THE LAWS OF SOME NILPOTENT GROUPS OF SMALL RANK

Dedicated to the memory of Hanna Neumann

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We shall take for granted the basic terminology currently in use in the theory of varieties of groups. Kovács, Newman, Pentony [2] and Levin [3] prove that if m is an integer greater than 2, then the variety $\mathbf{N}_m$ of all nilpotent groups of class at most m is generated by its free group $F_{m-1}(\mathbf{N}_m)$ of rank $m-1$ but not by its free group $F_{m-2}(\mathbf{N}_m)$ of rank $m-2$. That is, the free groups $F_k(\mathbf{N}_m)$, $2 \leqq k \leqq m-2$, do not generate $\mathbf{N}_m$. In general little is known of the varieties generated by them. The purpose of the present paper is to record the varieties of the free groups $F_k(\mathbf{N}_m)$ of the nilpotent varieties $\mathbf{N}_m$ of all nilpotent groups of class at most m for $2 \leqq k \leqq m-2$ and $5 \leqq m \leqq 6$. This is done by describing a basis for the laws in these groups, that is a set of laws the fully invariant closure of which is the set of all laws for $F_k(\mathbf{N}_m)$. The set of laws, which, together with the appropriate nilpotency law, form a basis for the relevant groups $F_k(\mathbf{N}_m)$ are listed below:

$F_3(\mathbf{N}_5)$: $[[x_4,x_1,x_5],[x_3,x_2]]^{-1}[[x_4,x_2,x_5],[x_3,x_1]]\,[[x_4,x_3,x_5],[x_2,x_1]]^{-1}$
$[[x_3,x_1,x_5],[x_4,x_2]]\,[[x_3,x_2,x_5],[x_4,x_1]]^{-1}[[x_2,x_1,x_5],[x_4,x_3]]^{-1}$.

$F_2(\mathbf{N}_5)$: $[[x_2,x_1,x_5],[x_4,x_3]]\,[[x_4,x_3,x_5],[x_2,x_1]]^{-1}$,
$[[x_2,x_1,x_5],[x_4,x_3]]\,[[x_3,x_2,x_5],[x_4,x_1]]\,[[x_3,x_1,x_5],[x_4,x_2]]^{-1}$.

$F_4(\mathbf{N}_6)$: (i) $[[x_6,x_5],[x_2,x_1],[x_4,x_3]]^{-2}[[x_6,x_5],[x_3,x_1],[x_4,x_2]]^{2}$
$[[x_6,x_5],[x_3,x_2],[x_4,x_1]]^{-2}[[x_4,x_3],[x_2,x_1],[x_6,x_5]]$
$[[x_4,x_2],[x_3,x_1],[x_6,x_5]]^{-1}[[x_4,x_1],[x_3,x_2],[x_6,x_5]]$
$[[x_5,x_4],[x_3,x_2],[x_6,x_1]]^{-1}[[x_5,x_3],[x_4,x_2],[x_6,x_1]]$
$[[x_5,x_2],[x_4,x_3],[x_6,x_1]]^{-1}[[x_5,x_4],[x_3,x_1],[x_6,x_2]]$
$[[x_5,x_3],[x_4,x_1],[x_6,x_2]]^{-1}[[x_5,x_1],[x_4,x_3],[x_6,x_2]]$

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$$[[x_5,x_4],[x_2,x_1],[x_6,x_3]]^{-1}[[x_5,x_2],[x_4,x_1],[x_6,x_3]]$$
$$[[x_5,x_1],[x_4,x_2],[x_6,x_3]]^{-1}[[x_5,x_3],[x_2,x_1],[x_6,x_4]]$$
$$[[x_5,x_2],[x_3,x_1],[x_6,x_4]]^{-1}[[x_5,x_1],[x_3,x_2],[x_6,x_4]],$$

(ii) $[[x_6,x_4],[x_2,x_1],[x_5,x_3]]^{-1}[[x_6,x_4],[x_3,x_1],[x_5,x_2]]$
$[[x_6,x_4],[x_3\,x_2],[x_5,x_1]]^{-1}[[x_6,x_5],[x_2,x_1],[x_4,x_3]]$
$[[x_6,x_5],[x_3,x_1],[x_4,x_2]]^{-1}[[x_6,x_5],[x_3,x_2],[x_4,x_1]]$
$[[x_5,x_4],[x_3,x_2],[x_6,x_1]]\,[[x_5,x_4],[x_3,x_1],[x_6,x_2]]^{-1}$
$[[x_5,x_4],[x_2,x_1],[x_6,x_3]]$.

$F_3(\mathbf{N}_6)$: (i) $[[x_4,x_3,x_6],[x_2,x_1,x_5]]\,[[x_4,x_3,x_5],[x_2,x_1,x_6]]^{-1}$
$[[x_4,x_2,x_6],[x_3,x_1,x_5]]^{-1}[[x_4,x_2,x_5],[x_3,x_1,x_6]]$
$[[x_4,x_1,x_6],[x_3x_2x_5]]\,[[x_4,x_1,x_5],[x_3,x_2,x_6]]^{-1}$,

(ii) $[[x_4,x_3,x_5,x_6],[x_2,x_1]]\,[[x_4,x_2,x_5,x_6],[x_3,x_1]]^{-1}$
$[[x_4,x_1,x_5,x_6],[x_3,x_2]]\,[[x_3,x_2,x_5,x_6],[x_4,x_1]]$
$[[x_3,x_1,x_5,x_6],[x_4,x_2]]^{-1}[[x_2,x_1,x_5,x_6],[x_4,x_3]]$,

(iii) $[[x_6,x_5],[x_2,x_1],[x_4,x_3]]^2[[x_6,x_5],[x_3,x_1],[x_4,x_2]]^{-2}$
$[[x_6,x_5],[x_3,x_2],[x_4,x_1]]^2[[x_4,x_3],[x_2,x_1],[x_6,x_5]]^{-1}$
$[[x_4,x_2],[x_3,x_1],[x_6,x_5]]\,[[x_4,x_1],[x_3,x_2],[x_6,x_5]]^{-1}$.

(iv) $[[x_5,x_4],[x_3,x_2],[x_6,x_1]]^{-1}[[x_5,x_3],[x_4,x_2],[x_6,x_1]]$
$[[x_5,x_2],[x_4,x_3],[x_6,x_1]]^{-1}[[x_5,x_4],[x_3,x_1],[x_6,x_2]]$
$[[x_5,x_3],[x_4,x_1],[x_6,x_2]]^{-1}[[x_5,x_1],[x_4,x_3],[x_6,x_2]]$
$[[x_5,x_4],[x_2,x_1],[x_6,x_3]]^{-1}[[x_5,x_2],[x_4,x_1],[x_6,x_3]]$
$[[x_5,x_1],[x_4,x_2],[x_6,x_3]]^{-1}[[x_5,x_3],[x_2,x_1],[x_6,x_4]]$
$[[x_5,x_2],[x_3,x_1],[x_6,x_4]]^{-1}[[x_5,x_1],[x_3,x_2],[x_6,x_4]]$,

(v) same as (ii) unde. $F_4(\mathbf{N}_6)$.

$F_2(\mathbf{N}_6)$: (i) $[[x_4,x_3,x_5],[x_2,x_1,x_6]]\,[[x_4,x_3,x_6],[x_2,x_1,x_5]]$,

(ii) $[[x_2,x_1,x_5,x_\epsilon],[x_4,x_3]]\,[[x_4,x_3,x_5,x_6],[x_2x_1]]^{-1}$,

(iii) $[[x_4,x_3,x_6],[x_2,x_1,x_5]]\,[[x_4,x_2,x_6],[x_3,x_1,x_5]]^{-1}$
$[[x_4,x_1,x_6],[x_3,x_2,x_5]]$,

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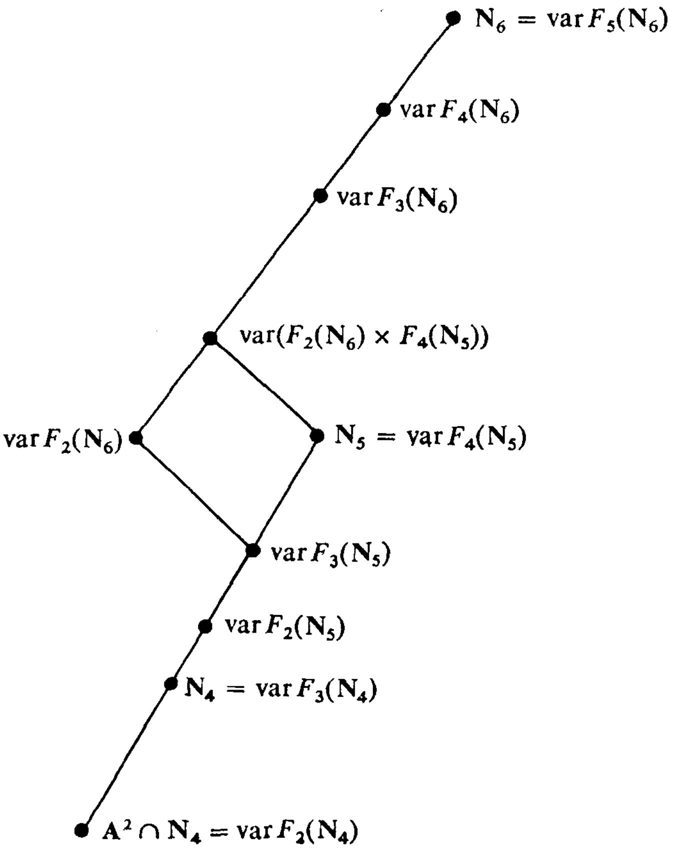
(iv) $[[x_3,x_2,x_5,x_6], [x_4,x_1]]^{-1}[[x_3,x_1,x_5,x_6], [x_4,x_2]]$
$[[x_2,x_1,x_5,x_6], [x_4,x_3]]^{-1}$,

(v) $[[x_1,x_2], [x_3,x_4], [x_5,x_6]]$,

(vi) $[[x_6,x_4,x_5], [x_2,x_1,x_3]]^{-1}[[x_5,x_3,x_6], [x_2,x_1,x_4]]$,

(vii) same as that under $F_3(\mathbf{N}_5)$.

We exhibit the lattice formed by the varieties generated by the above sets o laws.



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