The Absolute Summability (A) of Fourier Series

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§ 1. In a recent paper[1] Dr J. M. Whittaker has shown that the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

of a function $f(\theta)$ which has a Lebesgue integral in $(-\pi, \pi)$, is[2] *absolutely summable* (A) to sum l, if

$$(\alpha) \qquad \int_0^{\delta} |\phi(t)| \, t^{-1} \, dt$$

exists, where

$$2\phi(t) = f(\theta + 2t) + f(\theta - 2t) - 2l.$$

In this paper two other forms of criterion for absolute summability (A) of a Fourier series are obtained. In § 2, it is shown that the series is absolutely summable (A), if

(β) $\qquad$ $\phi(t)$ *is absolutely continuous in* $(0, \delta)$.

In § 3, another criterion is found, *viz.*

(γ) $\qquad$ *the existence of the integral* $\int_0^{\delta} |\Phi(t)| \, t^{-2} \, dt$,

where

$$\Phi(t) = \int_0^t \phi(u) \, du.$$

In § 4, the mutual relations of these three criteria are discussed, where it is shown that, while (β) is independent of (α) and (γ), (γ) includes (α).

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[1] *Proc. Edinburgh Math. Soc.* (2), 2 (1930), 1–5.

[2] A series

$$\sum_{n=0}^{\infty} a_n,$$

has been defined to be *absolutely summable* (A), if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is convergent in $(0 \leqslant x < 1)$ and if $f(x)$ is of bounded variation in $(0, 1)$.

In the last article it is proved that a Fourier series may be absolutely summable (A) at a point, without being convergent in the ordinary sense at that point.

§ 2. From the Poisson's series[1] (convergent for $0 \leqslant x < 1$)

$$P(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} x^n (a_n \cos n\theta + b_n \sin n\theta),$$

we get

$$\begin{aligned} \tfrac{1}{2}\pi Q(x) &= \tfrac{1}{2}\pi \{P(x) - l\} \\ &= \int_0^{\pi/2} \phi(t) \frac{1-x^2}{1-2x\cos 2t + x^2} dt, \\ &= \int_0^{\delta} + \int_{\delta}^{\pi/2} \phi(t) \frac{1-x^2}{1-2x\cos 2t + x^2} dt, \\ &= Q_1(x) + Q_2(x), \text{ say,} \end{aligned}$$

where δ is a constant such that $0 < \delta < \dfrac{\pi}{2}$.

It is easy to prove that

$$\int_0^{x_1} | Q'_2(x) | \, dx,$$

where $0 < x_1 < 1$, is less than a constant and hence $Q_2(x)$ is of bounded variation in $(0, 1)$.

Now suppose $\phi(t)$ is *absolutely continuous* in $(0, \delta)$. Hence, integrating by parts

$$\begin{aligned} Q_1(x) &= \phi(\delta) \tan^{-1}\left\{\frac{1+x}{1-x}\tan\delta\right\} - \int_0^{\delta} \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\}\left(\frac{d\phi(t)}{dt}\right) dt \\ &= J(x) - K(x). \end{aligned}$$

Here $J(x)$ is obviously a function of bounded variation in $(0, 1)$, whilst

$$\begin{aligned} \int_0^{x_1} |K'(x)| dx &= \int_0^{x_1} \left| \int_0^{\delta} \frac{d\phi(t)}{dt} \cdot \frac{d}{dx}\left[\tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\}\right] dt \right| dx \\ &\leqslant \int_0^{x_1} dx \int_0^{\delta} \left|\frac{d\phi(t)}{dt}\right| \left| \frac{\sin 2t}{1-2x\cos 2t + x^2} \right| dt \\ &= \int_0^{\delta} \left| \frac{d\phi(t)}{dt} \right| U(x_1, t)\, dt \end{aligned}$$

inverting the order of integration[2]; here

[1] E. W. Hobson, *Theory of Functions of a Real Variable*, **2** (1926), 629.

[2] *Ibid.*, **1** (1927), 630.

$$U(x_1, t) = \int_0^{x_1} \left| \frac{\sin 2t}{1 - 2x \cos 2t + x^2} \right| dx$$

$$= \tan^{-1}\left\{\frac{1 + x_1}{1 - x_1} \tan t\right\} - t \leqslant \frac{\pi}{2}.$$

Therefore $$\int_0^{x_1} |K'(x)| \, dx \leqslant \frac{\pi}{2} \int_0^{\delta} \left| \frac{d\phi(t)}{dt} \right| dt < c,$$

where c is a constant.[1] Hence $K(x)$ is a function of bounded variation in (0, 1). Therefore $Q_1(x)$ and consequently $Q(x)$ from (2) is a function of bounded variation in (0, 1), so that the series (1) is absolutely summable (A) at θ, if it converges in virtue of $\phi(t)$ being absolutely continuous in $(0, \delta)$.

§ 3. Let

$$W(x, t) = \frac{1 - x^2}{1 - 2x \cos 2t + x^2},$$

and $$\Phi(t) = \int_0^t \phi(u) \, du.$$

Taking $\delta = \frac{\pi}{4}$, we have

$$\int_0^{x_1} |Q'_1(x)| \, dx = \int_0^{x_1} \left| \int_0^{\pi/4} \phi(t) \, . \, \frac{\partial W(x, t)}{\partial x} \, dt \right| dx$$

$$= \int_0^{x_1} \left| \left[\Phi(t) \frac{\partial W(x, t)}{\partial x} \right]_0^{\pi/4} - \int_0^{\pi/4} \Phi(t) \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \, dt \right| dx$$

$$\leqslant A + \int_0^{x_1} \int_0^{\pi/4} |\Phi(t)| \, . \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dt \, dx,$$

where A is a constant. Inverting the order of integration, we have

$$\int_0^{x_1} |Q'_1(x)| \, dx \leqslant A + \int_0^{\pi/4} |\Phi(t)| \left\{ \int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx \right\} dt.$$

Now

$$\int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx = \int_0^{x_1} \left| 4 \sin 2t \, \frac{1 + x^4 - 6x^2 + 2x(1 + x^2) \cos 2t}{(1 - 2x \cos 2t + x^2)^3} \right| dx$$

$$= \int_0^{x_1} |V(x, t)| \, dx,$$

where

$$V(x, t) = 4 \sin 2t \, \frac{1 + x^4 - 6x^2 + 2x(1 + x^2) \cos 2t}{(1 - 2x \cos 2t + x^2)^3}.$$

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[1] *Ibid.*, 593.

Let us write

$$t_1 = \sin^{-1} \frac{1 - x_1}{\sqrt{2}\sqrt{(1 + x_1^2)}}, \qquad (0 < t_1 < \pi/4).$$

Then, if $0 \leqslant t \leqslant t_1$, we have

$$\int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| = \int_0^{x_1} V(x, t) \, dx$$

$$= \frac{4 \sin 2t \,.\, x_1 (1 - x_1^2)}{(1 - 2x_1 \cos 2t + x_1^2)^2}.$$

Now since

$$(1 - 2x_1 \cos 2t + x_1^2)^2 = (1 - x_1)^4 + 8x_1 (1 + x_1^2) \sin^2 t - 16 x_1^2 \sin^2 t \cos^2 t$$
$$\geqslant 8x_1 (1 - x_1)^2 \sin^2 t,$$

it follows that, for $0 \leqslant t \leqslant t_1$,

$$\int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx \leqslant \frac{(1 + x_1) \cos t}{(1 - x_1) \sin t} < \frac{\pi^2}{2t^2}.$$

If $t_1 \leqslant t \leqslant \frac{1}{4}\pi$, and x', such that $(\sqrt{2} - 1) \leqslant x' < 1$, be given by

$$\cos 2t = \frac{6x'^2 - 1 - x'^4}{2x'(1 + x'^2)},$$

we see that

$$\int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx \leqslant \int_0^1 \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx$$

$$= \int_0^{\sqrt{2}-1} V(x, t) \, dx + \int_{\sqrt{2}-1}^{x'} V(x, t) \, dx - \int_{x'}^1 V(x, t) \, dx$$

$$= \frac{8 \sin 2t \,.\, x' (1 - x'^2)}{(1 - 2x' \cos 2t + x'^2)^2}$$

$$< D \cos t \operatorname{cosec}^2 t < \tfrac{1}{4} \pi^2 D t^{-2},$$

where D is a constant.

Thus we have

$$\int_0^{x_1} | Q'_1(x) | \, dx \leqslant A + \left(\int_0^{t_1} + \int_{t_1}^{\pi/4} \right) | \Phi(t) | \left\{ \int_0^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \, \partial x} \right| dx \right\} dt$$

$$< A + \tfrac{1}{2} \pi^2 \int_0^{t_1} | \Phi(t) | \, t^{-2} \, dt + \tfrac{1}{4} \pi^2 D \int_{t_1}^{\pi/4} | \Phi(t) | \, t^{-2} \, dt$$

$$< A + B \int_0^{\pi/4} | \Phi(t) | \, t^{-2} \, dt,$$

where B is a constant.

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Hence $Q_1(x)$ will be of bounded variation in (0, 1) and consequently the Fourier series will be absolutely summable (A), provided that

$$\int_0^\delta |\Phi(t)|\, t^{-2}\, dt$$

exists.

§ 4. *The criterion* (γ) *includes* (α).

The proof of this is quite straightforward and is therefore omitted.

(γ) *is not included in* (α).

Take

$$\phi(t) = \rho t^{\rho-1} \sin \frac{1}{t} - \frac{1}{t^{2-\rho}} \cos \frac{1}{t}, \quad (1 < \rho < 2),$$

so that

$$\Phi(t) = t^\rho \sin \frac{1}{t}.$$

Then (γ) exists, but (α) does not exist.

(β) *is included neither in* (α) *nor in* (γ).

Thus

$$\phi(t) = \left(\log \frac{1}{t}\right)^{-1}$$

satisfies (β), but neither (α) nor (γ).

Again (β) being an especial case of Jordan's test, cannot include (α) or (γ).[1]

§ 5. The existence of the integral

$$\int_0^\delta |\Phi(t)|\, t^{-2}\, dt$$

is not a sufficient condition for the convergence of the corresponding Fourier series.

For if we take

$$\phi(t) = r t^{r-1} \sin \frac{1}{t} - \frac{1}{t^{2-r}} \cos \frac{1}{t}, \quad (1 < r < \tfrac{3}{2}),$$

so that

$$\Phi(t) = t^r \sin \frac{1}{t}.$$

Then we have

$$\begin{aligned} I &= \frac{1}{\pi} \int_0^\epsilon \left(r t^{r-1} \sin \frac{1}{t} - \frac{1}{t^{2-r}} \cos \frac{1}{t} \right) . \frac{\sin (2n+1) t}{t} dt \\ &= \frac{1}{\pi} \int_0^\epsilon r \sin \frac{1}{t} . \frac{\sin (2n+1) t}{t^{2-r}} dt - \frac{1}{\pi} \int_0^\epsilon \frac{1}{t^{2-r}} \cos \frac{1}{t} . \frac{\sin (2n+1) t}{t} dt \\ &= I_1 - I_2. \end{aligned}$$

[1] See G. H. Hardy, *Messenger of Math.*, **49** (1919–20), 150.

Now $\lim_{n \to \infty} I_1$ is zero, but by means of results due to Du Bois-Reymond,[1] it can be proved that I_2 does not tend to any definite limit, as $n \to \infty$. Hence the corresponding Fourier series will not converge at θ, although

$$\int_0^\delta |\Phi(t)| \, t^{-2} \, dt$$

exists.

Thus it has been shown that a Fourier series may be non-convergent at a point, but nevertheless absolutely summable (A) at that point.

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[1] *Abhand. d. Bayer. Akad.* (1876), II, 37.
See also G. H. Hardy, *Quarterly Journal*, **44** (1913), 242–263.