since $\mathcal{J}^1$ does not vanish, this ODE does not have contact singularities near $\pi^2(e^2)$. It follows from standard Hamilton-Jacobi theory that this ODE is locally $C - \mathrm{Diff}^\infty(G^1)$-equivalent to $p = 0$. If we prolong this diffeomorphism to $G^2(\mathbf{R}^2)$ then our second order ODE must have FL represented by $p = 0$, $s = 0$ at the origin. The normal form of part i) follows directly. For the existence part we differentiate this normal form to write $0 = y''\big(1 - 2y'''a + y''(a_x + a_y y' + a_s y''')\big)$ which is the same as $y'' = 0$ or $y''' = 1 - y''(a_x + y'a_y)/(2a + y''a_s)$. Hence for initial data

$x = 0 = y, y' = 0 = y''$ we have two C^∞-solutions $f(x) = 0$ and $g(x) = x^3\bar{g}(x)$ where $\bar{g}(0) = 1$.

For part ii) we need only observe that since the π^1-fiber is not tangent to $\pi^2(\mathrm{FL})$ the solutions to the first order ODE can be described as the image of the curves, $y =$ constant, under the map $(x,y) \longrightarrow \big(x, h(x,y)\big)$, $h_y(0,0) \neq 0$. Hence the C-equivalence of part i) is given by the prolongation of this π_d-fiber preserving map. For part iii) we may assume that e^2 is represented by the origin in the (8)-coordinates. Since the fold locus is now represented by $\bar{s} = 0 = \bar{p}$ the normal form follows. Observe that the prolongation of the diffeomorphism $(x,y) \longrightarrow (y,x)$ yields C-equivalences between the models in ii) and iii). Thus the solutions $y = 0$ and $y = x^3\bar{g}(x)$ for part ii) correspond to $x = 0$ and $x = y^3\bar{g}(y)$ for part iii). We are finished.

EXAMPLE. Consider the ODE

$$0 = xy' - (y'')^2 - y, \\ = xp - s^2 - y. \tag{20}$$

The fold locus is given by the relations $s = 0 = xp - y$ in the (5)-coordinates. This fold locus is a Möbius strip whose generating circle π^2-projects to a generating circle of $G^1(\mathbf{R}^2) \cong \mathbf{R}^2 \times S^1$. So if we locally parameterize FL by $(u,v) \longrightarrow (u, uv, v, 0) = (x, y, p, s)$ then the pull back of $\mathcal{K}$ and $\mathcal{J}^1$ are $dp - sdx = dv$ and $dy - pdx = udv$. So the two ideals $\mathcal{K}$ and $\mathcal{J}^1$ are dependent when pulled-back to FL, and the pull-back of $\mathcal{J}^1$ is nonzero when $u \neq 0$. It follows that the $\pi^1 \circ \pi^2$-projection of the singular solutions consists of all lines through the origin of finite slope. (The locus $u = 0$ corresponds to a "secondary contact singularity" in $\pi^2(\mathrm{FL})$ viewed as a first order ODE for the singular solutions). The other family of solutions which intersect FL transversaly in Σ^3 are the simultaneous solutions of $y''' = x/2$ and (20). The ODE (20) admits a completion to $IG^1 - IG^2$ given by

$$0 = \bar{p}^5 x - \bar{s}^2 - \bar{p}^6 y, \tag{21}$$

in (8)-coordinates. Here the fold locus is represented by the relations $\bar{s} = 0$ and $x - \bar{p}y = 0$. This fold locus intersects ig^1 on the curve $s = 0 = p, x = 0$ which $\pi^1 \circ \pi^2$-projects to the vertical line through the origin. The other family of solutions which intersects $\mathrm{FL} \cap ig^1 = \{x = 0\}$ is given by $y = (12x)^{\frac{1}{3}}$ and its vertical translates. The completion Σ^3 of the ODE (20) satisfies $\Sigma^3 \cap (IG^2 - IG^1) = \emptyset$, and $\Sigma^3 \cap (IG^2 \cap IG^1) = \{x = 0, \tilde{\bar{s}} = 0 = \bar{p}\}$. However the 3-variety fails to be regular on this latter intersection.

EXAMPLE. Consider the ODE

$$0 = y - (y'')^2 - (y')^2 - y'x \\ = y - s^2 - p^2 - px, \tag{22}$$

whose fold locus is given by the relations $s = 0 = y - p^2 - px$ in the (5)-coordinates. This fold locus, viewed as a first order ODE, is the Clariut equation. Its solutions consist of all lines tangent to the parabola $y = x^2/4$. This parabola corresponds to another type of "secondary contact singularity" in $\pi^2(\mathrm{FL})$. Thus, if we relax the conditions that IG^1 not vanish, when pulled back to FL, then we have a diverse collection of *C-inequivalent* ODE with secondary singular solutions.

It should be clear to the reader that if an ODE admits a two dimensional group of C-symmetries whose isotropy subgroup is everywhere trivial then the ODE cannot have generic contact singularities as in Section 3. However, if a second order ODE satisfies: the submersion with fold condition; FL is not tangent to the π^2-fiber; the ODE admits a C-symmetry group as above; and there exists a smooth single valued solution $f: U \subset \mathbf{R}_d \longrightarrow \mathbf{R}_r$ whose 2-jet $j^2f: U \longrightarrow J^2(\mathbf{R}_d, \mathbf{R}_r)$ intersects the fold locus transversally in Σ^3, then the ODE has singular solutions as in Theorem 6 i). The proof follows from the observation that $R_x(0,0,0,0) = 0$ in the (5)-coordinates patch. Hence every point of FL is a point of contact singularity. Thus $\mathrm{Ann}^2 \subset T\Sigma^3$ at every point of FL. The condition on the 2-jet intersecting FL implies that $\mathrm{Ann}^2 \cap TFL$ is one dimensional. Now by homogeneity, the ideal $\mathcal{J}^1$ pulled back to FL either vanishes nowhere or everywhere on FL. The latter is impossible since FL is not tangent to the π^2-fiber.

b) $IG^2 \cap IG^1$ contain the fold locus. As observed in Section 3 the fold locus of an ODE typically intersects $IG^2 \cap IG^1$ at an isolated point. At the other extreme we have the following.

THEOREM 8. *Given a smooth ODE with $\Sigma^3 \longrightarrow G^2(\mathbf{R}^2) \longrightarrow G^1(\mathbf{R}^2)$ a submersion with fold, if* FL $\subset IG^2 \cap IG^1$ *then the ODE is locally* FP*-equivalent to*

$$0 = (y'')^2 - (y')^7 a\Big(x, y, \frac{y''}{(y')^3}\Big)$$
$$= \bar{p} - \bar{\bar{s}}^2 a(x, y, \bar{\bar{s}}),$$

where $a(x, y, \bar{\bar{s}})$ is a smooth function in the (10)-coordinates with $a(0,0,0) \neq 0$. Furthermore, every smooth immersive solution $c: \mathbf{R} \longrightarrow \Sigma^3$ with $c(0) \in$ FL locally defines a single valued function as in Observation 2. The singularity in such a solution is stable under perturbation of initial conditions.

PROOF. The proof is similar to that of Theorem 6 i) and details are left to the reader.

EXAMPLE. Consider the ODE

$$(y'')^2 = (y')^7. \tag{23}$$

The solutions are given by $y = \frac{2}{3}(-\frac{5}{2}x + \alpha)^{\frac{3}{5}} + \beta$, α, $\beta \in \mathbf{R}$ and the singular solutions are the π_d-fibers. The G^2-resolution of the former intersects the latter when $x = 2\alpha/5$. Clearly the singularities in these solutions persist under perturbation in initial conditions.

5.2 *ODE with Morse singularities.* Consider an ODE such that $R = 0$ and $dR = 0$ at e^2. If we assume that the hessian of R at e^2, Hess(R) is nondegenerate and indefinite then the existence problem for smooth immersive solutions $c: \mathbf{R} \longrightarrow \{R = 0\}$ with $c(0) = e^2$ becomes tractable. We may assume that e^2 is the origin in the (5)-coordinates. Notice that the vector field L is well defined on J^2 with an *isolated* zero at the Morse singularity. However, the linear part of L is highly degenerate at this point.

First observe that the existence of such a solution requires that the quadratic cone $\mathrm{Hess}(R) = 0$ in $T_{e^2}G^2(\mathbf{R}^2)$ intersect the 2-plane $\mathrm{Ann}^2(e^2)$ non-trivially. We will assume that $\mathrm{Hess}(R)$ restricted to this plane is nondegenerate and indefinite. By the Morse Lemma there exists local coordinates (u, v, r, s) at e^2 such that: $\mathrm{Ann}^2(e^2)$ is represented by $\partial r \oplus \partial s$; $R = 0$ is represented by $u^2 \pm v^2 - rs = 0$; $y = \alpha u + \beta v \,\mathrm{MOD}\, \mathcal{M}^2(u, v, r, s)$, $p = \gamma u + \delta v \,\mathrm{MOD}\, \mathcal{M}^2(u, v, r, s)$, $\alpha\delta - \beta\gamma \neq 0$. Here, $\mathrm{MOD}\, \mathcal{M}^2(u, v, r, s)$ means we ignore terms of order ≥ 2 in (u, v, r, s). We may locally parameterize the Morse Cone $R = 0$ as $(\lambda, a, b) \longrightarrow \lambda(a, b, a^2 \pm b^2, 1) = (u, v, r, s)$. Now pull back the 1-forms $dy - pdx$, $dp - sdx$ over this parametrization and observe the following. Since $\alpha\delta - \beta\gamma \neq 0$ these 1-forms annihilate a unique smooth vector field (defined on a neighborhood of the origin) of the form $-\lambda\partial\lambda + A(\lambda, a, b)\partial a + B(\lambda, a, b)\partial b$ with,

$$\begin{pmatrix} y_a & y_b \\ p_a & p_b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \equiv \begin{pmatrix} y \\ p \end{pmatrix} \mathrm{MOD}\, \mathcal{M}^2(\lambda, a, b). \tag{24}$$

Hence, $A \equiv a \,\mathrm{MOD}\, \mathcal{M}^2(\lambda, a, b)$ and $B = b \,\mathrm{MOD}\, \mathcal{M}^2(\lambda, a, b)$. This vector field spans the canonical line field L. Moreover, its linearization at the origin has full rank and a unique eigenvector, with eigenvalue -1, which is transverse to the $\partial a \oplus \partial b$ plane. If we push forward the corresponding smooth integral curve then we have the following.

THEOREM 9. *Given a smooth ODE with $e^2 \in \{R = 0\}$ a Morse critical point for $R: G^2(\mathbf{R}^2) \longrightarrow \mathbf{R}$, such that the Hessian of R at e^2 is nondegenerate and indefinite then we have the following:*

If the subspace $\mathrm{Ann}^2(e^2)$ is nondegenerate and indefinite with respect to this hessian then there exist exactly two smooth immersed solutions c with $c(0) = e^2$ whose tangent vectors at $c(0)$ are the two null directions in $\mathrm{Ann}^2(e^2)$.

α) If $e^2 \in G^2(\mathbf{R}^2) - (IG^2 \cup IG^1)$ and a hessian-null vector is not tangent to the π^2-fiber at e^2 then the corresponding solution defines a local single valued solution as in Case 1a).

β) If $e^2 \in ig^1$ and a hessian-null vector is tangent to the π^2-fiber at e^2 then the corresponding solution defines a local single valued solution as in Observation 1.

γ) If $e^2 \in IG^2 \cap IG^1$ and a hessian-null vector is tangent to the π^2-fiber at e^2 then the corresponding solution defines a local single valued solution as in Observation 2.

δ) If $e^2 \in IG^2 \cap IG^1$ and a hessian-null vector is not tangent to the π^2-fiber or IG^2 at e^2 then the corresponding solution defines a local single valued solution as in Observation 3.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= x^2 - (y'')^2 + Ay^2 + Byy' + C(y')^2 \\ &= x^2 - s^2 + Ay^2 + Byp + Cp^2, \end{aligned} \tag{25}$$

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where A, B, C are smooth functions in the (5)-coordinates with $4AC - B^2 \neq 0$ at the origin. The origin is a Morse critical point with two hessian-null vectors $\partial x \pm \partial s$. It follows that there exist two local single valued solutions as in Case 1.

If we replace the leading term in (25) by xs then one of the smooth immersive solutions has 2-rank $= 1 =$ 1-rank at the origin. Hence we may apply part α) above. The other has 2-rank $= 0$. This latter example includes a class of quasi-linear ODE.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= -yy'' + Ax^2(y')^3 + Bx(y')^2 + Cy' \\ &= y\bar{s} + Ax^2 + B\bar{p}x + C\bar{p}^2, \end{aligned} \tag{26}$$

where A, B, C are smooth functions in the (8)-coordinates with $4AC - B^2 \neq 0$ at the origin. The origin is a Morse critical point in ig^1 with hessian-null vectors ∂y, and $\partial \bar{s}$. It follows that one of the smooth immersive solutions has 2-rank $= 1$, 1-rank $= 0$ and the other defines a local single valued solution as in Observation 1. This example includes a class of quasi-linear ODE.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= -(y')^2 + y''(Ax^2 + Bxy + Cy^2) \\ &= \tilde{\bar{s}}\bar{p} + Ax^2 + Bxy + Cy^2, \end{aligned} \tag{27}$$

where A, B, C are smooth functions in the (10)-coordinates with $4AC - B^2 \neq 0$. The origin is a Morse critical point such that one of the smooth immersive solutions defines a local single valued solution as in Observation 2. This example includes a class of quasi-linear ODE.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= (y'')^2 A - (y'')(y')^4 B + (y')^8 C + xy(y')^2(y'')^2 \\ &= A\bar{p}^2 + B\bar{p}\tilde{\bar{s}} + C\tilde{\bar{s}}^2 + xy, \end{aligned} \tag{28}$$

where A, B, C are smooth functions in the (10)-coordinates with $4AC - B^2 \neq 0$ and A, $C \neq 0$ at the origin. Then both of the smooth immersive solutions intersect $IG^2 \cap IG^1$ and are not tangent to either IG^2 or IG^1. Hence 2-rank $= 1$ and 1-rank $= 0$. It follows that these solutions define local single valued solutions as in Observation 2.

5.3 *Base completions.* We note in closing that as in the classical complex analytic case one can implement a "base completion" of $\mathbf{R}_r$ to $\mathbf{RP}^1 \simeq S^1$ in order to study "poles" in solutions. Without modification: the manifolds with projections $G^2(\mathbf{R} \times S^1) \longrightarrow G^1(\mathbf{R} \times S^1) \longrightarrow \mathbf{R} \times S^1$ and the subsets IG^1, IG^2, ig^1 can be defined as in Section 1. The map $(x,y) \longrightarrow (x, 1/y)$ prolongs to the (5)-coordinates as $(x,y,p,s) \longrightarrow \big(x, 1/y, -p/y^2, (2p^2 - ys)/y^3\big)$. So that given an ODE, $R = 0$, in the (5)-coordinates, if the pull back of the ODE over this map defines a smooth relation at $y = 0$, then the above results can be adapted to construct local single valued solutions $f(x)$ which become unbounded at an isolated point. Moreover the function $1/f(x)$ will have a C^∞, C^1 or C^0, G^2-resolution as in Observations 1, 2, 3.

EXAMPLE. Consider the ODE

$$0 = y^3 - 4y^4(y')^4 - 4y^5(y')^2y'' + y^6(y'')^2 - (y')^2 + y^2y'x$$
$$(29) \qquad = y^3 - 4y^4p^4 - 4p^2y^5s + y^6s^2 - p^2 + xpy^2,$$

which pulls back over the above prolongation to the ODE with singular solutions (22). It follows that all the singular solutions have poles of order 1 with $1/y$ smooth, and there exists a 1-parameter family of solutions for which an isolated order 3 pole bifurcates into a pair of order 1 poles.

6. **Global remarks.** Observe that both IG^1 and IG^2 are embedded hypersurfaces in $G^2(\mathbf{R}^2)$. If follows that given a smoothly immersed integral curve $c: \mathbf{R} \longrightarrow G^2(\mathbf{R}^2)$ there exist two oriented intersection numbers, $\cap\#(c, IG^1)$ and $\cap\#(c, IG^2)$. They can be identified with the degree of the Gauss map and Maslov index (respectively) of the curve $\pi^1 \circ \pi^2 \circ c$ in $\mathbf{R}^2$ they are both invariants for c up to a homotopy through integral curves. Naturally if an integral curve is a solution to an ODE then these invariants are restricted. For example if the ODE lies entirely in the (5)-coordinates (*i.e.*, its fiber completion has empty intersection with IG^1 or IG^2) then the corresponding homotopy invariant is trivial. For example $p = F(x, y, s)$ or $s = G(x, y, p)$ where F and G are smooth. In general these restrictions are difficult to deduce.

7. **Appendix.** The following local models illustrate the various cases in Theorem 6.

CASE α). Upon prolongation to $G^2(R^2)$ the π_d-fiber preserving diffeomorphisms of R^2 given by

$$(30) \qquad (x, y) \longrightarrow (x + \alpha, y + \beta + \gamma x + \delta x^2),$$

α, β, γ, $\delta \in \mathbf{R}$, are transitive on $J^2(\mathbf{R}_d, \mathbf{R}_r)$, *i.e.*, transitive on the (5)-coordinate patch. Thus, if $e^2 \in \Sigma^2 \longrightarrow G^2(\mathbf{R}^2)$ lies in this coordinate patch then Σ^3 at e^2 is locally FP-equivalent to $R(x, y, p, s) = 0$, at the origin. The origin is π^2-regular if and only if $R(0) = 0 \neq R_s(0)$. (Subscripts denote partial derivatives). The origin lies in the fold locus if and only if $R(0) = 0 = R_s(0)$, $R_{ss}(0) \neq 0$. In this case the fold locus is locally given by the relations $R = 0 = R_s$.

The line field L must satisfy the following differential relations on $G^2(\mathbf{R}^2)$,

$$(31) \qquad \begin{aligned} dR &= 0, \\ dy - p\,dx &= 0, \\ dp - s\,dy &= 0, \end{aligned}$$

and hence is spanned by

$$(32) \qquad R_s\partial x + pR_s\partial y + sR_s\partial p - (R_x + pR_y + sR_p)\partial s.$$

https://doi.org/10.4153/CJM-1996-043-9 Published online by Cambridge University Press

The origin is a contact singularity if and only if $R(0) = 0 = R_s(0)$, and $R_x(0) = 0$. In this case the linearization of L at the origin is given by

$$\begin{pmatrix} R_{sx} & * & * & R_{ss} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -R_{xx} & * & * & -R_{xs} - R_p \end{pmatrix} \tag{33}$$

The characteristic polynomial is given by

$$\lambda^2(\lambda^2 + \lambda R_p + R_{xx}R_{ss} - R_{xs}^2 - R_pR_{xs}). \tag{34}$$

Notice that $\mathrm{Ann}^2(0) = \partial x \oplus \partial s$ is invariant under this linearization and must contain all of the eigenvectors with nonzero eigenvalue. It follows that if the following C^∞-open conditions hold at the origin

$$\begin{aligned} R_{xx}R_{ss} - R_{xs}^2 - R_pR_{xs} &\neq 0, \\ dR \wedge dR_s \wedge d(R_x + pR_y + sR_p) &\neq 0, \\ R_{ss} &\neq 0, \end{aligned} \tag{35}$$

then the curve of contact singularities is locally embedded and transverse to this 2-plane of eigenvectors. Observe that such an eigenvector cannot be tangent to FL.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= y' - c(x,y) + y''b(x,y) - (y'')^2a(x,y)/2 \\ &= p - c(x,y) + sb(x,y) - s^2a(x,y)/2 \end{aligned} \tag{36}$$

where a, b, c are smooth functions, $a(0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$. The fold locus is given by

$$\begin{aligned} s &= b/a \\ p &= (2ac - b^2)/2a, \end{aligned} \tag{37}$$

and the contact singularities satisfy the additional relation

$$0 = c_x + pc_y - s(b_x + pb_y + 1) + s^2(a_x + pa_y)/2. \tag{38}$$

So if $c_x(0) = 0$ but $c_{xx}(0) \neq 0$, the curve of contact singularities is transverse to $\mathrm{Ann}^2(0)$.

CASE β). Upon prolongation, the maps (30) are transitive on $IG^1 - IG^2$. Hence if $e^2 \in \Sigma^3 \cap I^1 \cap I^2$ then it is locally FP-equivalent to $R(x,y,p,\tilde{s}) = 0$ at the origin. Now the origin is π^2-regular if and only if $R(0) = 0 \neq R_{\tilde{s}}(0)$. The origin lies in the fold locus if and only if $R(0) = 0 = R_{\tilde{s}}(0)$, $R_{\tilde{s}\tilde{s}}(0) \neq 0$. The line field L is spanned by

$$\tilde{s}R_{\tilde{s}}\partial x + p\tilde{s}R_{\tilde{s}}\partial y + R_{\tilde{s}}\partial p - (\tilde{s}R_x + p\tilde{s}R_y + R_p)\partial\tilde{s}. \tag{39}$$

The origin is a contact singularity if and only if $R(0) = 0 = R_{\tilde{s}}(0)$, and $R_p(0) = 0$. In this case the linearization of L at the origin is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & R_{\tilde{s}p} & R_{\tilde{s}\tilde{s}} \\ * & * & -R_{pp} & -R_{p\tilde{s}} - R_x \end{pmatrix}. \tag{40}$$

The characteristic polynomial is given by

$$\lambda^2(\lambda^2 + \lambda R_x + R_{pp}R_{\tilde{s}\tilde{s}} - R_{\tilde{s}p}^2 - R_x R_{p\tilde{s}}), \tag{41}$$

and the nonzero eigenvectors must lie in $\mathrm{Ann}^2(0) = \partial p \oplus \partial\tilde{s}$. For a C^∞-open set of ODE germs the intersection $\mathrm{CS} \cap IG^2$ will be a generic point of contact singularity and $\mathrm{FL} \cap IG^2$ will be an immersed curve.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= (y'')^2[x - c(y,y')] + y''b(y,y') - a(y,y')/2 \\ &= s^2[x - c(y,p)] + sb(y,p) - a(y,p)/2, \end{aligned} \tag{42}$$

where a, b, c are smooth functions, $a, (0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$. The pull back of this ODE over the map (7) can be rewritten

$$0 = x - c(y,p) + \tilde{s}b(y,p) - \tilde{s}^2 a(y,p). \tag{43}$$

The fold locus is given by

$$\begin{aligned} \tilde{s} &= b/a, \\ x &= (2ac - b^2)/2a, \end{aligned} \tag{44}$$

If $c_p(0,0) = 0$ then $\mathrm{CS} \cap IG^2 \neq \emptyset$ and if $c_{pp}(0,0) \neq 0$ then the curve of contact singularities is transverse to $\mathrm{Ann}^2(0)$. Also if b is the zero function then $\mathrm{FL} \subset IG^2$.

CASE γ). Upon prolongation the maps (30) are transitive on $ig^1 \subset IG^1 - IG^2$. Hence if $e^2 \in \Sigma^3 \cap ig^1$ then it is locally FP-equivalent to $R(x,y,\bar{p},\bar{s}) = 0$ at the origin. The origin is π^2-regular if and only if $R(0) = 0 \neq R_{\bar{s}}(0)$ and it lies in the fold locus if and only if $R(0) = 0 = R_{\bar{s}}(0)$, $R_{\bar{s}\bar{s}}(0) \neq 0$. The line field L is spanned by

$$\bar{p}R_{\bar{s}}\partial x + R_{\bar{s}}\partial y + \bar{s}R_{\bar{s}}\partial\bar{p} - (\bar{p}R_x + R_y + \bar{s}R_{\bar{p}})\partial\bar{s}. \tag{45}$$

The origin is a contact singularity if and only if $R(0) = 0 = R_s(0)$ and $R_y(0) = 0$. In this case the linearization of L at the origin is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & R_{\bar{s}y} & * & R_{\bar{s}p} \\ 0 & 0 & 0 & 0 \\ * & -R_{yy} & * & -R_{y\bar{s}} - R_{\bar{p}} \end{pmatrix}. \tag{46}$$

The characteristic polynomial is given by

$$\lambda^2(\lambda^2 + \lambda R_{\bar{p}} + R_{yy}R_{\bar{s}\bar{p}} - R^2_{y\bar{s}} - R_{\bar{p}}R_{y\bar{s}}), \tag{47}$$

and all nonzero eigenvectors must lie in $\mathrm{Ann}^2(0) = \partial y \oplus \partial\bar{s}$. For a C^∞-open set of ODE germs the intersection $\mathrm{CS} \cap IG^1$ will be a stable point of contact singularity $\mathrm{FL} \cap IG^1$ will be an immersed curve and $\mathrm{FL} \cap ig^1$ will be an isolated point in $\mathrm{FL} - \mathrm{CS}$.

EXAMPLE. Consider the ODE

$$\begin{aligned} 0 &= (y')^5 - (y')^6 c(x,y) - (y')^3 y'' b(x,y) - (y'')^2 a(x,y)/2 \\ &= p^5 - p^6 c(x,y) - sp^3 b(x,y) - s^2 a(x,y)/2 \end{aligned} \tag{48}$$

where a, b, c are smooth functions $a(0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$. The pull back of this ODE over the map (9) can be written

$$0 = \bar{p} - c(x,y) + \bar{s}b(x,y) + \bar{s}^2 a(x,y)/2. \tag{49}$$

the fold locus is given by

$$\begin{aligned} \bar{s} &= b/a \\ \bar{p} &= (2ac - b^2)/2a, \end{aligned} \tag{50}$$

and the contact singularities satisfy the additional relation

$$0 = c_y + \bar{p}c_x - \bar{s}(b_y + \bar{p}b_x + 1) + s^2(a_y + \bar{p}a_x)/2. \tag{51}$$

So if $c_y(0,0) = 0$ but $c_{yy}(0,0) \neq 0$ the curve of contact singularities is transverse to $\mathrm{Ann}^2(0)$. Also if b is the zero function then $\mathrm{FL} \subset IG^1$. If both b and c are the zero functions then $\mathrm{FL} \subset ig^1$.

CASE δ). Finally, upon prolongation the maps (12) are transitive on $IG^2 \cap IG^1$. Hence if $e^2 \in \Sigma^3 \cap IG^2 \cap IG^1$ then it is locally FP-equivalent to $R(x,y,\bar{p},\tilde{\bar{s}}) = 0$ at the origin. The origin is π^2-regular if and only if $R(0) = 0 \neq R_{\tilde{\bar{s}}}(0)$ and it lies in the fold locus if and only if $R(0) = 0 = R_{\tilde{\bar{s}}}(0)$, $R_{\tilde{\bar{s}}\tilde{\bar{s}}}(0) \neq 0$. The line field L is spanned by

$$\bar{p}\tilde{\bar{s}}R_{\tilde{\bar{s}}}\partial x + \tilde{\bar{s}}R_{\tilde{\bar{s}}}\partial y + R_{\tilde{\bar{s}}}\partial\bar{p} - (\bar{p}\tilde{\bar{s}}R_x + \tilde{\bar{s}}R_y + R_{\bar{p}})\partial\tilde{\bar{s}}. \tag{52}$$

The origin is a contact singularity if and only if $R(0) = 0 = R_{\tilde{\bar{s}}}(0)$ and $R_y(0) = 0$. In this case the linearization of L at the origin is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & R_{\tilde{\bar{s}}\bar{p}} & R_{\tilde{\bar{s}}\tilde{\bar{s}}} \\ * & * & -R_{\bar{p}\bar{p}} & -R_{\bar{p}\tilde{\bar{s}}} - R_y \end{pmatrix}. \tag{53}$$

The characteristic polynomial is given by

$$\lambda^2(\lambda^2 + \lambda R_y + R_{\bar{p}\bar{p}}R_{\tilde{\bar{s}}\tilde{\bar{s}}} - R^2_{\bar{p}\tilde{\bar{s}}} - R_y R_{\bar{p}\tilde{\bar{s}}}), \tag{54}$$

and all nonzero eigenvectors must lie in $\mathrm{Ann}^2(0) = \partial\bar{p} \oplus \partial\tilde{\bar{s}}$. For a C^∞-open set of ODE germs the intersection $\mathrm{CS} \cap IG^2$ will be a stable point of contact singularity and $\mathrm{FL} \cap IG^2 \cap IG^1$ will be an isolated point in $\mathrm{FL} - \mathrm{CS}$.

EXAMPLE. Consider the ODE

$$
\begin{aligned}
0 &= (y'')^2\Big[y - c\Big(x, \frac{1}{y'}\Big)\Big] - y''(y')^3 b\Big(x, \frac{1}{y'}\Big) - (y'')^2 a\Big(x, \frac{1}{y'}\Big)/2 \\
&= s^2\Big[y - c\Big(x, \frac{1}{p}\Big)\Big] - sp^3 b\Big(x, \frac{1}{p}\Big) - s^2 a\Big(x, \frac{1}{p}\Big)/2,
\end{aligned} \tag{55}
$$

where a, b, c are smooth functions. The pull back of this ODE over the composite of (9) and (11) can be rewritten

$$
0 = y - c(x,\bar{p}) + \tilde{\bar{s}}b(x,\bar{p}) - (\tilde{\bar{s}})^2 a(x,\bar{p})/2. \tag{56}
$$

So that the a, b, c as functions of $(x,\bar{p})$ are smooth at $\bar{p} = 0$. If these functions satisfy $a(0,0) \neq 0$, $b(0,0) = 0 = c(0,0)$. Then the ODE intersects $IG^2 \cap IG^1$ nontrivally. The fold locus is given by

$$
\begin{aligned}
\tilde{\bar{s}} &= b/a, \\
\bar{p} &= (2ac - b^2)/2a.
\end{aligned} \tag{57}
$$

The contact singularities satisfy the additional relation

$$
0 = -c_{\bar{p}} + \tilde{\bar{s}}(b_{\bar{p}} - c_y - \bar{p}c_x) + \tilde{\bar{s}}^2(b_y - a_{\bar{p}}/2 + \bar{p}b_x) - \tilde{\bar{s}}^3(\bar{p}a_x + a_y)/2. \tag{58}
$$

So if $c_{\bar{p}}(0,0) = 0$ but $c_{\bar{p}\bar{p}}(0,0) \neq 0$ the curve of contact singularities is transverse to $\mathrm{Ann}^2(0)$. Also if b is the zero function then $\mathrm{FL} \subset IG^2$. If both b and c are the zero function then FL lie in $IG^2 \cap IG^1$.

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https://doi.org/10.4153/CJM-1996-043-9 Published online by Cambridge University Press