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Semi-Classical Wavefront Set and Fourier Integral Operators

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Abstract. Here we define and prove some properties of the semi-classical wavefront set. We also define and study semi-classical Fourier integral operators and prove a generalization of Egorov's theorem to manifolds of different dimensions.

1 Introduction

In this article we define the semi-classical wavefront set and Fourier integral operators and establish some of their properties.

Robert [9], defined semi-classical Fourier integral operators through oscillatory integrals. He has proven a composition formula for a general class of semi-classical Fourier integral operators, while for the unitary group $U(t) = e^{-\frac{i}{h}tA(h)}$ of certain semi-classical pseudodifferential operators $A(h)$, he has established an oscillatory integral representation and has investigated the propagation of the frequency set of a distribution. Composition rules for semi-classical Fourier integral operators have been proven in [3].

Another good reference for the theory of classical Fourier integral operators is the book by Duistermaat [4], in which Fourier integral operators and some of their properties and applications are again studied through their oscillatory integral representations. Here we take a different approach and make a coordinate-invariant definition the starting point of our investigation of semi-classical Fourier integral operators.

This paper is organized as follows. The results on the propagation of the semi-classical wave front set which we prove in Section 3 are the analogs of the ones in the classical case ($h = 1$) [6, Ch. 7]. In this article we make the distinction between the finite and the infinite wavefront sets. In Section 4, we define semi-classical Fourier integral operators in a way similar to Melrose's definition in the classical case (see [7, Definition 25.1.1]). We then prove that, as in the classical case, one can also express semi-classical Fourier integral distributions in terms of oscillatory integrals. The proof from the classical case, however, does not completely carry over to the situation discussed here, because the Lagrangian submanifolds which we consider are no longer restricted away from the zero section in $T^*\mathbb{R}^n$. Section 5 is devoted to a generalization of the semi-classical Egorov theorem to manifolds of unequal dimensions.

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We remark that the results of this paper have been applied to the study of the structure of the scattering amplitude [1, 2]. In [1], we studied smooth compactly supported perturbations of the Euclidean Laplace–Beltrami operator and proved that the scattering amplitude is a semi-classical Fourier integral operator quantizing the scattering relation. In [2], we extended this result to the case of short range perturbations of the Laplacian when the scattering amplitude is restricted away from the diagonal in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. In both cases we have first applied the generalization of Egorov's Theorem given in Section 5 to reduce the problem to that of showing that the appropriately cut-off resolvent is a semi-classical Fourier integral operator associated to the perturbed Hamiltonian flow relation. In proving the latter, we have made use of the propagation of the semi-classical wave front set results proved in Section 3. Lastly, the characterization of semi-classical Fourier integral operators in terms of oscillatory integrals which we prove in Section 4.2 below has allowed us, under a certain geometric condition, to express the scattering amplitude in the form $e^{\frac{i}{h}S}a$, where S is a modified action along non-trapped trajectories and a is a suitably chosen symbol.

2 Preliminaries

In this section we recall some of the elements of semi-classical analysis which we will use here. First we define two classes of symbols

$$S^m_{2n}(1) = \left\{ a \in C^\infty(\mathbb{R}^{2n} \times (0, h_0]) : \forall \alpha, \beta \in \mathbb{N}^n, |\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \leq C_{\alpha,\beta} h^{-m} \right\}$$

and

$$S^{m,k}(T^*\mathbb{R}^n) = \Big\{ a \in C^\infty(T^*\mathbb{R}^n \times (0, h_0]) : \forall K \Subset \mathbb{R}^n, \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in K} |\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \leq C_{\alpha,\beta} h^{-m} \langle \xi \rangle^{k-|\beta|} \Big\},$$

where $h_0 \in (0, 1]$ and $m, k \in \mathbb{R}$. For $a \in S^m_{2n}(1)$ or $a \in S^{m,k}(T^*\mathbb{R}^n)$, we define the corresponding semi-classical pseudodifferential operator of class $\Psi^m_h(1, \mathbb{R}^n)$ or $\Psi^{m,k}_h(\mathbb{R}^n)$, respectively, by setting

$$Op_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i\langle x-y, \xi \rangle}{h}} a(x, \xi; h) u(y)\, dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

and extending the definition to $\mathcal{S}'(\mathbb{R}^n)$ by duality. Below we shall work only with symbols which admit asymptotic expansions in h and with pseudodifferential operators which are (left) quantizations of such symbols. For $A \in \Psi^k_h(1, \mathbb{R}^n)$ or $A \in \Psi^{m,k}_h(\mathbb{R}^n)$, we shall use $\sigma_0(A)$ and $\sigma(A)$ to denote its principal symbol and its complete symbol, respectively. If a is a symbol of the above classes, we shall say that a has compact support if there exists a compact set $K \Subset T^*\mathbb{R}^n$ such that $\operatorname{supp} a(\,\cdot\,; h) \subset K$ for every $h \in (0, h_0]$.

For a symbol $a \in S^{m,k}(T^*\mathbb{R}^n)$ or $a \in S^m_{2n}(1)$, we now define its (semi-classical) essential support $\text{ess-supp}_h\, a$. For that, we first let $\hat{T}^*\mathbb{R}^n = T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n$, where we set $S^*\mathbb{R}^n = (T^*\mathbb{R}^n\backslash 0)/\mathbb{R}_+$ with the $\mathbb{R}_+$ action given by multiplication on the fibers $(x,\xi)\mapsto(x,t\xi)$. As in [5], the points in $T^*\mathbb{R}^n$ will be called finite and the points in $S^*\mathbb{R}^n$ will be called infinite.

Definition 2.1 Let $(x_0,\xi_0)\in\hat{T}^*(\mathbb{R}^n)$. We shall say that (x_0,ξ_0) does not belong to the essential support of a if

- If (x_0,ξ_0) is finite, then there exist open neighborhoods U and V of x_0 and ξ_0, respectively, such that for all $\alpha,\beta\in\mathbb{N}^n$,

$$\partial_x^\alpha\partial_\xi^\beta a(x,\xi)=\mathcal{O}(h^\infty)\text{ uniformly in }(x,\xi)\in U\times V.$$

  We shall denote the complement of the set of all such points by $\text{ess-supp}_h^f a$.
- If (x_0,ξ_0) is infinite, then there exist an open neighborhood U of x_0, an open conic neighborhood V of ξ_0, and a constant $K>0$ such that for all $\alpha,\beta\in\mathbb{N}^n$,

$$\partial_x^\alpha\partial_\xi^\beta a(x,\xi)=\mathcal{O}(h^\infty\langle\xi\rangle^{-\infty})\text{ uniformly in }(x,\xi)\in U\times(\{\xi:\|\xi\|>K\}\cap V).$$

  We shall denote the complement of the set of all such points by $\text{ess-supp}_h^i a$.

We also let $\text{ess-supp}_h a=\text{ess-supp}_h^f a\sqcup\text{ess-supp}_h^i a$ denote the semi-classical essential support of a.

For $A\in\Psi_h^{m,k}(\mathbb{R}^n)$ or $A\in\Psi_h^m(1,\mathbb{R}^n)$, we then define its (semi-classical) wavefront set as

$$WF_h^f(A)=\text{ess-supp}_h^f a,\quad WF_h^i(A)=\text{ess-supp}_h^i a,\quad WF_h(A)=\text{ess-supp}_h a,$$

where $A=Op_h(a)$. We shall say that $WF_h(A)$ is compact if $WF_h(A)\equiv WF_h^f(A)$ and $WF_h^f(A)$ is compact.

We also define the class of semi-classical distributions $\mathcal{D}_h'(\mathbb{R}^n)$, with which we will work here

$$\mathcal{D}_h'(\mathbb{R}^n)=\big\{u\in C_h^\infty((0,1];\mathcal{D}'(\mathbb{R}^n)):\forall\chi\in C_c^\infty(\mathbb{R}^n)\exists\, N\in\mathbb{N}\text{ and }C_N>0: \\ |\mathcal{F}_h(\chi u)(\xi)|\le C_N h^{-N}\langle\xi\rangle^N\big\},$$

where $\mathcal{F}_h(\chi u)(\xi)=\langle e^{-\frac{i}{h}\langle\,\cdot\,,\xi\rangle},\chi u\rangle$, with $\langle\,\cdot\,,\,\cdot\,\rangle$ denoting the distribution pairing. The class of semi-classical distributions of compact support $\mathcal{E}_h'(\mathbb{R}^n)$ consists of the distributions $u\in\mathcal{D}_h'(\mathbb{R}^n)$ with supp $u(h)$ contained in a fixed compact set independent of h. The set $\mathcal{S}_h'(\mathbb{R}^n)$, on the other hand, consists of the elements of $\mathcal{D}_h'(\mathbb{R}^n)$ which, for every h, extend to continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. (We remind the reader that the distributions $\mathcal{D}'(\mathbb{R}^n)$ form the dual space to the space $C_c^\infty(\mathbb{R}^n)$ with respect to the inductive limit topology. See [7, vol. 1] for a presentation of the theory of distributions.)

We shall work with the L^2-based semi-classical Sobolev spaces $H_h^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, which consist of the distributions $u \in \mathcal{S}_h'(\mathbb{R}^n)$ such that $\mathcal{F}_h(u)$ is locally square integrable and

$$\|u\|^2_{H_h^s(\mathbb{R}^n)} \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\mathcal{F}_h(u)(\xi)|^2 d\xi < \infty.$$

We shall say that $u = v$ *microlocally* near an open set $U \subset T^*\mathbb{R}^n$, if there exists an open set $\widetilde{U} \Subset T^*\mathbb{R}^n$ with $\overline{U} \Subset \widetilde{U}$ such that for every $P \in \Psi_h^0(1, \mathbb{R}^n)$ satisfying

$$WF_h(P) \subset \widetilde{U}, \tag{1}$$

we have that $P(u-v) = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^n)$. We remark that this definition implicitly requires that $WF_h(P)$ have no infinite part.

We shall further say that u satisfies a property $\mathcal{P}$ *microlocally* near an open set $U \subset T^*\mathbb{R}^n$ if there exists $v \in \mathcal{D}_h'(\mathbb{R}^n)$ such that $u = v$ microlocally near U and v satisfies property $\mathcal{P}$.

To extend these notions to compact manifolds, we recall the following definitions of semi-classical distributions and pseudodifferential operators on compact manifolds. Let M be a smooth compact manifold and $\kappa_j \colon M_j \to X_j$, $j = 1, \dots, N$, a set of local charts. A semi-classical distribution u on M, $u \in \mathcal{D}_h'(M)$, is defined by a set of representatives $u_j \in \mathcal{D}_h'(X_j)$, $j = 1, \dots, N$, with $u_j \circ (\kappa_j \circ \kappa_k^{-1}) = u_k$ in $\kappa_k(M_j \cap M_k)$.

A linear continuous operator $A \colon C^\infty(M) \to \mathcal{D}_h'(M)$ belongs to $\Psi_h^m(1, M)$ or $\Psi_h^{m,k}(M)$ if for all $j \in \{1, \dots, N\}$ and $u \in C_c^\infty(M_j)$ we have $Au \circ \kappa_j^{-1} = A_j(u \circ \kappa_j^{-1})$ with $A_j \in \Psi_h^m(1, X_j)$ or $A_j \in \Psi_h^{m,k}(X_j)$, respectively, and

$$\chi_1 A \chi_2 \colon \mathcal{D}_h'(M) \to h^\infty C^\infty(M)$$

if $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \varnothing$.

3 The Semi-Classical Wavefront Set of a Semi-Classical Distribution

In this section we discuss the different notions of semi-classical wavefront set used in the literature and show that they are equivalent. We also establish some of their properties.

We begin by proving the following.

Lemma 3.1 *A distribution $u \in C_h^\infty((0, 1]; \mathcal{D}_h'(\mathbb{R}^n))$ is an element of $\mathcal{D}_h'(\mathbb{R}^n)$ if and only if for every $\chi \in C_c^\infty(\mathbb{R}^n)$ there exist $m, k_m, C_m \in \mathbb{R}$ such that $\|\chi u\|_{H_h^m(\mathbb{R}^n)} \leq C_m h^{-k_m}$, for every $h \leq h_0$ for some $h_0 \in (0, 1)$.*

Proof The first implication is clear. For the second implication, let $\chi \in C_c^\infty(\mathbb{R}^n)$ and let $m, k_m \in \mathbb{R}$ be such that $\|\chi u\|_{H_h^m(\mathbb{R}^n)} \leq C_m h^{-k_m}$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $\operatorname{supp} \chi$. Lastly, let $s \in \mathbb{Z}$ be such that $s < \min\{0, \lfloor m \rfloor\}$. Then

$$\begin{aligned} |\mathcal{F}_h(\chi u)| = \left| \left\langle e^{-\frac{i}{h}\langle \cdot, \xi \rangle} \psi, \chi u \right\rangle \right| &\leq \left\| e^{-\frac{i}{h}\langle \cdot, \xi \rangle} \psi \right\|_{H_h^{-s}(\mathbb{R}^n)} \|\chi u\|_{H_h^s(\mathbb{R}^n)} \\ &\leq C_{m,s} \langle \xi \rangle^{-s} h^{s-k_m}. \end{aligned}$$

■

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We now state the two alternative definitions of semi-classical wavefront set given in the literature. The first one is as in [5].

Definition 3.2 Let $u \in \mathcal{D}_h'(\mathbb{R}^n)$ and let $(x_0, \xi_0) \in \hat{T}^*(\mathbb{R}^n)$. We shall say that (x_0, ξ_0) does not belong to the semi-classical wavefront set of u if either of the following holds:

- (x_0, ξ_0) is finite and there exist $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and an open neighborhood U of ξ_0, such that $\forall N \in \mathbb{N}$, $\forall \xi \in U$, $|\mathcal{F}_h(\chi u)(\xi)| \leq C_{N,U} h^N$. We shall denote the complement of the set of all such points by $WF_h^f(u)$.
- (x_0, ξ_0) is infinite and there exist $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and a conic neighborhood U of ξ_0, such that $\forall N \in \mathbb{N}$, $\forall \xi \in U \cap \{\|\xi\| \geq \frac{1}{K}\}$, for some $K > 0$,

$$|\mathcal{F}_h(\chi u)(\xi)| \leq C_{N,U,K} h^N \langle \xi \rangle^{-N}.$$

We shall denote the complement of the set of all such points by $WF_h^i(u)$.

We shall further use $WF_h(u) = WF_h^f(u) \sqcup WF_h^i(u)$ to denote the semi-classical wavefront set of u.

The definition of semi-classical wavefront set given in [10] is as follows.

Definition 3.3 Let $u \in \mathcal{D}_h'(\mathbb{R}^n)$. Set

$$\widetilde{WF}_h(u) = \Big\{ (x, \xi) : \exists A \in \Psi_h^{0,0}(\mathbb{R}^n), \text{ with } WF(A) \text{ compact and } \sigma_o(A)(x, \xi) \neq 0, \\ \text{such that } Au \in h^\infty C^\infty((0, 1]_h; C^\infty(\mathbb{R}^n)) \Big\}^c.$$

We can now prove the following.

Lemma 3.4 *If $u \in \mathcal{D}_h'(\mathbb{R}^n)$, then $WF_h^f(u) \equiv \widetilde{WF}_h(u)$.*

Proof Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash WF_h^f(u)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi(x_0) \neq 0$ and $\mathcal{F}_h(\phi u)(\xi) = \mathcal{O}(h^\infty)$ uniformly in ξ in a bounded open neighborhood $V \subset \mathbb{R}^n$ of ξ_0. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp} \chi \subset V$. Consider

$$Au(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}\langle x-y, \xi \rangle} \varphi(x)\varphi(y)\chi(\xi)u(y)\, dy d\xi.$$

We clearly have that $A \in \Psi_h^{0,0}(\mathbb{R}^n)$ with $\sigma_0(A)(x_0, \xi_0) \neq 0$ and $Au = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^n)$.

Let $(x_0, \xi_0) \in T^*\mathbb{R}^n$ be such that there exists $A \in \Psi_h^{0,0}(\mathbb{R}^n)$ with $\sigma_0(A)(x_0, \xi_0) \neq 0$ and $Au = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^n)$. Let $\varphi, \chi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi(x_0) \neq 0$, $\chi(\xi_0) \neq 0$, and $\chi(hD)\varphi = BA + R$, where $B \in \Psi_h^{0,0}(\mathbb{R}^n)$, $R \in \Psi_h^{-\infty,-\infty}(\mathbb{R}^n)$. Then

$$\begin{aligned} \chi(hD)\varphi u(x) &= \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}\langle x-y, \xi\rangle} \chi(\xi)\varphi(y)u(y)\, dy d\xi \\ &= \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h} x\cdot\xi} \chi(\xi)\mathcal{F}_h(\varphi u)(\xi)\, d\xi = \mathcal{O}(h^\infty) \text{ in } C^\infty(\mathbb{R}^n). \end{aligned}$$

Hence, $\chi(\xi)\mathcal{F}_h(\varphi u)(\xi) = \mathcal{O}(h^\infty)$ uniformly in ξ, and therefore $\mathcal{F}_h(\varphi u)(\xi) = \mathcal{O}(h^\infty)$ uniformly in ξ in a bounded open set containing ξ_0, which implies that $(x_0, \xi_0) \notin WF_h^f(u)$. ■

Similary, one can prove the following.

Lemma 3.5 *Let $u \in \mathcal{D}_h'(\mathbb{R}^n)$. Then $(x_0, \xi_0) \notin WF_h^i(u)$ if and only if $\exists A \in \Psi_h^{0,0}(\mathbb{R}^n)$ with $\sigma_0(A)(x, \xi) \geq C > 0$ in a conic neighborhood of (x_0, ξ_0) such that $Au = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^n)$.*

Proof The proof is analogous to the one in the classical case — see [6, Proposition 7.4]. ■

We note that Lemmas 3.4 and 3.5 are stated without proof in [5, Proposition A.I.12].

3.1 Properties of the Semi-classical Wavefront Set

In this section, we establish some properties of semi-classical wavefront sets; namely, we show how semi-classical wavefront sets behave under compositions and tensor products of distributions.

For $u \in \mathcal{D}_h'(\mathbb{R}^n)$, we set $-WF_h(u) = \{(x, -\xi) : (x, \xi) \in WF_h(u)\}$.

We have the following preparatory lemma.

Lemma 3.6 *Let $u_1 \in \mathcal{D}_h'(\mathbb{R}^n)$ and $u_2 \in \mathcal{E}_h'(\mathbb{R}^n)$ satisfy $WF_h(u_1) \cap (-WF_h(u_2)) = \varnothing$. Then $\langle u_1, u_2 \rangle \stackrel{\text{def}}{=} \int u_1 u_2 = \mathcal{O}(h^\infty)$, where the integral is defined as in [6, Proposition 7.6].*

Proof For $u \in \mathcal{D}_h'(\mathbb{R}^n)$, let

$$\Sigma_h = \{\xi \in \mathbb{R}^n \sqcup \mathbb{S}^{n-1} : \exists\, x \in \mathbb{R}^n, (x, \xi) \in WF_h(u)\},$$
$$\Sigma_h^i = \{\xi \in \mathbb{S}^{n-1} : \exists\, x \in \mathbb{R}^n, (x, \xi) \in WF_h^i(u)\},$$
$$\Sigma_h^f = \{\xi \in \mathbb{R}^n : \exists\, x \in \mathbb{R}^n, (x, \xi) \in WF_h^f(u)\},$$
$$\Sigma_h^x = \{\xi \in \mathbb{R}^n \sqcup \mathbb{S}^{n-1} : (x, \xi) \in WF_h(u)\}.$$

We have that $\Sigma_h^x(u) = \lim_{\operatorname{supp}\phi \to \{x\}} \Sigma_h(\phi u)$. The proof is the same as in the classical case ($h = 1$) (see [7, §8.1]). For every $x_0 \in \mathbb{R}^n$, we can then find $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi(x_0) \neq 0$ and $\Sigma_h(\varphi u_1) \cap (-\Sigma_h(\varphi u_2)) = \varnothing$. By [6, Proposition 7.6], we have that

$$\int \varphi u_1 \varphi u_2 = \frac{1}{(2\pi h)^n} \int \mathcal{F}_h(\varphi u_1)(\xi)\mathcal{F}_h(\varphi u_2)(-\xi)\, d\xi. \tag{2}$$

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Now, since $\Sigma_h^i(\varphi u_1) \cap \Sigma_h^i(\varphi u_2) = \varnothing$, for every $\xi_0 \in \Sigma_h^i(\varphi u_1)$ we can find an open conic neighborhood U_{ξ_0} of ξ_0 such that $\mathcal{F}_h(\varphi u_2)(-\xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ uniformly in $U_{\xi_0} \cap \{\xi : \|\xi\| \geq \frac{1}{C}\}$ for some $C > 0$. Since $u_1 \in \mathcal{D}_h'(\mathbb{R}^n)$, it follows that there exist $N \in \mathbb{N}$ and $C' > 0$ such that $|\mathcal{F}_h(\varphi u_1)(\xi)| \leq C' h^{-N} \langle \xi \rangle^N$ and therefore $\mathcal{F}_h(\varphi u_1)(\xi)\mathcal{F}_h(\varphi u_2)(-\xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ uniformly in $U_{\xi_0} \cap \{\xi : \|\xi\| \geq \frac{1}{C_1}\}$. The compactness of $\mathbb{S}^{n-1}$ implies that we can find finitely many such neighborhoods $(U_l^1)_{l=1}^{L_1}$ and $(U_l^2)_{l=1}^{L_2}$ and a constant $C_1 > 0$ satisfying $\Sigma_h^i(\varphi u_1) \subset \bigcup_{l=1}^{L_2} U_l^2$ and $\Sigma_h^i(\varphi u_2) \subset \bigcup_{l=1}^{L_1} U_l^1$ and such that $\mathcal{F}_h(\varphi u_1)(\xi)\mathcal{F}_h(\varphi u_2)(-\xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ uniformly in $U_l^j \cap \{\xi : \|\xi\| \geq \frac{1}{C_1}\}$, $l = 1, \dots, L_j$, $j = 1, 2$. We can further arrange to have $(\bigcup_{l=1}^{L_1} U_l^1) \cap (\bigcup_{l=1}^{L_2} U_l^2) = \varnothing$. Lastly, we choose finitely many sets $(U_l)_{l=1}^{L_3}$ such that $\mathbb{S}^{n-1} \backslash (\bigcup_{k=1}^{2} \bigcup_l^{L_k} U_l^k) \subset \bigcup_{l=1}^{L_3} U_l$ and a constant $C_2 > 0$ such that $\mathcal{F}_h(\varphi u_j)(\xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ uniformly in $\xi \in \{\xi : \|\xi\| > \frac{1}{C_2}\} \cap U_l$, $j = 1, 2$, $l = 1, \dots, L_3$. With $C = \min\{C_1, C_2\}$, we then have

$$\int_{\{\xi : \|\xi\| > \frac{1}{C}\}} \mathcal{F}_h(\varphi u_1)(\xi)\mathcal{F}_h(\varphi u_2)(-\xi)\, d\xi = \mathcal{O}(h^\infty).$$

The same argument applied now to $\Sigma_h^f(\varphi u_j) \cap \{\xi \in \mathbb{R}^n : \|\xi\| \leq \frac{1}{C}\}$, $j = 1, 2$, gives that

$$\int_{\{\xi : \|\xi\| \leq \frac{1}{C}\}} \mathcal{F}_h(\varphi u_1)(\xi)\mathcal{F}_h(\varphi u_2)(-\xi)\, d\xi = \mathcal{O}(h^\infty)$$

and therefore $\int \varphi u_1 \varphi u_2 = \mathcal{O}(h^\infty)$.

Choosing a locally finite partition of unity $\sum_{j=1}^\infty \varphi_j^2 = 1$ with each function φ_j chosen as φ above, we have that $\langle u_1, u_2 \rangle = \sum_{j=1}^\infty \langle \varphi_j u_1, \varphi_j u_2 \rangle = \mathcal{O}(h^\infty)$. ■

We now consider two position spaces $\mathbb{R}^{d_2+d_1}$ and $\mathbb{R}^{d_3+d_2}$ split into $\mathbb{R}^{d_2} \times \mathbb{R}^{d_1}$ and $\mathbb{R}^{d_3} \times \mathbb{R}^{d_2}$, respectively. This leads to the corresponding splitting of the phase spaces $T^*\mathbb{R}^{d_2+d_1}$ and $T^*\mathbb{R}^{d_3+d_2}$ into $T^*\mathbb{R}^{d_2} \times T^*\mathbb{R}^{d_1}$ and $T^*\mathbb{R}^{d_3} \times T^*\mathbb{R}^{d_2}$. For sets $A \subset T^*\mathbb{R}^{d_2+d_1}$, $B \subset T^*\mathbb{R}^{d_3+d_2}$, and $a \subset T^*\mathbb{R}^{d_1}$, we define

$$B \circ A = \{(m_3, m_1) \in T^*\mathbb{R}^{d_3+d_1} : \exists m_2 \in T^*\mathbb{R}^{d_2}, (m_2, m_1) \in A, (m_3, m_2) \in B\},$$

$$A(a) = \{m_2 \in T^*\mathbb{R}^{d_2} : \exists m_1 \in T^*\mathbb{R}^{d_1}, (m_2, m_1) \in A, m_1 \in a\}.$$

For $U \in \mathcal{D}_h'(\mathbb{R}^{d_{j+1}+d_j})$, $j = 1, 2$, we let

$$(WF_h^k)'(U) = \{(x, \xi; y, \eta) : (x, \xi; y, -\eta) \in WF_h^k(U)\}, k \in \{f, i\},$$

$$(WF_h^i)'_{\mathbb{R}^{d_j}}(U) = \{(y, \eta) \in S^*\mathbb{R}^{d_j} : \exists x \in \mathbb{R}^{d_{j+1}}, (x, 0; y, \eta) \in (WF_h^i)'(U)\},$$

$$(WF_h^i)'_{\mathbb{R}^{d_{j+1}}}(U) = \{(x, \xi) \in S^*\mathbb{R}^{d_{j+1}} : \exists y \in \mathbb{R}^{d_j}, (x, \xi; y, 0) \in (WF_h^i)'(U)\}.$$

Thus $(WF_h^i)'_{\mathbb{R}^{d_j}}(U)$ selects the points of $WF_h^i(U)$ which point to a direction in $\mathbb{R}^{d_j}$, reverses their directions, and projects them onto $S^*\mathbb{R}^{d_j}$, while $(WF_h^i)'_{\mathbb{R}^{d_{j+1}}}(U)$ selects

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the points of $WF_h^i(U)$ which point to a direction in $\mathbb{R}^{d_{j+1}}$ and projects them onto $S^*\mathbb{R}^{d_{j+1}}$.

Lastly, we shall say that U is properly supported if there exists $0 < h_0 \ll 1$ such that for every $0 < h < h_0$, $\operatorname{supp} U(h)$ is proper in the sense that for all compact sets $K_j \subset \mathbb{R}^{d_j}$ and $K_{j+1} \subset \mathbb{R}^{d_{j+1}}$ the sets

$$\operatorname{supp} U(h)(K_j) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^{d_{j+1}} : \exists y \in K_j, (x, y) \in \operatorname{supp} U(h)\},$$
$$(\operatorname{supp} U(h))^{-1}(K_{j+1}) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^{d_j} : \exists x \in K_{j+1}, (x, y) \in \operatorname{supp} U(h)\}$$

are compact.

We now have the following results on the behavior of the semi-classical wavefront set under composition and tensor products.

Lemma 3.7 *Let $u \in \mathcal{D}_h'(\mathbb{R}^{d_2})$, $v \in \mathcal{D}_h'(\mathbb{R}^{d_1})$, $V \in \mathcal{D}_h'(\mathbb{R}^{d_2+d_1})$, $W \in \mathcal{D}_h'(\mathbb{R}^{d_3+d_2})$. Then*

(i) *$WF_h^f(Av) \subset WF_h^f(A) \cap WF_h^f(v)$ for $A \in \Psi_h^m(1, \mathbb{R}^{d_1})$ with compact wavefront set or $A \in \Psi_h^{m,k}(\mathbb{R}^{d_1})$, and $WF_h^i(Av) \subset WF_h^i(A) \cap WF_h^i(v)$ for $A \in \Psi_h^{m,k}(\mathbb{R}^{d_1})$.*

(ii) *$u \otimes v \in \mathcal{D}_h'(\mathbb{R}^{d_2+d_1})$, $WF_h^f(u \otimes v) \equiv WF_h^f(u) \times WF_h^f(v)$, and*

$$WF_h^i(u \otimes v) \subset (WF_h^i(u) \times WF_h^i(v)) \cup \big((\operatorname{supp} u \times \{0\}) \times WF_h^i(v)\big) \cup \big(WF_h^i(u) \times (\operatorname{supp} v \times \{0\})\big).$$

(iii) *If V is properly supported, $v \in \mathcal{E}_h'(\mathbb{R}^{d_1})$, $WF_h^i(v) \cap (WF_h^i)'_{\mathbb{R}^{d_1}}(V) = \varnothing$, and $WF_h^f(v)$ is compact, then*

$$Vv \in \mathcal{D}_h'(\mathbb{R}^{d_2}),$$
$$WF_h^f(Vv) \subset (WF_h^f)'(V)(WF_h^f(v)),$$
$$WF_h^i(Vv) \subset (WF_h^i)'(V)(WF_h^i(v)) \cup (WF_h^i)'_{\mathbb{R}^{d_2}}(V),$$

where Vv is the distribution obtained by applying to v the continuous linear operator from $C_c^\infty(\mathbb{R}^{d_1})$ into $\mathcal{D}_h'(\mathbb{R}^{d_1})$ with Schwartz kernel V.

(iv) *If at least one of V and W is properly supported,*

$$(WF_h^i)'_{\mathbb{R}^{d_2}}(V) \cap (WF_h^i)'_{\mathbb{R}^{d_2}}(W) = \varnothing,$$

and the set

$$\big\{ p \in T^*\mathbb{R}^{d_2} : \exists\, (q, r) \in T^*\mathbb{R}^{d_3} \times T^*\mathbb{R}^{d_1}, (q, p) \in (WF_h^f)'(W), (p, r) \in (WF_h^f)'(V)\big\}$$

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is compact, then

$$W \circ V \in \mathcal{D}_h'(\mathbb{R}^{d_3+d_1}),$$

$$(WF_h^f)'(W \circ V) \subset (WF_h^f)'(W) \circ (WF_h^f)'(V),$$

and

$$(WF_h^i)'(W \circ V) \subset (WF_h^i)'(W) \circ (WF_h^i)'(V) \cup (WF_h^i)'_{\mathbb{R}^{d_3}}(W) \times (\mathbb{R}^{d_1} \times \{0\}) \cup (\mathbb{R}^{d_3} \times \{0\}) \times (WF_h^i)'_{\mathbb{R}^{d_1}}(V),$$

where $W \circ V$ denotes the Schwartz kernel of the operator which is the composition of the operators with Schwartz kernels W and V.

Remark Part (iii) of this lemma is proved in [5, Proposition A.I.13] without the assumption on $WF_h^f(v)$. In our proof, however, we also show that all estimates can be made uniformly in a neighborhood of $WF_h(v)$.

Proof We begin by proving (i). Let $(x_0, \xi_0) \notin WF_h^f(A)$. The proof is the same for the two classes of operators; we only treat the case $A \in \Psi_h^{0,0}(\mathbb{R}^{d_1})$. Let $B \in \Psi_h^{0,0}(\mathbb{R}^{d_1})$ satisfy $\sigma(B)(x_0, \xi_0) \neq 0$, $\sigma(B) \in C_c^\infty(\mathbb{R}^{d_1})$, and $WF_h^f(B) \cap WF_h^f(A) = \varnothing$. Then $BA \in \Psi_h^{-\infty,-\infty}(\mathbb{R}^{d_1})$ and therefore $BAu = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^{d_1})$.

Now let $(x_0, \xi_0) \notin WF_h^f(u)$. Let $c \in C_c^\infty(\mathbb{R}^{d_1})$ satisfy $c(x_0, \xi_0) \neq 0$ and let $d \in S^0_{2d_1}(1)$ be such that $d\#_h c = 1$ in a neighborhood $W \subset (WF_h^f(u))^c$ of (x_0, ξ_0). Further, let $\chi \in C_c^\infty(T^*\mathbb{R}^{d_1})$ have support in an open set $V \Subset W$ and be equal to 1 on an open subset $U \Subset V$. Then the operator $T = Op_h(\chi d \#_h c)$ has symbol $\sigma(T) \equiv 1 \bmod h^\infty$ in $S^0_{2d_1}(1)$ in U and supported in W, and therefore $Tu = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^{d_1})$. Let B be elliptic at (x_0, ξ_0) with $WF_h(B) \subset U$. Then we have that $BA \equiv BAT \bmod \Psi_h^{-\infty,-\infty}(\mathbb{R}^{d_1})$ and hence $BAu \equiv BATu = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^{d_1})$. Therefore, $(x_0, \xi_0) \notin WF_h^f(Au)$.

The proof in the infinite case is analogous to the proof in the classical case (see [6, Lemma 7.2]).

We now turn to proving (ii). It is trivial to check that $u \otimes v \in \mathcal{D}_h'(\mathbb{R}^{d_2+d_1})$. We shall now prove that $WF_h^f(u \otimes v) \subset WF_h^f(u) \times WF_h^f(v)$. Let $(x_0, \xi_0; y_0, \eta_0) \notin WF_h^f(u) \times WF_h^f(v)$. Without loss of generality, we can assume that $(x_0, \xi_0) \notin WF_h^f(u)$. Then there exists $\chi_2 \in C_c^\infty(\mathbb{R}^{d_2})$ with $\chi_2(x_0) \neq 0$ and a bounded open set $O_2 \subset \mathbb{R}^{d_2}$ with $\xi_0 \in O_2$ such that $|\mathcal{F}_h(\chi_2 u)(\xi)| = \mathcal{O}(h^\infty)$ uniformly for $\xi \in O_2$. Now let $\chi_1 \in C_c^\infty(\mathbb{R}^{d_1})$ be such that $\chi_1(y_0) \neq 0$. Then $|\mathcal{F}_h(\chi_1 v)(\eta)| \leq Ch^{-m}\langle \eta \rangle^m$ for some $C > 0$, $m > 0$, and therefore $|\mathcal{F}_h(\chi_2 u \otimes \chi_1 v)(\xi, \eta)| = \mathcal{O}(h^\infty)$ uniformly in $(\xi, \eta) \in O_2 \times O_1$ for any bounded open set $O_1 \subset \mathbb{R}^{d_1}$ with $\eta \in O_1$. Therefore $(x_0, \xi_0; y_0, \eta_0) \notin WF_h^f(u \otimes v)$.

The opposite inclusion can be proved similarly.

The proof of the second assertion in (ii) is as in the C^∞ case. See [7, Theorem 8.2.9].

We now turn to proving (iii). For every $0 < h \ll 1$, we define the distribution $V(h)v(h)$ in the following way. For $\varphi \in C_c^\infty(\mathbb{R}^{d_2})$ we let

$$\langle V(h)v(h), \varphi\rangle \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^{d_1+d_2}} \int \mathcal{F}_h(V(h))(\xi,\eta)\mathcal{F}_h(\varphi \otimes v(h))(-\xi,-\eta)\, d\xi d\eta.$$

(See also [6, Theorem 7.8].) The fact that $Vv \in \mathcal{D}'_h(\mathbb{R}^{d_2})$ is proved in [5, Proposition A.I.13].

We shall now prove that $(WF_h^f)'(V)(WF_h^f(v))$ is a closed set. Let $((x_n,\xi_n))_{n\in\mathbb{N}} \subset (WF_h^f)'(V)(WF_h^f(v))$ converge to (x_0,ξ_0). For every $n \in \mathbb{N}$, let $(y_n,\eta_n) \in WF_h^f(v)$ be such that $(x_n,\xi_n;y_n,\eta_n) \in (WF_h^f)'(V)$. Since $WF_h^f(v)$ is compact, after passing to a subsequence, we can assume that $(y_n,\eta_n) \to (y_0,\eta_0) \in WF_h^f(v)$. Therefore $(x_n,\xi_n;y_n,\eta_n) \to (x_0,\xi_0;y_0,\eta_0)$ and since $(WF_h^f)'(V)$ is closed, it follows that $(x_0,\xi_0;y_0,\eta_0) \in (WF_h^f)'(V)$. This implies that $(x_0,\xi_0) \in (WF_h^f)'(V)(WF_h^f(v))$, and therefore $(WF_h^f)'(V)(WF_h^f(v))$ is closed.

Now let $(x_0,\xi_0) \in ((WF_h^f)'(V)(WF_h^f(v)))^c$, and let $O, O' \subset \mathbb{R}^{d_2}$ be open neighborhoods of x_0 and ξ_0, respectively, such that $O \times O' \subset ((WF_h^f)'(V)(WF_h^f(v)))^c$ and O' is bounded. Let $\chi \in C_c^\infty(\mathbb{R}^{d_2})$ with $\chi(x_0) \neq 0$ have support in O and let $\xi \in O'$. By the proof of Lemma 3.6, we have that

$$\begin{aligned}&\langle V(\cdot,\cdot\cdot), \chi(\cdot)e^{-\frac{i}{h}\langle\cdot,\xi\rangle} \otimes v(\cdot\cdot)\rangle \\ &\quad = \frac{1}{(2\pi h)^{d_1+d_2}} \int \mathcal{F}_h(V)(\eta,\tau)(\mathcal{F}_h(\chi(\cdot)e^{-\frac{i}{h}\langle\cdot,\xi\rangle}) \otimes \mathcal{F}_h(v))(\eta,\tau)d\eta d\tau = \mathcal{O}(h^\infty)\end{aligned}$$

uniformly in $\xi \in O'$.

The proof in the case of the infinite wavefront set is the same as in the C^∞ case (see [6, Theorem 7.8]).

Lastly, to prove (iv), we first define the distribution $W \circ V$ as follows. Let $\chi_j \in C_c^\infty(\mathbb{R}^{d_j})$, $j = 1,3$. Then

$$\begin{aligned}&\langle W \circ V, \chi_3 \otimes \chi_1\rangle \stackrel{\text{def}}{=} \\ &\qquad \frac{1}{(2\pi h)^{d_1+d_2+d_3}} \int \mathcal{F}_h(W \otimes \chi_1)(\xi,\eta,\tau)\mathcal{F}_h(\chi_3 \otimes V)(-\xi,-\eta,-\tau)\, d\xi d\eta d\tau\end{aligned}$$

and extend the definition to $C_c^\infty(\mathbb{R}^{d_3+d_1})$ by continuity (see [6, Theorem 7.10].) The fact that $W \circ V \in \mathcal{D}'_h(\mathbb{R}^{d_3+d_1})$ follows as in the proof of [5, Proposition A.I.13].

We shall now prove that $(WF_h^f)'(W) \circ (WF_h^f)'(V)$ is closed. For that, let

$$((x_n,\xi_n\,;y_n,\eta_n))_{n\in\mathbb{N}} \subset (WF_h^f)'(W) \circ (WF_h^f)'(V)$$

converge to $(x_0,\xi_0\,;y_0,\eta_0)$. Let $((z_n,\zeta_n))_{n\in\mathbb{N}} \subset \mathbb{R}^{d_2}$ be such that $(x_n,\xi_n\,;z_n,\zeta_n) \in (WF_h^f)'(W)$, $(z_n,\zeta_n\,;y_n,\eta_n) \in (WF_h^f)'(V)$, $n \in \mathbb{N}$. By the hypothesis, we can assume

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that, after passing to a subsequence, $(z_n, \zeta_n) \to (z_0, \zeta_0)$. Since then $(x_n, \xi_n; z_n, \zeta_n) \to (x_0, \xi_0; z_0, \zeta_0)$ and $(WF_h^f)'(W)$ is closed, it follows that $(x_0, \xi_0; z_0, \zeta_0) \in (WF_h^f)'(W)$. Similarly, $(y_0, \eta_0; z_0, \zeta_0) \in (WF_h^f)'(V)$, and therefore

$$(x_0, \xi_0; y_0, \eta_0) \in (WF_h^f)'(W) \circ (WF_h^f)'(V).$$

Now let $(x, \xi; y, \eta) \in ((WF_h^f)'(W) \circ (WF_h^f)'(V))^c$. Let $O_1, O_1' \subset \mathbb{R}^{d_1}$, $O_3, O_3' \subset \mathbb{R}^{d_3}$ be open neighborhoods of y, η, x, and ξ, respectively, such that $O_3 \times O_3' \times O_1 \times O_1' \subset (WF_h'(W) \circ WF_h'(V))^c$ and O_1' and O_3' are bounded. Let $\varphi \in C_c^\infty(\mathbb{R}^{d_1})$ and $\psi \in C_c^\infty(\mathbb{R}^{d_3})$ have supports inside O_1 and O_3, respectively, and satisfy $\varphi(y_0) \neq 0$ and $\psi(x_0) \neq 0$. Then, by the proof of Lemma 3.6, we have that

$$\langle W(\cdot, \cdot\cdot) \otimes \varphi(\cdot\cdot\cdot) e^{-\frac{i}{h}\langle \cdot\cdot\cdot, \eta\rangle}, \psi(\cdot) e^{-\frac{i}{h}\langle \cdot, \xi\rangle} \otimes V(\cdot\cdot, \cdot\cdot\cdot)\rangle = \mathcal{O}(h^\infty)$$

uniformly in $(\xi, \eta) \in O_3' \times O_1'$. Therefore $(x, \xi; y, \eta) \notin WF_h^f(W \circ V)$.

The proof in the infinite case is the same as in the C^∞ case, see [6, Theorem 7.10]. ■

4 Microlocal Semi-Classical Fourier Integral Operators

In this section we define semi-classical Fourier integral operators as microlocal objects and prove a characterization of the Schwartz kernels of these operators, which is a semi-classical analog of Melrose's characterization of Lagrangian distributions in [7, Definition 25.1.1]. We then use this characterization to describe the finite wavefront set of these semi-classical Fourier integral distributions.

4.1 Parameterizing Lagrangian Submanifolds

We first review some facts from symplectic geometry relating non-degenerate phase functions and Lagrangian submanifolds.

Let $V \subset \mathbb{R}^n \times \mathbb{R}^m$, $m \in \mathbb{N}_0$, be an open set and let $\varphi = \varphi(x, \theta) \in C_b^\infty(V; \mathbb{R})$. For $m > 0$, let φ also be a phase function in the sense of [8, Section 2.4]. If $a \in S^r_{n+m}(1)$, $r \in \mathbb{R}$, we define the oscillatory integral

$$I(a, \varphi) = \int_{\mathbb{R}^m} e^{\frac{i}{h}\varphi(\cdot, \theta)} a(\cdot, \theta)\, d\theta \quad \text{if } m > 0,$$

as in [8, Section 2.4], and set $I(a, \varphi) = e^{\frac{i}{h}\varphi} a$ if $m = 0$.

We further let

$$C_\varphi = \{(x, \theta) \in V : \varphi_\theta'(x, \theta) = 0\} \quad \text{and} \quad \Lambda_\varphi = \{(x, \varphi_x'(x, \theta)) : (x, \theta) \in C_\varphi\}$$

for $m > 0$, and

$$\Lambda_\varphi = \{(x, \varphi'(x)) : x \in V\}$$

for $m = 0$. We also recall that a phase function φ is non-degenerate if

(3) $\varphi'_\theta(x,\theta) = 0$ implies that $(\varphi''_{\theta x}\,\varphi''_{\theta\theta})$ has maximum rank at (x,θ).

If $m = 0$, it is a standard fact from symplectic geometry that Λ_φ is a Lagrangian submanifold of $T^*\mathbb{R}^n$. If $m > 0$, (3) implies that C_φ is a smooth n-dimensional manifold. Let $j_\varphi\colon C_\varphi \ni (x,\theta) \mapsto (x, \varphi'_x(x,\theta)) \in \Lambda_\varphi$. Then, after shrinking V around any fixed point $(x',\theta') \in C_\varphi$, we can assume that Λ_φ is a Lagrangian submanifold of $T^*\mathbb{R}^n$ and j_φ is a diffeomorphism. For a proof, we refer the reader to [6, Lemmas 11.2, 11.3].

If $\Lambda \subset T^*\mathbb{R}^n$ is a Lagrangian submanifold such that for some open subset $U \subset T^*\mathbb{R}^n$ the map $\pi_\xi\colon \Lambda \cap U \ni (x,\xi) \mapsto \xi \in \mathbb{R}^n$ is a local diffeomorphism, then there exist an open set $W \subset \mathbb{R}^n\backslash\{0\}$ and a function $H \in C_b^\infty(W;\mathbb{R})$ satisfying

(4) $$\Lambda \cap U = \{(H'(\xi),\xi) : \xi \in W\}.$$

For a proof, see [6, Section 9].

If $\Lambda \subset T^*\mathbb{R}^n$ is any Lagrangian submanifold and $\gamma \in \Lambda$, then there exists an open set $U \subset T^*\mathbb{R}^n$, $\gamma \in U$, and a non-degenerate phase function $\varphi \in C^\infty(V)$, $V \subset \mathbb{R}^{n+m}$ open, $m \in \mathbb{N}_0$, such that

(5) $$\Lambda \cap U = \Lambda_\varphi.$$

We include the proof of this well-known result here for completeness and to introduce some notation. Let $\mu = T_\gamma\Lambda$ be identified in a natural way with a subspace of $T^*\mathbb{R}^n$. By [6, Lemma 9.5] we have that after a canonical change of coordinates we may assume that

(6) $$\mu = \{(0, x''; \xi', Bx'')\},$$

for a splitting of the coordinates $x = (x', x'')$ and $\xi = (\xi', \xi'')$, where $x' = (x_1, \dots, x_k)$, $k \in \{0, \dots, n\}$, and B is a real symmetric matrix. This implies that the differential of the projection $\pi\colon \Lambda \to (x'', \xi')$ is bijective at γ and therefore this map is a local diffeomorphism from a neighborhood of γ in Λ to the (x'',ξ')-space. Therefore there exists a function $S \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ and an open neighborhood $U \subset T^*\mathbb{R}^n$ of γ such that $\Lambda \cap U = \{(\frac{\partial S}{\partial \xi'}, x''; \xi', -\frac{\partial S}{\partial x''})\} \cap U$. From this it easily follows that
$$\varphi(x,\xi') = \langle x', \xi'\rangle - S(x'',\xi')$$
is a non-degenerate phase function such that $\Lambda \cap U = \Lambda_\varphi \cap U$.

4.2 Semi-Classical Fourier Integral Operators

In this section we define semi-classical Fourier integral distributions and characterize them in terms of oscillatory integrals.

The main definition is as follows.

Definition 4.1 Let M be a smooth n-dimensional manifold and let $\Lambda \subset T^*M$ be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on T^*M. Let $r \in \mathbb{R}$. Then the space $I_h^r(M, \Lambda)$ of semi-classical Fourier integral distributions of order r associated to Λ is defined as the set of all $u \in \mathcal{D}_h'(M)$ such that

$$\Big(\prod_{j=0}^{N} A_j\Big)(u) = \mathcal{O}_{L^2(M)}(h^{N-r-\frac{n}{4}}), \quad h \to 0, \tag{7}$$

for all $N \in \mathbb{N}_0$ and for all $A_j \in \Psi_h^0(1, M)$, $j = 0, \dots, N-1$, with compact wavefront set near Λ and principal symbols vanishing on Λ, and any $A_N \in \Psi_h^0(1, M)$ with compact wavefront set near Λ.

We are now naturally led to the following:

Definition 4.2 Let M_1 and M_2 be smooth manifolds, let $\pi_j\colon T^*M_2 \times T^*M_1 \to T^*M_j$, $j = 1, 2$, denote the canonical projection and σ_j the canonical symplectic form on T^*M_j, $j = 1, 2$. A continuous linear operator $C_c^\infty(M_1) \to \mathcal{D}_h'(M_2)$, whose Schwartz kernel is an element of $I_h^r(M_2 \times M_1, \Lambda)$ for some Lagrangian submanifold Λ of $(T^*M_2 \times T^*M_1, \pi_2^*\sigma_2 + \pi_1^*\sigma_1)$ and some $r \in \mathbb{R}$ will be called a global semi-classical Fourier integral operator of order r associated to Λ. We denote the space of these operators by $\mathcal{I}_h^r(M_2 \times M_1, \Lambda)$.

Remark The exotic looking numerology for the order needs to be explained. We follow the same convention as that in the classical case and require that pseudodifferential operators with compactly supported symbols in $S^0_{2n}(1)$ have Schwartz kernels in $I_h^0(\mathbb{R}^n \times \mathbb{R}^n, N^*\Delta)$, where Δ is the diagonal in $\mathbb{R}^{2n}$ and $N^*\Delta = \{(x, \xi; x, -\xi) : (x, \xi) \in T^*\mathbb{R}^n\}$ is the conormal bundle over Δ. Explicitly, suppose that

$$K(x, y) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} c(x, \xi)\, d\xi,$$

where $c \in S^0_{2n}(1)$ has compact support. Then

$$\|K\|_{L^2(\mathbb{R}^{2n})} = \frac{1}{(2\pi h)^{\frac{n}{2}}} \|\mathcal{F}_h^y K\|_{L^2(\mathbb{R}^{2n})} = \frac{1}{(2\pi h)^{\frac{n}{2}}} \|c\|_{L^2(\mathbb{R}^{2n})},$$

where $\mathcal{F}_h^y$ is the semi-classical Fourier transform in the y variable, which is consistent with (7) with $N = 0$ and the order $r = 0$ ($M = \mathbb{R}^{2n}$ here).

We now have the following semi-classical analog of [7, vol. IV, Lemma 25.1.2].

Lemma 4.3 *If $u \in I_h^r(M, \Lambda)$, then $Au \in I_h^r(M, \Lambda)$ for every $A \in \Psi_h^0(1, M)$ with compact wavefront set or $A \in \Psi_h^{0,k}(T^*M)$, $k \in \mathbb{R}$.*

If $u \in \mathcal{D}_h'(M)$ is such that for every $(x_0, \xi_0) \in \Lambda$ there exists $A \in \Psi_h^0(1, M)$ elliptic at (x_0, ξ_0) with compact wavefront set and $Au \in I_h^r(M, \Lambda)$, then $u \in I_h^r(M, \Lambda)$. The same conclusion holds if $A \in \Psi_h^{0,k}(T^*M)$, $k \in \mathbb{R}$.

Proof To prove the first statement, let $u \in I_h^r(M,\Lambda)$, and let $A \in \Psi_h^0(1,M)$ have compact wavefront set or $A \in \Psi_h^{0,k}(T^*M)$, $k \in \mathbb{R}$. Further, let $A_j \in \Psi_h^0(1,M)$, $j = 0,\dots,N$, $N \in \mathbb{N}$, have compact wavefront sets near Λ and principal symbols vanishing on Λ. Then $A_N A \in \Psi_h^0(1,M)$ and has compact wavefront set. Thus, using Definition 4.1, we obtain that

$$\Big(\prod_{j=0}^{N} A_j\Big)(Au) = \Big(\prod_{j=0}^{N-1} A_j\Big)(A_N A u) = \mathcal{O}_{L^2(M)}(h^{N-r-\frac{n}{4}}), \quad h \to 0.$$

Therefore $Au \in I_h^r(M,\Lambda)$.

To prove the converse, let $B \in \Psi_h^0(1,M)$ have compact wavefront set and satisfy $(x_0,\xi_0) \notin WF_h(BA-I)$. Then $(x_0,\xi_0) \notin WF_h(BAu-u)$. From the first part of the proof, we have that $BAu \in I_h^r(M,\Lambda)$. Now let $P \in \Psi_h^0(1,M)$ have wavefront set contained in a sufficiently small neighborhood of $(x_0,\xi_0) \in \Lambda$ so that $PBAu - Pu = \mathcal{O}(h^\infty)$ in $C^\infty(M)$. Since again $PBAu \in I_h^r(M,\Lambda)$, we have that $(\prod_{l=0}^N A_l)(Pu) = \mathcal{O}_{L^2}(h^{N-r-\frac{n}{4}})$, $h \to 0$, for any set of operators $(A_l)_{l=0}^N$, $N \in \mathbb{N}_0$, as in (7). Thus $Pu \in I_h^r(M,\Lambda)$ for every $P \in \Psi_h^0(1,M)$ with wavefront set contained in a sufficiently small neighborhood of any point $(x_0,\xi_0) \in \Lambda$. The compactness of the wavefront sets of the operators $(A_l)_{l=0}^N$, $N \in \mathbb{N}$, now allows us to find $P_j \in \Psi_h^0(1,M)$, $j = 1,\dots,J$, $J \in \mathbb{N}$, such that $P_j u \in I_h^r(M,\Lambda)$, $j = 1,\dots,J$, and $\sum_{j=1}^J P_j \equiv 1$ near $\bigcup_{l=0}^N WF_h(A_l)$. Using the calculus of semi-classical pseudodifferential operators, we further obtain

$$\mathcal{O}_{L^2(M)}(h^{N-r-\frac{n}{4}}) = \Big(\prod_{l=0}^{N} A_k\Big)\Big(\sum_{j=1}^{J} P_j\Big)u = \Big(\prod_{l=0}^{N} A_k\Big)u + \mathcal{O}_{L^2(M)}(h^\infty),$$

which completes the proof.

The proof in the case of an operator $A \in \Psi_h^{0,k}(T^*M)$, $k \in \mathbb{R}$, is analogous. ∎

Lemma 4.3 reduces the study of distributions $u \in I_h^r(M,\Lambda)$, $r \in \mathbb{R}$, to the case where $WF_h^f(u)$ is contained in a small neighborhood of a point $(x_0,\xi_0) \in \Lambda$ and the support of u is close to x_0. In this case, we can locally trivialize T^*M and assume that $M = \mathbb{R}^n$. Our next theorem characterizes semi-classical Fourier integral distributions microlocally and, owing to the remark just made, it suffices to state it in the setting of $\mathbb{R}^n$.

Theorem 4.4 *Let $\Lambda \subset T^*\mathbb{R}^n$ be a Lagrangian submanifold and let $\gamma \in \Lambda$. Let φ be a non-degenerate phase function in an open set $V \subset \mathbb{R}^{n+m}$, $m \in \mathbb{N}_0$, such that $\Lambda = \Lambda_\varphi$ in a neighborhood of γ. If $a \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ is such that $\operatorname{supp} a \Subset V$, then $I(a,\varphi) \in I_h^r(\mathbb{R}^n,\Lambda)$.*

Conversely, if $u \in I_h^r(\mathbb{R}^n,\Lambda)$, then for every non-degenerate phase function φ defined in an open set $V \subset \mathbb{R}^{n+m}$, $m \in \mathbb{N}_0$, such that $\Lambda = \Lambda_\varphi$ near γ, there exists $a \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ with $\operatorname{supp} a \Subset V$ such that $u = I(a,\varphi)$ microlocally near γ.

Proof Let γ have canonical coordinates (x_0, ξ_0) and let us first assume that Λ is transverse to the section $\xi = \xi_0$ at γ. Then there exists an open neighborhood $U \subset T^*\mathbb{R}^n$ of γ such that $\pi_\xi \colon \Lambda \cap U \ni (x, \xi) \mapsto \xi \in \mathbb{R}^n$ in canonical coordinates is a local diffeomorphism. Let $H \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ be chosen such that, perhaps after adjusting V, $\Lambda_\varphi = \{(H'(\xi), \xi) : \xi \in W\}$ for some bounded open set $W \subset \mathbb{R}^n$. For $\xi \in \mathbb{R}^n$ consider

$$\mathcal{F}_h(I(a,\varphi))(\xi) = \iint e^{\frac{i}{h}(\varphi(x,\theta) - \langle x, \xi\rangle)} a(x,\theta)\, d\theta dx. \tag{8}$$

For $\xi \notin W$, integration by parts in (x, θ) gives

$$\mathcal{F}_h(I(a,\varphi)) = \mathcal{O}(h^\infty) \text{ in } C_c^\infty(W^c). \tag{9}$$

Now let $\bar\xi \in W$. Then the function $\Phi(x, \theta; \bar\xi) = \varphi(x, \theta) - \langle x, \bar\xi\rangle$ has a critical point at $(\bar x(\bar\xi), \bar\theta(\bar\xi))$, which is the inverse image in C_φ under j_φ of the point $(H'(\bar\xi), \bar\xi)$.

To prove that the critical point is non-degenerate, let v be in the kernel of

$$\Phi''_{x\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi); \xi_0) = \begin{bmatrix} \varphi''_{xx}(\bar x(\bar\xi), \bar\theta(\bar\xi)) & \varphi''_{x\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi)) \\ \varphi''_{\theta x}(\bar x(\bar\xi), \bar\theta(\bar\xi)) & \varphi''_{\theta\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi)) \end{bmatrix}.$$

Then $v \in \ker(\varphi''_{\theta x}(\bar x(\bar\xi), \bar\theta(\bar\xi))\, \varphi''_{\theta\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi)))$ and therefore $v \in T_{(\bar x(\bar\xi), \bar\theta(\bar\xi))} C_\varphi$. We also have that $v \in \ker(\varphi''_{xx}(\bar x(\bar\xi), \bar\theta(\bar\xi))\, \varphi''_{x\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi)))$, and since j_φ and π_ξ are diffeomorphisms, it follows that $v = 0$. Hence

$$\text{the matrix } \Phi''_{x\theta}(\bar x(\bar\xi), \bar\theta(\bar\xi); \bar\xi) \text{ is non-singular.} \tag{10}$$

We can therefore apply the method of stationary phase to the integral (8) and obtain

$$\mathcal{F}_h(I(a,\varphi))(\bar\xi) \sim e^{\frac{i}{h}\Phi(\bar x(\bar\xi), \bar\theta(\bar\xi); \bar\xi)} \sum_{k=0}^\infty h^{k + \frac{n}{2} + \frac{m}{2}} (A_{2k}(D_{x,\theta})a)(\bar x(\bar\xi), \bar\theta(\bar\xi)), \tag{11}$$

where $A_{2k}(D)$ are differential operators of orders $\leq 2k$, respectively.

The implicit function theorem and (10) now imply that, perhaps after shrinking W around $\bar\xi$, $\bar x \in C^\infty(W; \mathbb{R}^n)$ and $\bar\theta \in C^\infty(W; \mathbb{R}^m)$. We further adjust W so that $\bar x \in C_b^\infty(W; \mathbb{R}^n)$ and $\bar\theta \in C_b^\infty(W; \mathbb{R}^m)$. Thus $\Phi'_\xi(\bar x(\xi), \bar\theta(\xi); \xi) = -H'(\xi)$, $\xi \in W$, and therefore, by adding a constant to H if necessary, we can assume that $\Phi(\bar x(\xi), \bar\theta(\xi); \xi) = -H(\xi)$ for $\xi \in W$. We also have that for every k,

$$A_{2k}(D_{x,\theta})(a)(\bar x(\,\cdot\,), \bar\theta(\,\cdot\,)) \in S_n^{r + \frac{m}{2} + \frac{n}{4}}(1).$$

Thus, with $A \in S_n^{r - \frac{n}{4}}(1)$, $A \sim \sum_{k=0}^\infty h^{k + \frac{n}{2} + \frac{m}{2}} (A_{2k}(D_{x,\theta})a)(\bar x(\,\cdot\,), \bar\theta(\,\cdot\,))$, $A = \mathcal{O}(h^\infty)$ in $S_n^{r - \frac{n}{4}}(1)$ outside W, we obtain from (9), (8), and (11) that

$$\mathcal{F}_h(I(a,\varphi))(\xi) = e^{-\frac{iH(\xi)}{h}} A(\xi). \tag{12}$$

Now, the ideal of smooth functions vanishing on Λ_φ is generated by the symbols $a_j(x,\xi) = x_j - H'_{\xi_j}(\xi)$, $j = 1,\dots,n$. Since $I(a,\varphi)$ has compact support, by adjusting V without changing $I(a,\varphi)$ we can assume that Λ_φ is compact and we can choose $\chi \in C_c^\infty(T^*\mathbb{R}^n)$ equal to 1 on a neighborhood of Λ_φ. Then $\widetilde{a}_j = \chi a_j \in S_{2n}(1)$, $j = 1,\dots,n$, vanish on Λ_φ. By the calculus of pseudodifferential operators, we have that $Op_h(\widetilde{a}_j)I(a,\varphi) = Op_h(a_j)I(a,\varphi) + E_jI(a,\varphi)$, where $E_jI(a,\varphi) = \mathcal{O}(h^\infty)$, $h \to 0$, in $C_c(\mathbb{R}^n)$. Therefore

$$\|(Op_h(\widetilde{a}_j))^\alpha(I(a,\varphi))\|_{L^2(\mathbb{R}^n)} = \|(x - H'(hD))^\alpha(I(a,\varphi))\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty).$$

Using Plancherel's theorem, we now obtain

$$\|(Op_h(\widetilde{a}_j))^\alpha(I(a,\varphi))\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi h)^{\frac{n}{2}}}\|(-hD - H')^\alpha \mathcal{F}_h(I(a,\varphi))\|_{L^2(\mathbb{R}^n)} \tag{13}$$

and substituting (12) into (13), we further have that

$$\|(Op_h(\widetilde{a}_j))^\alpha(I(a,\varphi))\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{|\alpha| - r - \frac{n}{4}}), \alpha \in \mathbb{N}^n, h \to 0.$$

Thus $I(a,\varphi) \in I_h^r(\mathbb{R}^n, \Lambda)$.

We remark here that the same argument will allow us in similar situations to use symbols in condition (7) which do not belong to the class $S_{2n}^0(1)$ but vanish on the Lagrangian submanifold Λ, and below we will do so without repeating this argument.

We now turn to proving the converse. Let U, H, W, and V be chosen so that (4) and (5) hold and W is bounded. Extend H to a function in $C_b^\infty(\mathbb{R}^n;\mathbb{R})$. Let $P \in \Psi_h^0(1,\mathbb{R}^n)$ satisfy (1) and set $\widetilde{u} = Pu$. The symbols $a_j(x,\xi) = x_j - H'_{\xi_j}(\xi)$, $j = 1,\dots,n$, vanish on $\Lambda_\varphi \cap U$, and therefore we obtain from (7) that

$$\|(x - H'(hD))^\alpha(\widetilde{u})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{|\alpha| - r - \frac{n}{4}}), \quad \alpha \in \mathbb{N}^n, h \to 0$$

and hence, after taking the Fourier transform,

$$\|(-hD - H')^\alpha \mathcal{F}_h(\widetilde{u})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{|\alpha| + \frac{n}{4} - r}), \quad \alpha \in \mathbb{N}^n, h \to 0. \tag{14}$$

Let $v(\xi) = e^{\frac{iH(\xi)}{h}}\mathcal{F}_h(\widetilde{u})(\xi)$. Substitution into (14) then gives

$$\|(hD)^\alpha v\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{|\alpha| + \frac{n}{4} - r}), \quad \alpha \in \mathbb{N}^n, h \to 0,$$
$$\|D^\alpha v\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{\frac{n}{4} - r}), \quad \alpha \in \mathbb{N}^n.$$

Therefore, $v \in S_n^{r - \frac{n}{4}}(1)$. Let $\Phi(x,\theta;\xi) = \varphi(x,\theta) - \langle x, \xi\rangle$, $\xi \in W$, $(x,\theta) \in V$. Choose $\bar{\xi} \in W$ and let $(\bar{x}(\bar{\xi}), \bar{\theta}(\bar{\xi})) \in C_\varphi$ be the critical point of $\Phi(\cdot,\cdot\cdot;\bar{\xi})$. Let $M \Subset V$ be a neighborhood of $(\bar{x}(\bar{\xi}), \bar{\theta}(\bar{\xi}))$ such that $\operatorname{sgn}\Phi''$ is constant on M, and let $\psi \in C^\infty(M;\mathbb{R}^n)$ be such that $\psi(x,\theta) = \varphi'_x(x,\theta)$ on C_φ. Define

$$\widetilde{a}_0(x,\theta) = \frac{1}{(2\pi h)^{\frac{n+m}{2}}}(e^{-\frac{i\pi}{4}\operatorname{sgn}\Phi''_{x\theta}}|\det \Phi''_{x\theta}|^{\frac{1}{2}} v) \circ \psi(x,\theta) \text{ for } (x,\theta) \in M.$$

Then by the first part of the proof, we have that

$$e^{-\frac{iH}{h}}v - \mathcal{F}_h(I(\widetilde{a}_0,\varphi)) = \mathcal{O}(h), \quad h \to 0, \text{ in } S_n^{r-\frac{n}{4}}(1).$$

Iterating this process, we obtain a sequence of symbols $\widetilde{a}_l \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ with $\operatorname{supp}\widetilde{a}_l \subset M$, $l \in \mathbb{N}_0$, such that if we denote $U_s = I(\sum_{l=0}^{s} h^l\widetilde{a}_l,\varphi)$, $s \in \mathbb{N}_0$, we have that

$$e^{-\frac{iH}{h}}v - \mathcal{F}_h(U_s) = \mathcal{O}(h^{s+1}), \quad h \to 0, \text{ in } S_n^{r-\frac{n}{4}}(1).$$

Therefore, if we choose an asymptotic sum $\widetilde{a} \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ of $\sum_{k=0}^{\infty} h^k\widetilde{a}_k$, we obtain $\widetilde{u} = I(\widetilde{a},\varphi)+\mathcal{O}(h^\infty)$, $h \to 0$, in $C_c^\infty(\mathbb{R}^n)$. It then follows that for some $a \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ with $\operatorname{supp} a \Subset V$ we have $u = I(a,\varphi)$ microlocally near γ.

We now consider the case of a general Lagrangian submanifold $\Lambda \subset T^*\mathbb{R}^n$. Our strategy is to reduce this case to the one just studied by applying a symplectic transformation to Λ that transforms it into a Lagrangian submanifold transverse to the constant section through a point on Λ. More precisely, let $\gamma \in \Lambda$ and assume that the coordinates have been chosen in such a way that $\mu = T_\gamma^*\Lambda$ has the form (6). Choose a real symmetric matrix $A_\Lambda = \begin{bmatrix} 0_{k\times k} & 0 \\ 0 & D_{\Lambda(n-k)\times(n-k)} \end{bmatrix}$ such that

$$\det(B + D_\Lambda) \neq 0. \tag{15}$$

Let $\widetilde{\Lambda} = \{(x,\xi + A_\Lambda x) : (x,\xi) \in \Lambda\}$ and let $\widetilde{\gamma} = (x_0,\xi_0 + A_\Lambda x_0) = (x_0,\eta_0)$, where (x_0,ξ_0) are the coordinates of γ. Then, if $\varphi \in C^\infty(V;\mathbb{R})$, $V \subset \mathbb{R}^{n+m}$, $m \in \mathbb{N}_0$, is a non-degenerate phase function which parameterizes Λ near γ, it is clear that $\widetilde{\varphi}(x,\theta) = \frac{1}{2}\langle A_\Lambda x, x\rangle + \varphi(x,\theta)$ is a non-degenerate phase function which parameterizes $\widetilde{\Lambda}$ near $\widetilde{\gamma}$.

Let $\widetilde{\mu} = T_{\widetilde{\gamma}}\widetilde{\Lambda}$. It is easy to see that $\widetilde{\mu} = \{(0,x'';\xi',(B+D_\Lambda)x'')\}$. It then follows from (15) that $\widetilde{\Lambda}$ is transverse to the constant section $\eta = \eta_0$ at $\widetilde{\gamma}$.

Let $u \in I_h^r(\mathbb{R}^n,\Lambda)$ and let $A_l \in \Psi_h^0(1,\mathbb{R}^n)$, $l = 0,\dots,N$, be such that $\sigma_0(A_l)|_\Lambda = 0$, $l = 0,\dots,N-1$, and $\text{ess-supp}_h\,\sigma(A_l)$, $l = 0,\dots,N$, is compact. From (7) we have that

$$\Big(\prod_{l=0}^{N} e^{\frac{i}{2h}\langle A_\Lambda\cdot,\cdot\rangle} A_l e^{-\frac{i}{2h}\langle A_\Lambda\cdot\cdot,\cdot\cdot\rangle}\Big) e^{\frac{i}{2h}\langle A_\Lambda\cdot\cdot,\cdot\cdot\rangle} u = \mathcal{O}_{L^2(\mathbb{R}^n)}(h^{N-r-\frac{n}{4}}), h \to 0.$$

We shall now prove that for $a \in S_{2n}^0(1)$ with $\text{ess-supp}_h\, a$ compact, the operator

$$B = e^{\frac{i}{2h}\langle A_\Lambda\cdot,\cdot\rangle} Op_h(a) e^{-\frac{i}{2h}\langle A_\Lambda\cdot\cdot,\cdot\cdot\rangle}$$

is a semi-classical pseudodifferential operator.

We let $\bar{a}$ denote the Weyl symbol of $Op_h(a)$, *i.e.*, $\bar{a} \in S_{2n}^0(1)$ is such that

$$Op_h^w(\bar{a}) \overset{\text{def}}{=} \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\langle x-y,\xi\rangle}\bar{a}\Big(\frac{x+y}{2},\xi\Big)\, d\xi = Op_h(a).$$

Making the change of variable $\eta = \xi + \frac{1}{2}A_\Lambda(x+y)$, we obtain that

$$B = e^{\frac{i}{2h}\langle A_\Lambda x,x\rangle} Op_h^w(\bar{a}) e^{-\frac{i}{2h}\langle A_\Lambda y,y\rangle} = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\langle x-y,\eta\rangle}\bar{a}\Big(\frac{x+y}{2}, \eta - A_\Lambda \frac{x+y}{2}\Big)\, d\eta.$$

Thus $B \in \Psi_h^0(1,\mathbb{R}^n)$ and its Weyl symbol $\bar{b}$ satisfies $\bar{b}(x,\xi) = \bar{a}(x,\xi - A_\Lambda x)$.

We now recall that

$$\bar{a}(x,\xi) = e^{-\frac{ih}{2}D_x D_\xi} a(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}h^{|\alpha|}}{(2i)^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_x^\alpha a(x,\xi) \tag{16}$$

in $S^{0,-\infty}(T^*\mathbb{R}^n)$ and in $S^0_{2n}(1)$ (see [8, Remark 2.7.3]), and therefore ess-supp$_h\,\bar{a}$ is compact. Setting $b = \sigma(B)$, we have similarly that ess-supp$_h\, b$ is compact. Furthermore, as in (16), we obtain that

$$b_0(x,\xi) = \bar{b}_0(x,\xi) = a_0(x,\xi - A_\Lambda x). \tag{17}$$

Thus $B_l = e^{\frac{i}{2h}\langle A_\Lambda \cdot,\cdot\rangle} A_l e^{-\frac{i}{2h}\langle A_\Lambda \cdot,\cdot\rangle} \in \Psi_h^0(1,\mathbb{R}^n)$, $l = 0,\dots,N$, and $WF_h(B_l)$, $l = 0,\dots,N$, is compact. From (17) we also have that $\sigma_0(B_l)|_{\widetilde{\Lambda}} = 0$, $l = 0,\dots,N-1$.

We can now apply the first part of this proof and we have that $Pe^{\frac{i}{2h}\langle A_\Lambda\cdot,\cdot\rangle}u = I(\widetilde{a},\widetilde{\varphi})$, $\widetilde{a} \in S^{r+\frac{m}{2}+\frac{n}{4}}_{n+m}(1)$, where $P \in \Psi_h^0(1,\mathbb{R}^n)$ satisfies (1). Therefore $u \equiv I(a,\varphi)$ near γ for some $a \in S^{r+\frac{m}{2}+\frac{n}{4}}_{n+m}(1)$.

The converse follows from reversing this argument. ■

Remark Let $u \in I_h^r(M,\Lambda)$. Then Theorem 4.4 and Lemma 4.3 imply that for any $P \in \Psi_h^0(1,M)$ with compact wavefront set contained in a sufficiently small neighborhood of a point $\rho \in \Lambda$, Pu is given by a finite sum of oscillatory integrals of the form $I(a,\varphi)$ for any non-degenerate phase function $\varphi \in C^\infty(\mathbb{R}^{n+m})$ for some $m \in \mathbb{N}_0$ such that $\Lambda = \Lambda_\varphi$ near $\rho \in \Lambda$ and some $a \in S^{r+\frac{m}{2}+\frac{n}{4}}_{n+m}(1)$. Applying an integration by parts argument to each of the oscillatory integrals $I(\alpha,\varphi)$ as in [6, Chapter 7], we then have the following lemma.

Lemma 4.5 *If $u \in I_h^r(M,\Lambda)$ and $P \in \Psi_h^0(1,M)$ has compact wavefront set contained in a sufficiently small neighborhood of a point in Λ, then $WF_h^f(Pu) \subset \Lambda$.*

5 Generalization of Egorov's Theorem

We now prove a generalization of Egorov's theorem to manifolds of unequal dimensions. To state our result, we need the following notion of microlocal equivalence of Fourier integral operators.

Definition 5.1 Let M_j, $j = 1,2$, be smooth manifolds, $\Lambda \subset T^*M_2 \times T^*M_1$ a Lagrangian submanifold, and $T, T' \in \mathcal{I}_h^r(M_2 \times M_1,\Lambda)$ for some $r \in \mathbb{R}$. For open

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or closed sets $U \subset T^*M_1$ and $V \subset T^*M_2$, the operators T and T' are said to be *microlocally equivalent* near $V \times U$ if there exist open sets $\widetilde{U} \Subset T^*M_1$ and $\widetilde{V} \Subset T^*M_2$ with $\overline{U} \Subset \widetilde{U}$ and $\overline{V} \Subset \widetilde{V}$ such that for any $A \in \Psi_h^0(1, M_1)$ and $B \in \Psi_h^0(1, M_2)$ with $WF_h(A) \subset \widetilde{U}$ and $WF_h(B) \subset \widetilde{V}$ we have that

$$B(T - T')A = \mathcal{O}(h^\infty)\colon \mathcal{D}_h'(M_1) \to C^\infty(M_2).$$

We shall also write $T \equiv T'$ near $V \times U$.

We remark again that this definition implicitly requires that $WF_h(A)$ and $WF_h(B)$ have no infinite parts.

Lastly, we introduce the map $\upsilon\colon T^*M \to T^*M$, $(x, \xi) \mapsto (x, -\xi)$, for a smooth manifold M.

Theorem 5.2 *Let M_j, $j = 1, 2$, be smooth manifolds with* $\dim M_j = n_j$, $j = 1, 2$. *Let σ_j be the canonical symplectic form on T^*M_j, and $\pi_j\colon T^*M_2 \times T^*M_1 \to T^*M_j$ the projection onto the j-th factor, $j = 1, 2$. Let $\Lambda \subset T^*M_2 \times T^*M_1$ be a Lagrangian submanifold of $(T^*M_2 \times T^*M_1, \pi_2^*\sigma_2 + \pi_1^*\sigma_1)$ such that $\pi_1|_\Lambda$ is an immersion. Let $F \in \mathcal{I}_h^r(M_2 \times M_1, \Lambda)$, $r \in \mathbb{R}$, have a non-empty finite semi-classical wavefront set near $(\rho_2, \rho_1) \in \Lambda$.*

Then for every $A \in \Psi_h^0(1, M_2)$ with wavefront set contained in a sufficiently small neighborhood of ρ_2 there exists $B \in \Psi_h^0(1, M_1)$ with wavefront set contained in a sufficiently small neighborhood of $\upsilon(\rho_1)$ such that

$$AF \equiv FB \text{ near } (\rho_2, \upsilon(\rho_1)) \quad \text{and} \quad i^{n_1}(\pi_1|_\Lambda)^*\upsilon^*\sigma_0(B) = i^{n_2}(\pi_2|_\Lambda)^*\sigma_0(A).$$

Remark Strictly speaking, we have not defined a symbol of a Fourier integral operator given in Definition 4.1. However, the proof of Theorem 4.4 shows that the non-vanishing of the amplitude given there is invariantly defined.

Proof By a partition of unity we can reduce the proof to the local case where $M_j \subset \mathbb{R}^{n_j}$, T^*M_j is trivial for $j = 1, 2$, $F = \int e^{\frac{i}{h}\varphi(x,z,\theta)} u(x, z, \theta; h)\, d\theta$, where φ is a non-degenerate phase function in a neighborhood of $(x_0, z_0, \theta_0) \in M_2 \times M_1 \times \mathbb{R}^m$ for some $m \in \mathbb{N}_0$ such that $\Lambda \cap U = \Lambda_\varphi$ for an open set U with $(x_0, \xi_0\,; z_0, \eta_0) \in \Lambda \cap U$, $u \in S_{n_1+n_2+m}^{\frac{n_1+n_2}{4}+\frac{m}{2}+r}(1)$ with compact support, $u \sim \sum_{k=0}^\infty h^{k-\frac{n_1+n_2}{4}-\frac{m}{2}-r} u_k$, $u_k \in S_{n_1+n_2+m}^0(1)$, and $A = \int_{\mathbb{R}^{n_2}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x, \xi)\, d\xi$ where $a \in S_{2n_2}^0(1)$ has compact support and $a \sim \sum_{k=0}^\infty h^k a_k$, $a_k \in S_{2n_2}^0(1)$, $k \in \mathbb{N}$.

Let $\Phi(y, \xi\,; x, z, \theta) = \langle x - y, \xi\rangle + \varphi(y, z, \theta)$. Then Φ has a critical point (with respect to the variables $p = (y, \xi)$)

$$p_0(x, z, \theta) = (y_0(x, z, \theta), \xi_0(x, z, \theta)) = (x, \varphi_x'(x, z, \theta)).$$

The Hessian of Φ is

$$\Phi''(y_0(x, z, \theta), \xi_0(x, z, \theta); x, z, \theta) = \begin{bmatrix} \varphi_{xx}''(x, z, \theta) & -I \\ -I & 0 \end{bmatrix},$$

and has determinant 1 and signature 0.

Let $\Psi(w,\eta;x,z,\theta) = \varphi(x,w,\theta) + \langle w-z,\eta\rangle$. Then Ψ has a critical point (with respect to the variables $q=(w,\eta)$)

$$q_0(x,z,\theta) = (w_0(x,z,\theta),\eta_0(x,z,\theta)) = (z,-\varphi'_z(x,z,\theta)).$$

The Hessian of Ψ is

$$\Psi''(w_0(x,z,\theta),\eta_0(x,z,\theta);x,z,\theta) = \begin{bmatrix} \varphi''_{zz}(x,z,\theta) & I \\ I & 0 \end{bmatrix},$$

and has determinant 1 and signature 0.

We define

$$g_{p_0(x,z,\theta)}(p) = \Phi(p;x,z,\theta) - \Phi(p_o(x,z,\theta);x,z,\theta) - \frac{\langle \Phi''(p_0(x,z,\theta);x,z,\theta)(p-p_0(x,z,\theta)),p-p_0(x,z,\theta)\rangle}{2}$$

and

$$f_{q_0(x,z,\theta)}(q) = \Psi(q;x,z,\theta) - \Psi(q_o(x,z,\theta);x,z,\theta) - \frac{\langle \Psi''(q_0(x,z,\theta);x,z,\theta)(q-q_0(x,z,\theta)),q-q_0(x,z,\theta)\rangle}{2}.$$

For $j \in \mathbb{N}_0$ and $c_l \in C_c^\infty(\mathbb{R}^{2n_2})$, set

$$(L_j^l(c_l u))(x,z,\theta) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu} \frac{\langle (\Phi''(p_0(x,z,\theta);x,z,\theta))^{-1}D,D\rangle^\nu}{i^j 2^\nu \mu!\nu!} \left(g^\mu_{p_0(x,z,\theta)}(\cdot,\cdot\cdot)c_l(x,\cdot\cdot)u(\cdot,z,\theta)\right)(p_0(x,z,\theta)).$$

For $j \in \mathbb{N}_0$ and $c_r \in C_c^\infty(\mathbb{R}^{2n_1})$, set

$$(L_j^r(uc_r))(x,z,\theta) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu} \frac{\langle (\Psi''(q_0(x,z,\theta);x,z,\theta))^{-1}D,D\rangle^\nu}{i^j 2^\nu \mu!\nu!} \left(f^\mu_{q_0(x,z,\theta)}(\cdot,\cdot\cdot)c_r(\cdot,\cdot\cdot)u(x,\cdot,\theta)\right)(q_0(x,z,\theta)).$$

Now, since $\pi_1|_\Lambda$ is an immersion, it follows from the inverse function theorem, that there exists $\kappa \in C^\infty(T^*\mathbb{R}^{n_1}; T^*\mathbb{R}^{n_2}\times T^*\mathbb{R}^{n_1})$, such that $\kappa\circ\pi_1|_\Lambda = \mathrm{id}|_\Lambda$. Now let $b_0 \in C_c^\infty(\mathbb{R}^{2n_1})$ be supported near $\upsilon(\rho_1)$ and satisfy $b_0 = i^{n_2-n_1}\upsilon^*\kappa^*(\pi_2|_\Lambda)^*a_0$. Then $i^{n_2} j_\varphi^*(\pi_2|_\Lambda)^*a_0 - i^{n_1} j_\varphi^*(\pi_1|_\Lambda)^*\upsilon^* b_0$ vanishes on C_φ. Since φ is a non-degenerate phase function, by the implicit function theorem we have that the functions $\{\varphi'_{\theta_1},\dots,\varphi'_{\theta_m}\}$ participate in a coordinate system $\{w_1,\dots,w_{n_1+n_2+m}\}$ on $\mathbb{R}^{n_1+n_2+m}$ near (x_0,z_0,θ_0) in

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which $C_\varphi \equiv \{(w_1, \dots, w_{n_1+n_2+m}) : w_1 = \dots = w_m = 0\}$. Therefore, by Taylor's theorem, it follows that there exist $c_j^0 \in S^0_{n_1+n_2+m}(1)$, $j = 1, \dots, m$, with compact support, such that

$$i^{n_2} j_\varphi^*(\pi_2|_\Lambda)^* a_0 - i^{n_1} j_\varphi^*(\pi_1|_\Lambda)^* \upsilon^* b_0 = \sum_{j=1}^m \varphi'_{\theta_j} c_j^0$$

near (x_0, z_0, θ_0).

For every $k > 0$ we now choose $b_k \in S^0_{2n_1}(1)$ with compact support in such a way that

$$\Big(i^{n_2} \sum_{\alpha+\beta+\gamma=k} L^l_\alpha(a_\gamma u_\beta) - i^{n_1} \sum_{\alpha+\beta+\gamma=k,} L^r_\alpha(u_\beta b_\gamma) - \sum_{l=1}^m D_{\theta_l}(c_l^{k-1} u_0) \Big)\Big|_{C_\varphi} = 0$$

and choose $c_l^k \in S^0_{n_1+n_2+m}(1)$, $l = 1, \dots, m$, with compact support such that

$$i^{n_2} \sum_{\alpha+\beta+\gamma=k} L^l_\alpha(a_\gamma u_\beta) - i^{n_1} \sum_{\alpha+\beta+\gamma=k,} L^r_\alpha(u_\beta b_\gamma) - \sum_{l=1}^m D_{\theta_l}(c_l^{k-1} u_0) = \sum_{l=1}^m \varphi'_{\theta_l} c_l^k u_0.$$

Lastly, let $b \in S^0_{2n_1}(1)$ be such that $b \sim \sum_{j=0}^\infty h^j b_j$ and set $B = Op_h(b)$. We shall prove that $AF \equiv FB$ near $(\rho_2, \upsilon(\rho_1))$.

For that, in the integrals

$$AF(x,z) = \frac{1}{(2\pi h)^{n_2}} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) e^{\frac{i}{h}\varphi(y,z,\theta)} u(y,z,\theta)\, d\theta dy d\xi,$$
$$FB(x,z) = \frac{1}{(2\pi h)^{n_2}} \iint e^{\frac{i}{h}\varphi(x,w,\theta)} u(x,w,\theta) e^{\frac{i}{h}\langle w-z,\eta\rangle} b(w,\eta)\, d\theta dw d\eta.$$

we apply the method of stationary phase (see [7, Theorem 7.7.5]) in the (y,ξ) and the (w,η) variables, respectively, to obtain

$$AF(x,z) \sim i^{n_2} \sum_{t=0}^\infty h^{t-\frac{n_1+n_2}{4}-\frac{m}{2}-r} \sum_{\alpha+\beta+\gamma=t} \int e^{\frac{i}{h}\varphi(x,z,\theta)} (L^l_\alpha(a_\gamma u_\beta))(x,z,\theta)\, d\theta,$$
$$FB(x,z) \sim i^{n_1} \sum_{t=0}^\infty h^{t-\frac{n_1+n_2}{4}-\frac{m}{2}-r} \sum_{\alpha+\beta+\gamma=t} \int e^{\frac{i}{h}\varphi(x,z,\theta)} (L^r_\alpha(u_\beta b_\gamma))(x,z,\theta)\, d\theta.$$

Therefore

$$(AF - FB)(x,z) \sim \sum_{t=0}^\infty h^{t-\frac{n_1+n_2}{4}-\frac{m}{2}-r} d_t(x,z)$$

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with

$$d_t(x,z)=\int e^{\frac{i}{h}\varphi(x,z,\theta)}\sum_{\alpha+\beta+\gamma=t}[i^{n_2}(L^l_\alpha(a_\gamma u_\beta))(x,z,\theta)-i^{n_1}(L^r_\alpha(u_\beta b_\gamma))(x,z,\theta)]\,d\theta,$$

for $t\in\mathbb{N}_0$.

By the choice of b_0 we now have

$$\begin{aligned}d_0(x,z)&=\int e^{\frac{i}{h}\varphi(x,z,\theta)}u_0(x,z,\theta)[i^{n_2}a_0(x,\varphi'_x(x,z,\theta))-i^{n_1}b_0(z,-\varphi'_z(x,z,\theta))]\,d\theta\\&=\int e^{\frac{i}{h}\varphi(x,z,\theta)}\sum_{j=1}^m\varphi'_{\theta_j}(x,z,\theta)c^0_j(x,z,\theta)u_0(x,z,\theta)\,d\theta\\&=-h\int e^{\frac{i}{h}\varphi(x,z,\theta)}\sum_{j=1}^m D_{\theta_j}(c^0_j(x,z,\cdot)u_0(x,z,\cdot))(\theta)\,d\theta,\end{aligned}$$

using integration by parts.

By the choice of the symbol b_1 and the functions c^1_l, $l=1,\dots,m$, we then obtain, as above,

$$d_0(x,z)+hd_1(x,z)=-h^2\int e^{\frac{i}{h}\varphi(x,z,\theta)}\sum_{j=1}^m D_{\theta_j}(c^1_j(x,z,\cdot)u_0(x,z,\cdot))(\theta)\,d\theta.$$

Iterating this argument, we have from the choice of the symbols b'_ks that

$$AF\equiv FB \text{ near } (\rho_2,\upsilon(\rho_1)).$$ ■

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