BULL. AUSTRAL. MATH. SOC. 58F23, 30D35
VOL. 71 (2005) [387–397]

UNIFORMLY PERFECT JULIA SETS OF MEROMORPHIC FUNCTIONS

SHENG WANG AND LIANG-WEN LIAO

Julia sets of meromorphic functions are uniformly perfect under some suitable conditions. So are Julia sets of the skew product associated with finitely generated semigroup of rational functions.

Pommerenke and Beardon [**5, 12, 13**] introduced and studied the uniformly perfect sets. Following Pommerenke, many authors have researched the topic, and many papers on the uniformly perfect sets have appeared in the literature. It was proved that the Julia sets of rational functions are uniformly perfect, see [**8, 9, 11**], and also [**13**]. However, unlike the case of rational functions, the Julia sets of a transcendental meromorphic function may not be uniformly perfect, see the example in Section 1. But, it is interesting to discuss when the Julia sets of transcendental meromorphic functions will be uniformly perfect. Zheng [**17, 18, 19**] studied uniformly perfect boundaries of stable domains in the iteration of meromorphic functions. In this paper, we shall study the Julia sets of functions from the class PM, which a is more general class of functions than meromorphic functions. We shall also prove that the Julia sets of the skew product associated with a finitely generated semigroup of rational functions, each generator having degree not less than 2, are uniformly perfect.

1. UNIFORMLY PERFECT JULIA SETS ON CLASS PM

Following Baker, Dominguez and Herring [**3**], let E be a compact totally disconnected set in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and $f(z)$ be a function meromorphic in $E^c = \widehat{\mathbb{C}} \backslash E$ and such that $C(f, E^c, z_0) = \widehat{\mathbb{C}}$ for $z_0 \in E$, where the cluster $C(f, E^c, z_0)$ is defined by

$$C(f, E^c, z_0) = \left\{w : w = \lim_{n \to \infty} f(z_n) \in \widehat{\mathbb{C}} : \text{ for some } z_n \in E^c \text{ with } z_n \to z_0\right\}.$$

Denote by PM the set of all functions satisfying the condition that there is a compact totally disconnected set $E = E(f)$ such that $f(z)$ is meromorphic in E^c and $C(f, E^c, z_0) = \widehat{\mathbb{C}}$ for all $z_0 \in E$.

Received 27thOctober, 2004

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/05 $A2.00+0.00.

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

The points in E may not be isolated. However, f has similar properties near a point in E as a transcendental entire function has near an essential singularity. It is clear that if $E = \emptyset$ then f is a rational function. The class PM is closed under composition of functions. In fact, if $f, g \in PM$, then

$$E(f \circ g) = E(g) \cup g^{-1}\big(E(f)\big).$$

For $f \in PM$, f^n denotes the n-th iterate of f, that is,

$$f^n = f(f^{n-1}), \quad f^0(z) \equiv z, \quad n = 1, 2, \ldots,$$

and so f^n is defined in $E(f^{n-1})^c$. The Fatou set $F(f)$ of f is the largest open set in which all $f^n(z)(n \in \mathrm{N})$ are meromorphic and $\{f^n\}$ is normal. The complement of $F(f)$ is called the Julia set $J(f)$ of f. $J(f)$ is perfect. Let $J_\infty(f) = \bigcup_{n=0}^{\infty} E(f^n)$. If $J_\infty(f) = \emptyset$, then f is a rational function. If $\#\big(J_\infty(f)\big) = 1$, then f is conjugate with a transcendental entire function under a Möbius transformation, where $\#\big(J_\infty(f)\big)$ denotes the cardinality of the set $J_\infty(f)$. If $\#(J_\infty)(f) = 2$, then f is conjugate with a holomorphic self-map of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. If $\#\big(J_\infty(f)\big) \geqslant 3$, then $J(f) = \overline{J_\infty(f)}$, see [**3**].

Denote by $\operatorname{sing}(f^{-t})$ the set of all singularities of the inverse function of f^t and the limit values of these singularities for some $t \geqslant 1$. Set

$$S_p(f) = \bigcup_{k=0}^{p} f^k\big(\operatorname{sing}(f^{-1})\backslash E(f^k)\big),$$

and

$$P(f) = \bigcup_{p=1}^{\infty} S_p(f).$$

A compact set D is uniformly perfect if there exists a constant $c > 0$ for all $r \in (0, \operatorname{diam} D)$ such that

$$\big\{z \in \widehat{\mathbb{C}} : cr < |z - \zeta| < r, \forall \zeta \in D\big\} \cap D \neq \emptyset,$$

where $\operatorname{diam} D$ denotes the diameter of D in the sense of the spherical metric. $d(A, B)$ denotes the Euclidean distance between two sets A and B. Now, we can state our first result as follows.

Theorem 1. *Let $f \in PM$. If $d\big(J(f), \overline{P}(f)\backslash J(f)\big) = \delta > 0$. Then $J(f)$ is uniformly perfect.*

A remark for Theorem 1. Consider

$$f_{p,\mu}(z) = \mu\Sigma_{n=p^2}^{\infty}((2z)/(n^{2p} - z^2)), p \in \mathrm{N}, \mu > 0.$$

It was shown in [**14**] that when $p \geqslant 6$ and $0 < \mu < (\Sigma_{n=p^2}^{\infty} 2 \times n^{-2p})^{-1}$, we have

$$\overline{P}(f_{p,\mu}) \cap J(f_{p,\mu}) = \emptyset.$$

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

Noting that $\overline{P}(f_{p,\mu})$ and $J(f_{p,\mu})$ are compact sets in $\widehat{\mathbb{C}}$, gives

$$d\Big(\overline{P}((f_{p,\mu})), J(f_{p,\mu})\Big) > 0.$$

It follows from Theorem 1 that $J(f_{p,\mu})$ is uniformly perfect. In fact, $J(f_{p,\mu}) \subset R$, see [**14**]. For a function $f \in PM$, if $d(\overline{P}(f), J(f)) = 0$, $J(f)$ may not be uniformly perfect. For example, consider

$$g(z) = c\Pi_{j=1}^{\infty}\Big\{1 + \Big(\frac{z}{r_j}\Big)^{k_j}\Big\},$$

where $c \in \Big(0, (1/4e^2)\Big)$, for some $n_0 > 0$ and r_1, $2^{n_0-1}c > 2r_1^{k_1}$, $k_1 \geqslant 5$, $n \geqslant n_0$,

$$\begin{aligned} r_{n+1} &= c\Big(1 + \Big(\frac{r_n}{r_1}\Big)^{k_1}\Big)\cdots\Big(1 + \Big(\frac{r_n}{r_n}\Big)^{k_n}\Big) \\ &> 2^{n-1}c\Big(\frac{r_n}{r_1}\Big)^{k_1} > 2r_n^{k_1} > 2r_n. \end{aligned}$$

It was proved in [**1**] that

$$c\Pi_{j=1}^{\infty}\Big\{1 + \Big(\frac{z}{r_j}\Big)^{k_j}\Big\}$$

converges uniformly in any compact domain and $g(A_n) \subset A_{n+1}$ for all sufficiently large n, where $A_n = \{z : r_n^2 \leqslant |z| \leqslant r_{n+1}^{1/2}\}$ and A_n contains in a multiply connected wandering component of $F(g)$. By Theorem 1 in [**20**], the limits of $g^k(z)$ are in $P'(g)$ for all $z \in A_n$, $k \to \infty$. We have $d(\overline{P}(g), J(g)) = 0$ for $P'(g) \subset J(g)$. But,

$$\operatorname{mod} A_n = \frac{1}{2\pi}\log\frac{r_{n+1}^{1/2}}{r_n^2} > \frac{1}{2\pi}\log\frac{r_n^{k_1/2}}{r_n^2} \to \infty,\ n \to \infty.$$

So, $J(g)$ is not uniformly perfect. From [**2**], $g(z)$ must have infinitely many critical points. In fact, by Theorem 1, $d(J(g), \overline{P}(g)\backslash J(g)) = 0$. In a more general case, let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$ with finitely many poles. If $J(f)$ has only bounded components, then $d(J(f), \overline{P}\backslash J(f)) = 0$, since $J(f)$ is not uniformly perfect by Theorem 1 in [**19**].

2. Proof of Theorem 1

In order to prove Theorem 1, we need the following lemma.

Lemma 1. *Let $f \in PM$. For any $z_0 \in J(f)$ and any neighbourhood U of z_0. If V is a compact subset of $\widehat{\mathbb{C}}$ not containing any exceptional value of $f(z)$, then there exists an integer $N > 0$, such that for all $n > N$,*

$$f^n(U) \supset V.$$

Proof: If f is entire in $\mathbb{C}$ or is a holomorphic self map of the punctured plane $\mathbb{C}\backslash\{0\}$, then the conclusion is true, see [**7, 10**].

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

Now, we consider $f \in PM$ but not being included in the above two cases. Since

$$J(f) = \overline{\bigcup_{n=0}^{\infty} E(f^n)} = \overline{\bigcup_{n=1}^{\infty} f^{-n+1}(E(f))}$$

and $J(f)$ is perfect, see [3], there exits $z_0 \in J(f) \cap U$ and $N > 0$ such that $z_0 \in E(f^N)$. Thus $C(f^N, E^C, z_0) = \widehat{\mathbb{C}}$. Noting that V does not contain any exceptional values of $f(z)$, we have

$$f^{N+k}(U) \supset V, \ k = 1, 2, \ldots.$$

The proof is complete. □

PROOF OF THEOREM 1: By contradiction, assume that $J(f)$ is not uniformly perfect. By the definition of a uniformly perfect set, there exists a sequence of annuli A_n separating $J(f)$ such that

$$A_n \cap J(f) = \emptyset, n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} \operatorname{mod}(A_n) = \infty.$$

This implies that for each $n \in \mathbf{N}$, there exists a component D_n of $A_n^c = \widehat{\mathbb{C}} \backslash A_n$ such that

$$\lim_{n\to\infty} \operatorname{diam} D_n = 0.$$

Denote by U_n the other connected component of A_n^c. It follows, see [11],

$$\inf_{n>0} \operatorname{diam} U_n > 0.$$

Let

$$g_n : \Delta \to A_n \cup D_n \subset \widehat{\mathbb{C}}$$

be a conformal mapping with $g_n(0) \in D_n \cap J(f)$, $n = 1, 2, \ldots$, where, throughout,

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Since a conformal mapping leaves modular invariant, we have

$$\operatorname{mod}(\Delta - g_n^{-1}(D_n)) = \operatorname{mod} A_n.$$

It follows, by $\lim_{n\to\infty} \operatorname{mod}(A_n) = \infty$,

$$\lim_{n\to\infty} \operatorname{diam}(g_n^{-1}(D_n)) = 0.$$

$\{g_n\}$ is normal in Δ, because

$$\inf_{n>0} \operatorname{diam}(g_n(\Delta))^c = \inf_{n>0} \operatorname{diam} U_n > 0$$

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

implies that $\{g_n\}$ is locally and uniformly bounded in Δ. Choose a sequence of numbers $\{\rho_n\}$, $\rho_n \to 0$ as $n \to \infty$, such that

$$0 < \rho_n < 1, \ n = 1, 2, \ldots.$$

Set

$$g_n^{-1}(D_n) \subset \Delta_n = \{z \in \mathbb{C} : |z| < \rho_n\},$$

for all sufficiently large n. ρ_n can be chosen so small that

$$\lim_{n\to\infty} \operatorname{diam} g_n(\Delta_n) = 0.$$

Make a double connected domain A_0, which separate Julia set of f, such that

$$0 < \min\Big\{d\big(z, J(f)\big) : z \in A_0\Big\} < \max\Big\{d\big(z, J(f)\big) : z \in A_0\Big\} < \delta.$$

For $z \in J(f)$ and a neighbourhood V of z, by Lemma 1, for all sufficiently large $l > 1$,

$$f^l(V) \supset A_0.$$

It is easy to see that the open set $g_n(\Delta_n)$ satisfies

$$g_n(\Delta_n) \cap J(f) \neq \emptyset.$$

And for all sufficiently large n,

$$g_n(\Delta_n) \cap \big(\overline{P} \backslash J(f)\big) = \emptyset.$$

Thus for all sufficiently large n, there exists connected domains $V_n \subset g_n(\Delta_n)$ and a smallest integer $l_n > 0$ such that

$$f^{l_n}(V_n) = A_0.$$

Take an un-contractible Jordan curve γ in A_0 such that $\max\Big\{d\big(z, J(f)\big) : z \in \gamma\Big\} < \delta$ and $d\big(\gamma, J(f)\big) > 0$. By the hypotheses of Theorem 1, there is a neighbourhood of γ contained by A_0, in which $f(z)$ has no singularities. So there is an un-contractible Jordan curve $\gamma_n \subset V_n$ such that

$$h_{l_n} : \gamma \to \gamma_n$$

is regular, where $h_{l_n}(z)$ is a univalent branch of f^{-l_n} such that

$$h_{l_n} \circ f^{l_n} : \gamma_n \to \gamma_n.$$

Because

$$J(f) \cap \operatorname{int}(\gamma_n) \neq \emptyset,$$

where $\operatorname{int}(\gamma_n)$ denotes the interior of γ_n, and as $\operatorname{diam}(\gamma_n) \to 0$ as $n \to \infty$, we conclude that $J(f)$ has an isolated point. However, $J(f)$ is perfect. This contradiction shows that $J(f)$ is a uniformly perfect set. Theorem 1 follows. ∎

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

3. Uniform perfectness of Julia sets of skew products

Given an integer $m \geqslant 1$, Σ_m denotes the one-sided word space, that is,

$$\Sigma_m = \{1, 2, \dots, m\} \times \{1, 2, \dots, m\} \times \dots = \prod_1^\infty \{1, 2, \dots, m\}.$$

Let f_j be rational in $\widehat{\mathbb{C}}$ of degree $\geqslant 2$, for $j = 1, 2, \dots, m$. For a word

$$w = (w_1, w_2, \dots, w_n, \dots) \in \Sigma_m,$$

$f_{w_n} \circ f_{w_{n-1}} \circ \dots \circ f_{w_1}$ denotes the composition of $f_{w_1}, f_{w_2}, \dots, f_{w_n}$. The Fatou set F_w is defined by

$$F_w = \left\{ z \in \widehat{\mathbb{C}} : \left\{ f_{w_n} \circ \dots \circ f_{w_1}(z) \right\}_n \text{ is normal in a neighbourhood of } z \right\},$$

and $J_w = \widehat{\mathbb{C}} \backslash F_w$. $\widetilde{f}$ is called the skew product associated with finitely many rational generators $\{f_1, \dots, f_m\}$, that is,

$$\begin{aligned} \widetilde{f} : \Sigma_m \times \widehat{\mathbb{C}} &\to \Sigma_m \times \widehat{\mathbb{C}} \\ (w, x) &\to \left(\sigma w, f_{w_1}(z)\right), \end{aligned}$$

where $\sigma w = (w_2, w_3, \dots, w_n, \dots)$. The Julia set $J(\widetilde{f})$ of $\widetilde{f}$ is defined by

$$J(\widetilde{f}) = \overline{\bigcup_{w \in \Sigma_m} \{w\} \times J_w},$$

and the Fatou set $F(\widetilde{f})$ of $\widetilde{f}$ is defined by

$$F(\widetilde{f}) = \Sigma_m \times \widehat{\mathbb{C}} \backslash J(\widetilde{f}).$$

The readers can find the above notations in [9]. We need to define another map π:

$$\begin{aligned} \pi : \Sigma_m \times \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ (w, x) &\to x. \end{aligned}$$

Then

$$\begin{aligned} \pi \circ \widetilde{f} : \Sigma_m \times \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ (w, x) &\to f_{w_1}(x). \end{aligned}$$

Mañé and Da Rocha [**11**] proved that the Julia set of a rational function of degree greater than 2 is uniformly perfect. See also [**8**]. Hinkkanen and Martin [**9**] extended this result to the case of a finitely generated rational semigroup G and proved that the Julia set of G is uniformly perfect. In this section, we shall consider the case of the skew

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

product associated to a finitely many rational generators and use Mañé and Da Rocha's method to prove the uniform perfectness of its Julia set, see also [**16**].

For a word $w \in \Sigma_m$ and a point $z_0 \in \widehat{\mathbb{C}}$, we define $O_w^-(z_0)$ and X_w as follows:

$$O_w^-(z_0) = \left\{ z \in \widehat{\mathbb{C}} : \pi \circ \widetilde{f}^n\big((w,z)\big) = z_0 \ \text{ for some integer } \ n > 0 \right\}$$

and

$$X_w = \left\{ z \in \widehat{\mathbb{C}} : \# O_w^- < \infty \right\}.$$

By [**14**, Proposition 3.2], $\pi\big(J(\widetilde{f})\big) = J(G)$, where $G = \langle f_1, f_2, \dots, f_m \rangle$ is a finitely generated rational semigroup. It is known that $J(G)$ is uniformly perfect. Considering

$$J(G) = \overline{\bigcup_{w \in \Sigma_m} J_w},$$

we define the uniform perfectness of $J(\widetilde{f})$ in the following.

Definition: $J(\widetilde{f})$ is called uniformly perfect, if for each word $w \in \Sigma_m$, J_w is uniformly perfect.

Now, let's state our another result below.

Theorem 2. *Let $\widetilde{f}$ be the skew product associated with the rational generators $\{f_1, f_2, \dots, f_m\}$, $m \geqslant 1$, and where*

$$deg f_j \geqslant 2, \quad j = 1, 2, \dots, m.$$

Then $J(\widetilde{f})$ is uniformly perfect.

4. Proof of Theorem 2

In this section we shall prove Theorem 2, mainly based on the method of Mañé and Da Rocha [**11**]. But the result is significant for the completeness of our study of the uniformly perfect Julia sets of iteration of rational functions. We need to introduce some lemmas first.

Lemma 2. *Let $\widetilde{f}$ be the skew product associated with the rational generators $\{f_1, \dots, f_m\}$, where each rational function f_j is of $deg f_j \geqslant 2, j = 1, 2, \dots, m; m \geqslant 1$. Then for each word $w \in \Sigma_m$, $\# X_w \leqslant 2$.*

Proof: Suppose $\# X_w \geqslant 3$. Then for each n, $\pi \circ \widetilde{f}^n$ can not take at least three values in $\{w\} \times \widehat{\mathbb{C}} \backslash X_w$. By Montel's Theorem, $\{\pi \circ \widetilde{f}^n\}$ is normal in $\{w\} \times \widehat{\mathbb{C}} \backslash X_w$, that is $\{f_{w_n} \circ \dots \circ f_{w_1}\}_n$ is normal in $\widehat{\mathbb{C}} \backslash X_w$, where $w = (w_1, w_2, \dots, w_n, \dots)$. So J_w is not perfect. This is impossible, see [**15**]. ∎

Lemma 3. *Under the hypotheses of Lemma 2, for each word*

$$w \in \Sigma_m, \ X_w \cap J_w = \emptyset.$$

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

PROOF: By Lemma 2, it is only necessary to consider the following two cases:

CASE ONE. If $X_w = \{a\}$, we may assume $X_w = \{\infty\}$ without loss of a generality, since we can make a suitable Möbius transformation which maps a to ∞. Then ∞ is an exceptional value of $f_{w_1}(z)$, and then f_{w_1} is a polynomial. Similarly, we also obtain that f_{w_2} is a polynomial. Inductively f_{w_j} is a polynomial, $j = 3, 4, \ldots$. It is easy to deduce that $\infty \in F_w$.

CASE TWO. If $X_w = \{a, b\}$, then similarly assume $X_w = \{0, \infty\}$, and then 0 and ∞ are both two exceptional values of f_{w_1}. Therefore f_{w_1} can be written in the form $c_1 z^{d_1}$, $c_1 \in \mathbb{C}$, $d_1 > 0$, if $f_{w_1}(0) = 0, f_{w_1}(\infty) = \infty$; or in the form $c_1 z^{-d_1}$, if $f_{w_1}(0) = \infty, f_{w_1}(\infty) = 0$. In any case, it follows that 0 and ∞ are both exceptional values of f_{w_n} in the similar way, $n = 2, 3, \ldots$. And now, we obtain that each f_{w_n} can be written in the form $c_n z^{d_n}$, if $f_{w_n}(0) = 0, f_{w_n}(\infty) = \infty$; or in the form $c_n z^{-d_n}$, if $f_{w_n}(0) = \infty, f_{w_n}(\infty) = 0$, $n = 2, 3, \ldots$. It follows $\{0, \infty\} \subset F_w$.

For the above two cases, we have $X_w \subset F_w$. So $J_w \cap X_w = \emptyset$. ∎

Following the similar proof of [4, Theorem 4.2.5(ii)], we immediately obtain the following lemma.

LEMMA 4. *Under the hypotheses of Lemma 2, for each word $w \in \Sigma_m$, any point $z \in J_w$ and its neighbourhood U, there exists an integer $N \geqslant 1$, such that for all $n > N$,*

$$f_{w_n} \circ \cdots \circ f_{w_1}(U) \supset J_w.$$

PROOF OF THEOREM 2: For any word

$$w = (w_1, w_2, \ldots, w_n, \ldots) \in \Sigma_m,$$

we only need to prove that $J_w(f)$ is uniformly perfect.

In fact, assume that for some word $w \in \Sigma_m$, $J_w(f)$ is not a uniformly perfect set. By definition, there exist a sequence of ring domains A_n separating $J_w(f)$ such that

$$A_n \cap J_w(f) = \emptyset, \ n = 1, 2, \ldots$$

and

$$\lim_{n \to \infty} \operatorname{mod} A_n = \infty.$$

Therefore, for each n, there exists a connected component U_n of A_n^c such that $\lim_{n\to\infty} \operatorname{diam} U_n = 0$. Let V_n denote another connected component of A_n^c. Then analogously to [**11**], we have $\inf_{n>0} \operatorname{diam} V_n > 0$. Let

$$g_n : \Delta \to A_n \cup U_n$$

be a conformal map that satisfies

$$g_n(0) \in U_n \cap J_w, n = 1, 2, \ldots \ .$$

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

Then

$$\operatorname{mod}\bigl(\Delta - g_n^{-1}(U_n)\bigr) = \operatorname{mod} A_n.$$

From the similar discussion of [**11**], $\{g_n\}$ is normal in Δ. Chose two sequences of positive numbers $\{r_n\}$, $\{\rho_n\}$, such that

$$2 \operatorname{diam} U_n < \rho_n < r_n < 1,$$

and

$$\frac{\rho_n}{r_n} \to 0, n \to \infty.$$

Set

$$\Delta_n = \bigl\{z \in \mathbb{C} : |z| < \rho_n\bigr\}.$$

Hence

$$\lim_{n\to\infty} \operatorname{diam} g_n(\Delta_n) = 0.$$

Since $g_n(\Delta_n)$ is open and

$$g_n(\Delta_n) \cap J_w(f) \neq \emptyset,$$

By Lemma 3 and Lemma 4, there exist integers $p_n > 0$, such that

$$f_{w_{p_n}} \circ \cdots \circ f_{w_1}\bigl(g_n(\Delta_n)\bigr) \supset J_w.$$

Let

$$\operatorname{diam} J_w(f) > c > 0,$$

and l_n is the minimum positive integer satisfying

$$\operatorname{diam}\Bigl(f_{w_{l_n}} \circ \cdots \circ f_{w_1}\bigl(g_n(\Delta_n)\bigr)\Bigr) \geqslant c.$$

When $n \to \infty$, $l_n \to \infty$. Because as $n \to \infty$,

$$\operatorname{diam}\bigl(g_n(\Delta_n)\bigr) \to 0.$$

On the other hand,

$$\operatorname{diam}\Bigl(f_{w_{l_n-1}} \circ \cdots \circ f_{w_1}\bigl(g_n(\Delta_n)\bigr)\Bigr) < c.$$

So

$$\operatorname{diam}\Bigl(f_{w_{l_n}} \circ f_{w_{l_n-1}} \circ \cdots \circ f_{w_1}\bigl(g_n(\Delta_n)\bigr)\Bigr) < cL,$$

where L is the Lipschitz constant of $f_{w_{l_n}}$. Suppose that cL is as small as we please, since c can be taken arbitrarily small. Let Y be a subset of $J_w(f)$ which contains only four distinct points and the spherical distance between any two points is bigger than or equal to cL. Then $f_{w_{l_n}} \circ \cdots \circ f_{w_1}\bigl(g_n(\Delta_n)\bigr)$ contains at most two points of Y. Define

$$\phi_n(z) = f_{w_{l_n}} \circ \cdots \circ f_{w_1}\bigl(g_n(r_n z)\bigr), n = 1, 2, \ldots .$$

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

Then again following [**11**], we deduce that $\{\phi_n\}$ is normal in Δ.

Thereafter, given an arbitrarily small $\varepsilon > 0$, for all sufficiently large n, there exists a neighbourhood W of 0 such that

$$\operatorname{diam} \phi_n(W) \leqslant \varepsilon. \tag{1}$$

But, when n is sufficiently large,

$$\left\{z : |z| < \frac{\rho_n}{r_n}\right\} \subset W$$

and

$$\begin{aligned} \operatorname{diam} \phi_n(W) &\geqslant \operatorname{diam}\left(f_{w_{l_n}} \circ \cdots \circ f_{w_1}\left(g_n\left(r_n\left\{z : |z| < \tfrac{\rho_n}{r_n}\right\}\right)\right)\right) \\ &= \operatorname{diam}\left(f_{w_{l_n}} \circ \cdots \circ f_{w_1}\left(g_n(\Delta_n)\right)\right) \geqslant cL. \end{aligned}$$

This contradicts (1), as ε may be chosen arbitrarily small. Theorem 2 follows. □

REFERENCES

[1] I.N. Baker, 'Some entire functions with multiply-connected wandering domains', *Ergodic Theory Dynam. Systems* **5** (1985), 163–169.

[2] I.N. Baker, J. Kotus and Y.N. Lü, 'Iterates of meromorphic functions IV: Critically finite functions', *Results Math.* **22** (1992), 651–656.

[3] I.N. Baker, P. Domínguez and M.E. Herring, 'Dynamics of functions meromorphic outside a small set', *Ergodic Theory Dynam. Systems* **21** (2001), 647–672.

[4] A.F. Beardon, *Iteration of rational functions* (Springer-Verlag, Berlin, 1991).

[5] A.F. Beardon and Ch. Pommerenke, 'The Poincaré metric of plane domains', *J. London Math. Soc.* **41** (1979), 475–483.

[6] L. Carleson and W. Gamelin, *Complex dynamics* (Springer-Verlag, New York, 1993).

[7] P. Fatou, 'Sur l'itération des fonctions transcendentes entiéres', *Acta Math.* **47** (1926), 337–370.

[8] A. Hinkkanen, 'Julia sets of rational functions are uniformly perfect', *Math. Proc. Cambridge Philos. Soc.* **113** (1993), 543-559.

[9] A. Hinkkanen and G.J. Martin, 'Julia sets of rational functions', *Math. Z.* **222** (1996), 161–169.

[10] X.H. Hua and C.C. Yang, *Dynamics of transcendental functions*, Asian Mathematics Series **1** (Gordon and Breach Science Publishers, Amsterdam, 1998).

[11] R. Mãñé and L.F. Da Rocha, 'Julia sets are uniformly perfect', *Proc. Amer. Math. Soc.* **62** (1992), 251–257.

[12] Ch. Pommerenke, 'Uniformly perfect sets and Poincaré metric', *Arch Math.* **32** (1979), 192–199.

[13] Ch. Pommerenke, 'On uniformly perfect sets and Fuchsian groups', *Analysis* **4** (1984), 299–321.

[14] G.M. Stallard, 'The Hausdorff dimension of Julia sets of meromorphic functions II', *J. London Math. Soc. 2* **60** (1999), 847–859.

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press

[15] H. Sumi, 'Skew product maps related to finitely generated rational semigroups', *Nonlinearity* **13** (2000), 995–1019.

[16] S. Wang, *On some properties of Fatou and Julia sets of meromorphic functions*, (in Chinese, Ph.D. Disssertation) (Tsinghua University, Beijing, 2002).

[17] J.H. Zheng, 'On uniformly perfect boundaries of stable domains in iteration of meromorphic functions', *Bull. London Math. Soc.* **32** (2000), 439–446.

[18] J.H. Zheng, 'Uniformly perfect sets and distortion of holomorphic functions', *Nagoya Math. J.* **164** (2001), 17–33.

[19] J.H. Zheng, 'On uniformly perfect boundaries of stable domains in iteration of meromorphic functions II', *Math. Proc. Cambridge Philos. Soc.* **132** (2002), 531–544.

[20] J.H. Zheng, 'Singularities and limit functions in iteration of meromorphic functions', *J. London Math. Soc.* **67** (2003), 195–207.

College of Economics and Management
South China Agricultural University
Guangzhou 510642
Guangdong
China
e-mail: wangsh@pub.zhaoqing.gd.cn

Department of Mathematics
Nanjing University
Nanjing 210093
China
e-mail: maliao@nju.edu.cn

https://doi.org/10.1017/S0004972700038387 Published online by Cambridge University Press