ON THE DUAL OF L^1

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(received February 9, 1965)

1. If (X, S, μ) is an arbitrary complemented measure space and X is σ-finite then $(L^1)^* = L^\infty$ or, more precisely, $(L^1)^*$ is isometric and isomorphic to L^∞ by the correspondence

$$G(f) = \int fg \, d\mu, \quad G \in (L^1)^*, \quad g \in L^\infty .$$

It is well known that there exist non σ-finite spaces with $(L^1)^* \neq L^\infty$.

In the Bourbaki theory of measure and integration it is always true that $(L^1)^* = L^\infty$ [2, p. 55]. However, measurability in the Bourbaki sense is a local property: a function is measurable if and only if its restriction to each compact set is measurable. For the non-topological general case a function is called locally measurable in [3] if its restriction to each measurable set of finite positive measure is measurable and L_ℓ^∞ denotes the analogue of L^∞ for locally measurable functions with norm $\eta_\ell^\infty(f) = \sup \{\eta^\infty(f\chi_e); \; e \in S, \; \mu(e) < \infty\}$. Always $(L^1)^* \supset L_\ell^\infty$ but, as was shown in [3], strict inequality may hold. In this note we extend [3] by proving

THEOREM 1. Every $L^1(X, S, \mu)$ is isometric and isomorphic to a space $\bar{L}^1 = L^1(\bar{X}, \bar{S}, \bar{\mu})$ with $(\bar{L}^1)^* = \bar{L}_\ell^\infty(\bar{X}, \bar{S}, \bar{\mu})$.

1) Partly supported by a National Research Council (Canada) Senior Research Fellowship.

2. When X is σ-finite, $X = \bigcup_{i=1}^{\infty} X_i$, with $0 < \mu(X_i) < \infty$, and $X_i \cap X_j = \emptyset$, (or $\mu(X_i \cap X_j) = 0$) $i \neq j$, and the general representation theorem is obtained as an easy extension of the finite theorem for each X_i. In [3], in studying the non σ-finite case, two decompositions of X were obtained using Zorn's lemma: the first into disjoint sets ((D) decomposition) and the second into null-disjoint sets (ND). For both

$$X = X_1 \cup X_2; \; X_1 \cap X_2 = \emptyset; \; X_2 = \bigcup_{a \in \mathcal{Q}} e_a, \quad 0 < \mu(e_a) < \infty;$$

$e \in S$, $e \subset X_1$ implies that $\mu(e) = 0$ or ∞.

In addition for

(D) $\quad e_a \cap e_{a'} = \emptyset, \quad a \neq a'$;

(ND) $\mu(e_a \cap e_{a'}) = 0, \quad a \neq a'$.

An additional property satisfied in the σ-finite case is

(*). For each $e \in S$ with $\mu(e) < \infty$,

$$\mu(e) = \sum_{a \in \mathcal{Q}} \mu(e \cap e_a) = \sup \Sigma \, \mu(e \cap e_a),$$

the supremum being taken over all finite sums of this form, as in [3]. For every (ND) decomposition (*) holds, but there exist measure spaces in which there is no (D) decomposition with (*) satisfied. In [3] it was noted that the existence of a (D*) (i.e., (D) plus (*)) decomposition implies that $(L^1)^* = L^{\infty}_{\ell}$. In the Bourbaki theory a (D*) decomposition always exists [2, § 1, 4].

To prove Theorem 1, let $X = X_1 \cup X_2$, $X_2 = \bigcup_{a \in \mathcal{Q}} e_a$ be an arbitrary (ND) decomposition of X and let

https://doi.org/10.4153/CMB-1965-061-x Published online by Cambridge University Press

$$\bar{X} = \bigcup_{a \in \mathcal{Q}} (\{a\} \times e_a) .$$

Let h denote the relation with domain $X - X_1$ and range $\bar{X}$ determined by the ordered pairs (x, y) where, for each $x \in X - X_1$, y runs through the points (a, x) of $\bar{X}$ with $x \in e_a$, $a \in \mathcal{Q}$. If the (ND)-decomposition is actually a (D)-decomposition, h will be a function, but in general this will not be the case. For a set $e \subset X - X_1$, $h(e)$ will denote the image set in $\bar{X}$.

We observe that $h(\bigcup e_\alpha) = \bigcup h(e_\alpha)$, $h(\bigcap e_\alpha) = \bigcap h(e_\alpha)$ for an arbitrary collection $\{e_\alpha\}$. For each a, $x \in e_a$, set $h_a(x) = (a, x)$. The mappings $h_a : e_a \to \{a\} \times e_a$ are bijective. If $e \subset e_a$, set $h_a(e) = \{a\} \times e$.

To illustrate the preceding definitions consider the example: $X = \{(x, y) : 0 \le x < 1,\ 0 \le y < 1\}$, S the smallest σ-algebra containing all the Lebesgue measurable subsets of every $I_x = \{(x, y) : 0 \le y < 1\}$ and every $I_y = \{(x, y) : 0 \le x < 1\}$. (Compare [3], p. 222.) Then

$$X = \Big(\bigcup_{0 \le x < 1} I_x \Big) \cup \Big(\bigcup_{0 \le y < 1} I_y \Big)$$

is an (ND)-decomposition of X, and $\bar{X}$ can be represented as

$$\bar{X} = \{(u, v) : 0 \le u < 2,\ 0 \le v < 1\} ,$$

where I_x is identified with I_u if $u = x < 1$ and I_y is identified with I_u if $y = u - 1$, $1 \le u < 2$. The image of (x, y) in X under the relation h consists of two points: (x, y) and $(y + 1, x)$ in $\bar{X}$; the image of $I_x \subset X$ consists of two intervals: I_u, $u = x$ and $\{(u, v) : 1 \le u < 2,\ v = x\}$.

To return to the general case, we shall determine an outer measure on $\bar{X}$ by means of a covering class C ([5], p. 91) consisting of $\bar{X}$ and, for each $a \in \mathcal{Q}$, all of the sets $h(e)$, $h_a(e)$ and $h(e) - h_a(e)$ for each measurable subset e of e_a. We set $\nu^1(\overline{X}) = \infty$ and

$$\nu^1(h(e)) = \nu^1(h_a(e)) = \mu(e), \quad \nu^1(h(e) - h_a(e)) = 0$$

for each such e in C, and define for each $A \subset \overline{X}$

$$\nu^*(A) = \inf \Sigma\, \nu^1(\overline{e}_i),$$

where the infimum is taken in the extended reals for all collections $\{\overline{e}_i\}$ of sets in C covering A.

We note that if A intersects more than a countable collection of sets $h(e_a)$, $a \in \mathcal{Q}$, then every covering must contain $\overline{X}$ so that $\nu^1(A) = \infty$. We note also that the definition makes every set of the form $h(e)-h_a(e)$ ν^*-null. For the example given above this implies that every subset of a horizontal line in $\overline{X}$ is ν^*-null.

Let $\overline{S}$ denote the ν^*-measurable subsets of $\overline{X}$, $\overline{\mu}$ the restriction of ν^* to $\overline{S}$. We shall show that Theorem 1 is true for the measure space $(\overline{X}, \overline{S}, \overline{\mu})$. The notation $\overline{x}$, $\overline{L}^1$, $\overline{\eta}^1$, $\overline{e}$, etc. will refer to the measure space $(\overline{X}, \overline{S}, \overline{\mu})$.

LEMMA. If $e \in S$, $e \subset e_a$, $a \in \mathcal{Q}$, then $h(e)$ and $h_a(e) \in \overline{S}$ with $\overline{\mu}(h(e)) = \overline{\mu}(h_a(e)) = \mu(e)$. If $\overline{e} \in \overline{S}$ and $\overline{e} \subset \{a\} \times e_a$, then there exists $e \subset e_a$ with $\overline{e} = h_a(e)$ and sets $e_1, e_2 \in S$ with $e_1 \subset e \subset e_2$, $\mu(e_2 - e_1) = 0$. In particular if the measure space $(\overline{X}, \overline{S}, \overline{\mu})$ is complete $e \in S$.

<u>Proof.</u> To show that $h(e) \in \overline{S}$ we must show that for every $A \subset \overline{X}$ with $\nu^*(A) < \infty$,

$$\nu^*(A) \geq \nu^*(A \cap h(e)) + \nu^*(A \cap \complement h(e)).$$

Assume that $e \subset e_a$ and that each covering set $\overline{e}_i$ is of the form $h(e_i)$. Then for each i,

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$e_i = (e_i \cap e) \cup (e_i \cap Ce)$, $h(e_i) = h(e_i \cap e) \cup h(e_i \cap Ce)$,

$\nu^1(h(e_i)) = \nu^1(h(e_i \cap e)) + \nu^1(h(e_i \cap Ce)) = \mu(e_i \cap e) + \mu(e_i \cap Ce)$,

since ν^1 is additive if $e_i \subset e_a$ and $\mu(e \cap e_i) = 0$ if $e_i \subset e_b$, $b \neq a$. Thus each covering of A can be replaced, without changing $\Sigma \nu^1(e_i)$, by a covering that splits into coverings of $A \cap h(e)$ and $A \cap Ch(e)$. A standard argument then shows that $h(e) \in \bar{S}$. We have assumed above that the covering sets were all of the form $h(e_i)$. In the general case, sets of the form $h_a(e_i)$ in the covering could be replaced by the sets $h(e_i)$ without changing the sum and sets of the form $h(e_i) - h_a(e_i)$ could be omitted by replacing A by $A' \subset A$, $\nu^*(A - A') = 0$.

To prove that $\bar{\mu}(h(e)) = \mu(e)$ we first observe that, since $h(e)$ covers itself,

$$\nu^*(h(e)) \leq \nu^1(h(e)) = \mu(e) .$$

Assume that $e \subset e_a$. Again there is no loss of generality in assuming that coverings of $h(e)$ are of the form $\{h(e_i)\}$. Since $h(e \cap e_i) = h(e) \cap h(e_i)$, the sets e_i can be replaced by sets in e_a. Given $\varepsilon > 0$ there is a covering $\{h(e_i)\}$ with

$$\nu^*(h(e)) \geq \Sigma_1^\infty \nu^1(h(e_i)) - \varepsilon .$$

Then

$$\Sigma_1^\infty \nu^1(h(e_i)) \geq \Sigma_1^\infty \nu^1(h(e \cap e_i)) = \Sigma_1^\infty \mu(e \cap e_i)$$

$$\geq \mu(\cup_1^\infty (e \cap e_i)) \geq \mu(e) .$$

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Since each set $h_a(e)$ differs from $h(e)$ by a null set each $h_a(e) \in \bar{S}$ with $\bar{\mu}(h_a(e)) = \mu(e)$.

The proof of the last part of the lemma is not difficult if we observe that covering sets for e can be assumed to be of the form $h_a(e_i)$ with $e_i \subset e_a$, $e_i \in S$.

Proof of Theorem 1. We identify $f \in \underline{L}^1$ and $\bar{f} \in \underline{\bar{L}}^1$ and write $f \sim \bar{f}$ if

$$\text{(i)} \quad \{x \in X : f(x) \neq 0\} = \bigcup_i e_i, \; e_i \in S, \; e_i \subset e_{a_i}, \; a_i \in \mathcal{Q},$$

$$a_i \neq a_j, \; e_i \cap e_j = \emptyset, \; i \neq j;$$

$$\text{(ii)} \quad \bar{f}(\bar{x}) = f(x), \; \bar{x} = (a_i, x), \; x \in e_i, \; i = 1, 2, \ldots;$$

$$= 0 \text{ elsewhere in } \bar{X}.$$

Then

$$\text{(iii)} \quad \{\bar{x} : \bar{f}(\bar{x}) = 0\} = \bigcup_i h_{a_i}(e_i), \; h_{a_i}(e_i) \cap h_{a_j}(e_j) = \emptyset, \; i \neq j.$$

We identify $\hat{f} \in L^1$ and $\hat{\bar{f}} \in \bar{L}^1$ and write $\hat{f} \sim \hat{\bar{f}}$ if these equivalence classes contain representatives f and $\bar{f}$ with $f \sim \bar{f}$. We shall show that this correspondence is bijective and preserves the vector operations and norm. That the scalar multiplication and norm are preserved is immediate from the definition. If $\hat{f}_i \sim \hat{\bar{f}}_i$, $f_i \in \hat{f}_i$, $\bar{f}_i \in \hat{\bar{f}}_i$, $i = 1, 2$ and $e_a \cap (\{x : f_1(x) \neq 0\} \cup \{x : f_2(x) \neq 0\})$, then clearly

$$(f_1 + f_2)\chi_{e_a} \sim (\bar{f}_1 + \bar{f}_2)\chi_{h(e_a)}.$$

Omitting at most a μ-null set e', $\{x \in X - e' : f_1(x) + f_2(x) \neq 0\}$ can be expressed in the form (i) and this implies that $\widehat{f_1 + f_2} \sim \widehat{\bar{f}_1 + \bar{f}_2}$.

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To show that the correspondence is bijective we first suppose that $\hat{f} \in L^1$, $f' \in \hat{f}$. Then if $e(f') = \{x \in X : f'(x) \neq 0\}$, $e(f') = \bigcup_{i=1}^{\infty} e_i$, with $e_i \cap e_j = \emptyset$, $i \neq j$, $\mu(e_i) < \infty$, $i = 1, 2, \ldots$, from integration theory. Since each set e_i can intersect at most a countable collection of the sets e_a, $a \in \mathcal{Q}$, $e(f') = e_0 \cup (\bigcup_1^{\infty} e_i')$, with $\mu(e_0) = 0$ and where each e_i' is contained in some e_a, $a \in \mathcal{Q}$. Forming unions of sets contained in the same sets e_a we can assume that the sets e_i' satisfy (i) above. Letting f denote the restriction of f' to $e(f') - e_0$, $f \in \hat{f}$ and determines $\bar{f}$ by (ii) with $f \sim \bar{f}$. Thus to each $\hat{f} \in L^1$ corresponds $\hat{\bar{f}} \in \bar{L}^1$ with $\hat{f} \sim \hat{\bar{f}}$. From the preceding paragraph it then follows that the correspondence is one-one, norm preserving but perhaps into $\bar{L}^1$.

Let $\hat{\bar{f}} \in \bar{L}^1$, $\bar{f} \in \hat{\bar{f}}$. Then if $\bar{e}(\bar{f}) = \{\bar{x} \in \bar{X} : \bar{f}(\bar{x}) = 0\}$, $\bar{e}(\bar{f}) = \bigcup_1^{\infty} \bar{e}_i$, with $\bar{e}_i \cap \bar{e}_j = \emptyset$, $i \neq j$, $\bar{e}_i \in \bar{S}$, $\bar{\mu}(\bar{e}_i) < \infty$, $i = 1, 2, \ldots$. There is no loss of generality in assuming that each set e_i is contained in a set $h(e_{a_i})$, $a_i \in \mathcal{Q}$ and that $a_i \neq a_j$ if $i \neq j$. If $\bar{e}_i' = \bar{e}_i \cap (\{a_i\} \times e_{a_i})$, $\bar{\mu}(\bar{e}_i - \bar{e}_i') = 0$. Thus we can assume that each e_i is contained in $\{a_i\} \times e_{a_i}$. By the last part of the lemma, again omitting at most a null set, we can suppose that each $\bar{e}_i$ is $h_{a_i}(e_i)$ with $e_i \in S$. Replacing the sets e_i by disjoint sets $e_1^* = e_1$, $e_i^* = e_i - \bigcup_{j<i} e_j^*$, $h_{a_i}(e_i)$ by $h_{a_i}(e_i^*)$, $\Sigma_1^{\infty} \bar{\mu}(h_{a_i}(e_i) - h_{a_i}(e_i^*)) = 0$ so that the restriction of $\bar{f}$ to $\bigcup_1^{\infty} h_{a_i}(e_i^*)$ is in $\hat{\bar{f}}$ and we can suppose that (iii) holds for $\bar{f}$ with the sets $\{e_i\}$ disjoint. There then exists f, defined on X and vanishing outside $\bigcup_{i=1}^{\infty} e_i$ with (i) and (ii)

holding and with $f \sim \bar{f}$. Thus to each $\hat{\bar{f}} \in \bar{L}^1$ corresponds $\hat{f} \in L^1$ with $\hat{f} \sim \hat{\bar{f}}$.

Finally the sets $\{a\} \times e_a$, $a \in \mathcal{Q}$, form a (D*)-decomposition for $\bar{X}$ so that $(L^1)^* = L_\ell^\infty(\bar{X}, \bar{S}, \bar{\mu})$.

3. The space $L^1(X, S, \mu)$ with the natural ordering, modulo null functions is a vector lattice that is an AL-space as defined by Kakutani [6]. In addition his Axiom IX is satisfied. This suggests the following extension of Theorem 1, the details of the proof being similar to Kakutani's proof of his Theorem 7.

THEOREM 2. Every $L^1(X, S, \mu)$ is isometric and lattice-isomorphic to a space $L^1(\bar{X}, \bar{S}, \bar{\mu})$ with $\bar{X}$ a locally compact, totally disconnected topological space such that $(\bar{L}^1)^* = L_\ell^\infty(\bar{X}, \bar{S}, \bar{\mu})$.

We outline the part of his argument needed here. If (X, S, μ) is a measure space with $\mu(X) < \infty$ and $\hat{S}$ is the space of equivalence classes of measurable sets modulo μ-null sets, $\hat{S}$ is a Boolean algebra with fundamental operations $\cup$, $\cap$ and complementation (-) (modulo μ-null sets). The Stone representation theorem [7, p. 22] then gives the existence of a compact topological space $\bar{X}$ with the points of $\bar{X}$ corresponding to the ultrafilters (maximal ideals) on $\hat{S}$ and with $\hat{S}$ corresponding to the Boolean algebra $\mathcal{J}$ of all sets of $\bar{X}$ that are both open and closed.

For each $e \in \hat{S}$ let $\bar{e}$ denote the image of e in $\mathcal{J}$ and define

$$\nu(\bar{e}) = \mu(e) .$$

Then ν is a countably additive measure on $\mathcal{J}$ that can be extended to a countably additive measure $\bar{\nu}$ on $\bar{S}$, the smallest σ-algebra containing $\mathcal{J}$ ([4], p. 54). Every $A \in \bar{S}$ can be written $A = \bar{e} \cup n$ with $\bar{e} \in \mathcal{J}$ and $\bar{\nu}(n) = 0$. Then $\bar{L}^1(\bar{X}, \bar{S}, \bar{\mu})$ is isometric and lattice isomorphic to $L^1(X, S, \mu)$.

In the general case fix an (ND) decomposition with $X_2 = \bigcup_{a \in \mathcal{Q}} e_a$ and let (e_a, S_a, μ_a) denote the measure space induced on e_a by (X, S, μ). The preceding two paragraphs then give the existence of a topological measure space $(\bar{e}_a, \bar{S}_a, \bar{\mu}_a)$ with $\bar{e}_a$ compact and with $L^1(\bar{e}_a)$ isometric and lattice isomorphic to $L^1(e_a)$ for each $a \in \mathcal{Q}$. Set

$$\bar{X} = \bigcup_{a \in \mathcal{Q}} (\{a\} \times \bar{e}_a)$$

and give $\bar{X}$ the topological set sum of the topologies on the sets $\bar{e}_a$ [1]. For this topology each $\bar{e}_a$ is compact and both open and closed so that $\bar{X}$ is locally compact. A measure space structure on $\bar{X}$ can then be introduced as in Theorem 1. It is easy to verify that the lattice operations are preserved.

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https://doi.org/10.4153/CMB-1965-061-x Published online by Cambridge University Press