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ROTUNDITY IN KÖTHE SPACES OF VECTOR-VALUED FUNCTIONS

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In this paper, Köthe spaces of vector-valued functions are considered. These spaces, which are generalizations of both the Lebesgue-Bochner and Orlicz-Bochner spaces, have been studied by several people (e.g., see [**1**], [**8**]). Perhaps the earliest paper concerning the rotundity of such Köthe space is due to I. Halperin [**8**]. In his paper, Halperin proved that the function spaces $E(X)$ is uniformly rotund exactly when both the Köthe space E and the Banach space X are uniformly rotund; this generalized the analogous result, due to M. M. Day [**4**], concerning Lebesgue-Bochner spaces. In [**20**], M. Smith and B. Turett showed that many properties akin to uniform rotundity lift from X to the Lebesgue-Bochner space $L^p(X)$ when $1 < p < \infty$. A survey of rotundity notions in Lebesgue-Bochner function and sequence spaces can be found in [**19**]. Further investigations showed that many of the theorems known in the context of Lebesgue-Bochner spaces still held true for Köthe spaces $E(X)$ where E is an Orlicz or a Musielak-Orlicz space ([**10**], [**12**]). Although the methods used in [**10**], [**12**], and [**20**] are different, it is possible to obtain several results analogous to those from [**10**] and [**20**] in the more general setting of Köthe spaces of vector-valued functions.

The question of when Köthe spaces of vector-valued functions posses certain rotundity properties is investigated here. In particular, Köthe spaces of vector-valued functions which are locally uniformly rotund, uniformly rotund in every direction, weakly uniformly rotund, or weakly uniformly rotund in the sense of Cudia are considered. With the exception of the last property, these properties are well-known. Weak uniform rotundity in the sense of Cudia is not a well-known property under this name, but it is well-known in several different senses; Theorem 9 below states several well-known geometric or approximation-theoretic conditions which are equivalent to Cudia's notion of weak uniformly rotundity.

Let us agree on some terminology. Let (T, Σ, μ) denote a measure space with a σ-finite and complete measure μ and $\mathcal{M}(T) = \mathcal{M}$ the space of Σ-measurable real-valued functions with functions equal μ-almost everywhere identified. We shall denote the support of a function u, that is, $\{t \in T : u(t) \neq 0\}$, by supp u. If $\mathcal{H}$ is a subset of $\mathcal{M}$, supp $\mathcal{H}$ will denote

$$\cup\{\text{supp } u : u \in \mathcal{H}\}.$$

A *Köthe space* E is a Banach subspace of $\mathcal{M}$ such that (i) if $|u| \leqq |v|$ μ – a. e. with $u \in \mathcal{M}$ and $v \in E$, then $u \in E$ and $\|u\| \leqq \|v\|$, and (ii) supp $E = T$. Köthe

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spaces are Banach lattices and detailed studies of Banach lattices and Köthe spaces may be found in [**2**], [**14**], [**15**], [**17**], and [**21**].

We recall a few known definitions and facts which will be useful in the sequel. The first result follows quickly from Theorem 1.2.1 in [**2**].

THEOREM 1. *Given a Köthe space E, there exists an increasing sequence (T_n) of subsets of T with*

$$\mu(T_n) < \infty \quad \textit{and} \quad \mu(T \setminus \cup T_n) = 0 \quad \textit{and}$$

$$L^\infty|_{T_n} \hookrightarrow E|_{T_n} \hookrightarrow L^1|_{T_n}$$

for all $n \in \mathbf{N}$. (The symbol $\hookrightarrow$ denotes a continuous embedding.)

A Banach lattice E is said to have an *order continuous norm* if, for every downward directed set $\{x_\alpha\}$ with $\inf \{x_\alpha\} = 0, \lim \|x_\alpha\| = 0$. Since the norm of a Köthe function space is σ-order complete, the norm of a Köthe function space is order continuous whenever every downward directed sequence with infimum 0 actually converges to 0 in norm [**15**, pp. 7, 29]. A Banach lattice E is a *KB – space* (Kantorovich-Banach) if each monotone sequence in the unit ball of E converges. In every KB-space, the norm is order continuous.

We state a few results concerning Banach lattices with order continuous norm and KB-spaces.

THEOREM 2. ([**2**], [**15**], [**17**], [**18**]). *Let E be a Banach lattice. Then:*

(i) *E has an order continuous norm if and only if E contains no subspace isomorphic to l^∞.*

(ii) *E is reflexive if and only if both E and E^* are KB-spaces.*

Now let us define the type of spaces to be considered in this paper. For a real Banach space X, denote by $\mathcal{M}(T,X)$, or just $\mathcal{M}(X)$, the family of all strongly measurable funcitons $f : T \longrightarrow X$ identifying functions which are μ-almost everywhere equal. For the Köthe space E, let

$$E(X) = \{f \in \mathcal{M}(X) : u(\cdot) = \|f(\cdot)\|_X \in E\}.$$

With the norm

$$\|f\| = \| \, \|f(\cdot)_X\| \, \|_E,$$

$E(X)$ is a Banach space.

Denote by $E^*_w(X^*)$, the set of weak* measurable functions $g : T \longrightarrow X^*$ such that $\|g(\cdot)\|_{X^*} \in E^*$ where two functions g_1 and g_2 in $E^*_w(X^*)$ are identified if

$$\langle x, g_1(\cdot)\rangle = \langle x, g_2(\cdot)\rangle$$

μ-almost everywhere for all $x \in X$. The following theorem characterizes the dual of some $E(X)$-spaces.

THEOREM 3 [1]. *Let E be a Köthe space with an order continuous norm. The map*

$$U : E_w^*(X^*) \longrightarrow E(X)^*$$

defined by

$$(Ug)(f) = \int \langle f, g \rangle \, d\mu \quad \text{for } f \in E(X)$$

is an isometry between the spaces $E_w^*(X^*)$ and $E(X)^*$.

Finally, two more definitions. A Köthe space E has the *Fatou property* if whenever $0 \leqq u_n \uparrow u$ and $u \in E$, then $\|u_n\| \uparrow \|u\|$. A Banach space X is *locally uniformly rotund* if, for each $\epsilon > 0$ and each $x \in X$ with $\|x\| = 1$, there exists $\delta(x, \epsilon) > 0$ such that if $y \in X$ with $\|y\| = 1$ and $\|x - y\| \geqq \epsilon$, then

$$\|(x + y)/2\| \leqq 1 - \delta(x, \epsilon).$$

LEMMA 4. *A locally uniformly rotund Köthe space with the Fatou property has an order continuous norm.*

Proof. In order to obtain a contradiction, let (f_n) be a sequence in the Köthe space E which decreases almost everywhere to 0 but with

$$\inf \|f_n\| > \delta \quad \text{for some } \delta > 0.$$

Setting

$$g_n = (f - f_n)/\|f_1\|$$

produces a sequence in the unit ball of E which increases almost everywhere to $f_1/\|f_1\|$. Thus, by the Fatou property, $\|g_n\|$ increases to 1. Since $g_n \leqq f_1/\|f_1\|$ almost everywhere,

$$1 \geqq \|(g_n + f_1/\|f_1\|)/2\| \geqq \|g_n\| \longrightarrow 1$$

and the local uniform rotundity of E implies that (g_n) converges to $f_1/\|f_1\|$ in norm. But

$$\|g_n - f_1/\|f_1\| \, \| = \|f_n\|/\|f_1\| > \delta/\|f_1\| > 0,$$

a contradiciton. Thus $\inf \|f_n\| = 0$ and the proof is complete.

We are now in position to characterize the locally uniformly rotund $E(X)$-space in the setting that E has the Fatou property.

THEOREM 5. *If E has the Fatou property then the space E(X) is locally uniformly rotund if and only if both E and X are locally uniformly rotund.*

Proof. Suppose the function space E and the range space X are both locally uniformly rotund. Let $\epsilon > 0$ and f in $E(X)$ with $\|f\| = 1$ be given. For each $n \in N$, set

$$A_n = \{t \in \operatorname{supp} f : \delta_X(f(t)/\|f(t)\|, \epsilon/8) \geqq 1/n\}.$$

By the upper semicontinuity of $\delta_X(\cdot, \epsilon/8)$ (see e.g. [**13**]), A_n is a measurable set. The local uniform rotundity of X implies that the sequence

$$(\|f(\cdot)\chi_{T\setminus A_n}\|)$$

decreases a.e. to 0. Lemma 4 then yields a natural number $n_0 \geqq 2$ such that

$$\|f\chi_{T\setminus A_{n_0}}\| \leqq \epsilon/64.$$

Let g be a norm-one function in $E(X)$ satisfying $\|f - g\| \geqq \epsilon$. Define

$$B = \{t \in T : \|f(t) - g(t)\| \geqq (\epsilon/4)\max\{\|f(t)\|, \|g(t)\|\}\}.$$

Since it is clear that

$$\|(f - g)\|_{T\setminus B}\| \leqq \epsilon/2,$$

it follows that

$$\|(f - g)\chi_B\| \geqq \epsilon/2.$$

Let

$$\eta = (1/2)\min\{\epsilon/8, 1/n_0\}.$$

Partition the set B into four sets:

$$B_1 = \{t \in B : \|g(t)\| < (1 - \eta)\|f(t)\|\}$$
$$B_2 = \{t \in B : (1 - \eta)\|f(t)\| \leqq \|g(t)\| \leqq \|f(t)\|\}$$
$$B_3 = \{t \in B : (1 - \eta)\|g(t)\| \leqq \|f(t)\| \leqq \|g(t)\|\}$$
$$B_4 = \{t \in B : \|f(t)\| < (1 - \eta)\|g(t)\|\}.$$

Then

$$\|(f - g)\chi_{B_i}\| \geqq \epsilon/8 \quad \text{for some } i = 1, 2, 3, 4.$$

For t in the sets B_1 or B_4, the norms of $f(t)$ and $g(t)$ differ by a "large" amount; here things are easy. For example, assume

$$\|(f-g)\chi_{B_1}\| \geqq \epsilon/8,$$

then, by the definition of B_1, $\|f\chi_{B_1}\| \geqq \epsilon/16$ and thus,

$$\|\,(\|f(\cdot)\| - \|g(\cdot)\|)\|_E \geqq \eta\epsilon/16.$$

The local uniform rotundity of E then yields that

$$\|(f+g)/2\| \leqq \|(\|f(\cdot)\| + \|(g(\cdot)\|)/2\|_E \leqq 1 - \delta_E(\|f(\cdot)\|, \eta\epsilon/16).$$

In the setting

$$\|(f-g)\|\chi_{B_4}\| \geqq \epsilon/8,$$

an analgous result occurs.

For t in the sets B_2 and B_3, the norms of $f(t)$ and $g(t)$ differ only by a "small" amount. Assume

$$\|(f-g)\|\chi_{B_2}\| \geqq \epsilon/8.$$

Then, for $t \in B_2 \cap A_{n_0}$,

$$\begin{aligned}\left\|\frac{f(t)+g(t)}{2}\right\| &\leqq \left(1 - \delta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\epsilon}{8}\right)\right)\|f(t)\| \\ &\leqq (1 - 1/n_0)\|f(t)\| \\ &\leqq ((1-1/n_0)/(1-\eta))\,\frac{\|f(t)\| + \|g(t)\|}{2} \\ &= (1-\alpha_1)\,\frac{\|f(t)\| + \|g(t)\|}{2}\end{aligned}$$

where

$$\alpha_1 = \left(\frac{1}{n_0} - \eta\right)\Big/(1-\eta) > 0.$$

If $\|(f-g)\chi_{B_3}\| \geqq \epsilon/8$, let $t \in B_3 \cap A_{n_0}$. Then

$$\begin{aligned}\left\|\frac{f(t)}{\|f(t)\|} - \frac{g(t)}{\|g(t)\|}\right\| &\geqq \left\|\frac{f(t)}{\|g(t)\|} - \frac{g(t)}{\|g(t)\|}\right\| - \left\|\frac{f(t)}{\|g(t)\|} - \frac{f(t)}{\|f(t)\|}\right\| \\ &\geqq \frac{\epsilon}{4} - \frac{\|g(t)\| - \|f(t)\|}{\|g(t)\|} \\ &\geqq \frac{\epsilon}{4} - \eta \\ &\geqq \frac{\epsilon}{8}.\end{aligned}$$

If follows from this that

$$\frac{1}{2}\left\|\frac{f(t)}{\|f(t)\|}+\frac{g(t)}{\|g(t)\|}\right\| \leqq 1-\delta_X\left(\frac{f(t)}{\|f(t)\|},\frac{\epsilon}{8}\right)$$

for t in $B_3 \cap A_{n_0}$. Thus, for t in $B_3 \cap A_{n_0}$,

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{f(t)}{\|f(t)\|}+\frac{g(t)}{\|f(t)\|}\right\| \\
&\leqq \frac{1}{2}\left(1-\delta_X\left(\frac{f(t)}{\|f(t)\|},\frac{\epsilon}{8}\right)\right)+\frac{1}{2}\left\|\frac{g(t)}{\|f(t)\|}-\frac{g(t)}{\|g(t)\|}\right\| \\
&\leqq \tfrac{1}{2}(1-1/n_0)+\frac{\|g(t)\|-\|f(t)\|}{2\|f(t)\|} \\
&=\frac{\|g(t)\|}{2\|f(t)\|}-\frac{1}{2n_0} \\
&\leqq \frac{1}{2}\left(\frac{1}{1-\eta}-\frac{1}{n_0}\right) \\
&=1-\alpha_2
\end{aligned}$$

where the last equality defines $\alpha_2 > 0$ since

$$\eta < \frac{1}{n_0} \leqq \frac{n_0+1}{2n_0+1}.$$

Therefore, if $t \in B_3 \cap A_{n_0}$,

$$\begin{aligned}
\|(f(t)+g(t))/2\| &\leqq (1-\alpha_2)\|f(t)\| \\
&\leqq (1-\alpha_2)\,\frac{\|f(t)\|+\|g(t)\|}{2}.
\end{aligned}$$

Thus if either

$$\|(f-g)\chi_{B_2}\| \geqq \epsilon/8 \quad \text{or} \quad \|(f-g)\chi_{B_3}\| \geqq \epsilon/8,$$

with $\alpha = \min\{\alpha_1, \alpha_2, \frac{1}{4}\}$ and C equaling the corresponding $B_2 \cap A_{n_0}$ or $B_3 \cap A_{n_0}$, we have

$$\left\|\frac{f(t)+g(t)}{2}\right\| \leqq (1-\alpha)\,\frac{\|f(t)\|+\|g(t)\|}{2}$$

for $t \in C$.

We claim that C is a "large" set in the sense that $\|f\chi_C\| \geqq \epsilon/64$. Assuming this momentarily, how does the proof finish? Note

$$\begin{aligned}\left\|\frac{f+g}{2}\right\| &\leqq \left\|\frac{\|f(\cdot)\|+\|g(\cdot)\|}{2}\chi_{T\setminus C}+(1-\alpha)\frac{\|f(\cdot)\|+\|g(\cdot)\|}{2}\chi_C\right\| \\ &\leqq \tfrac{1}{2}\|(\|f(\cdot)\|-\alpha\|f(\cdot)\|\chi_C)\|+\tfrac{1}{2} \\ &\leqq \frac{1}{2}\left\|\frac{\|f(\cdot)\|+(\|f(\cdot)\|-2\alpha\|f(\cdot)\|\chi_C)}{2}\right\|+\frac{1}{2} \\ &\leqq \tfrac{1}{2}(1-\delta_E(\|f(\cdot)\|,\alpha\epsilon/32))+\tfrac{1}{2} \\ &= 1-\tfrac{1}{2}\delta_E(\|f(\cdot)\|,\alpha\epsilon/32).\end{aligned}$$

Thus no matter which of the inequalities, $\|(f-g)\chi_{B_i}\| \geqq \epsilon/8$, $i=1,2,3,4$, holds,

$$\left\|\frac{f+g}{2}\right\| \leqq 1-\delta$$

where

$$\delta = \min\left\{\delta_E(\|f(\cdot)\|,\eta\epsilon/16),\tfrac{1}{2}\delta_E(\|f(\cdot)\|,\alpha\epsilon/32)\right\}.$$

Thus, once the claim is established, the proof of the theorem will be complete.

We now establish the claim. In the case $\|(f-g)\chi_{B_2}\| \geqq \epsilon/8$,

$$C = B_2 \cap A_{n_0}.$$

Since $\|g(t)\| \leqq \|f(t)\|$ for t in B_2,

$$\|f\chi_{B_2}\| \geqq \epsilon/16.$$

Then

$$\|f\chi_{B_2\cap A_{n_0}}\| \geqq \|f\chi_{B_2}\| - \|f\chi_{B_2\setminus A_{n_0}}\| \geqq \epsilon/16-\epsilon/64 \geqq \epsilon/64.$$

In the case $\|(f-g)\chi_{B_3}\| \geqq \epsilon/8$,

$$C = B_3 \cap A_{n_0}.$$

Here

$$\epsilon/8 \leqq \|f\chi_{B_3}\| + \|g\chi_{B_3}\| \leqq \frac{2}{1-\eta}\|f\chi_{B_3}\| \leqq 4\|f\chi_{B_3}\|$$

by the definitions of B_3 and η. Thus

$$\|f\chi_{B_3}\| \geqq \epsilon/32.$$

Then

$$\|f\chi_{B_3\cap A_{n_0}}\| \geqq \|f\chi_{B_3}\| - \|f\chi_{B_3\setminus A_{n_0}}\| \geqq \epsilon/64.$$

This establishes the claim and the theorem.

Although the above proof is similar in nature to the proof of the special case $E = L^p$, $1 < p < \infty$, given in [**20**], there is a difference. In [**20**], the assumption that μT is finite is actually used in the proof; in the Lebesgue-Bochner case, this is not really a restriction since, for any σ-finite measure μ, there exists a probability measure ν with $L^p(\mu, X)$ linearly isometric to $L^p(\nu, X)$. Since the properties considered in [**20**] are sequentially determined, knowing the theorems for σ-finite measures quickly yields the theorems in general. In the more general case considered above, it is unclear if knowing the theorem for all finite measures yields the theorem for a general measure. Thus the above proof avoids any dependence on the measure being finite. It should also be mentioned that in the case $E = L^p$, $1 < p < \infty$, a shorter and easier proof has been provided by M. Smith [**19**]. It is not apparent that his proof can be modified in the more general setting considered above.

For the special case $E = L_\Phi$ where L_Φ is a Musielak- Orlicz space, the analogous result concerning local uniform rotundity has been given in [**10**] but under the assumption of separability of X. By virtue of the above theorem, the assumption of separability of X may be avoided in [**10**].

A different sort of generalization of uniform rotundity is uniform rotundity in every direction, a property that characterizes those Banach spaces in which every bounded set has at most one Čebyšev center. A Banach space X is *uniformly rotund in every direction* if, for each $\epsilon > 0$ and nonzero $z \in X$, there exists $\delta_X(\rightarrow z, \epsilon) > 0$ such that if x and y are norm-one elements in X with $x - y = \alpha z$ for some α and $\|x - y\| \geqq \epsilon$, then

$$\|(x + y)/2\| \leqq 1 - \delta_X(\rightarrow z, \epsilon).$$

There is no guarantee that a Köthe space which is uniformly rotund in every direction has an order continuous norm. As an example, consider the space $L^\infty[0, 1]$ renormed with the norm

$$|||f||| = \sqrt{\|f\|_\infty^2 + \|f\|_2^2}.$$

It has been shown by Zizler (see e.g. [**4**], p. 1055) that $(L^\infty[0, 1], |||\cdot|||)$ is uniformly rotund in every direction and that $|||\cdot|||$ is equivalent to $\|\cdot\|_\infty$. Since

$$|||\chi_{[0,1/n]}||| \geqq 1 \quad \text{for all } n \text{ in } \mathbf{N},$$

$|||\cdot|||$ is not order continuous. However, if a Köthe space is uniformly rotund, then it does have an order continuous norm. This follows from the general fact that reflexive Banach lattices have order continuous norms (see e.g. [**15**, p. 27]). Using this, we have a sufficient condition for $E(X)$ to be uniformly rotund in every direction.

THEOREM 6. *If a Köthe space E is uniformly rotund and if X is uniformly rotund in every direction, then $E(X)$ is uniformly rotund in every direction.*

Proof. Let $f \neq 0$ belong to $E(X)$ and let $g \in E(X)$ be a norm-one element such that $\|f+g\| \leqq 1$. By Theorem 1 in [**5**], it suffices to show that

$$\| g + \tfrac{1}{2}f\| \leqq 1 - \gamma(f) \quad \text{where } \gamma(f) > 0.$$

Setting

$$A = \{t \in T : \|f(t)\| \geqq (\|f\|/4)\max\{\|f(t)+g(t)\|, \|g(t)\|\} > 0\},$$

we have

$$\|f\chi_A\| \geqq \|f\|/2.$$

Let

$$A_n = \{t \in \operatorname{supp} f : \delta_X(\rightarrow f(t), \|f\|/4) \geqq 1/n\}.$$

By [**13**], A_n is a measurable set. Since E has an order continuous norm, there exists $n_0 \in \mathbf{N}$ such that

$$\|f\chi_{T\setminus A_{n_0}}\| \leqq \|f\|/4.$$

So, for $B = A \cap A_{n_0}$, we have

$$\|f\chi_B\| \geqq \|f\|/4.$$

Let $0 < \theta < 1/n_0$ and define

$$B_1 = \{t \in B : |\ \|f(t)+g(t)\| - \|g(t)\|\ | \geqq \theta \max\{\|f(t)+g(t)\|, \|g(t)\|\}\}.$$

There are two possibilities: either

$$\|f\chi_{B_1}\| \geqq \|f\|/8 \quad \text{or} \quad \|f\chi_{B\setminus B_1}\| \geqq \|f\|/8.$$

If $\|f\chi_{B_1}\| \geqq \|f\|/8$, it is easy to see that

$$\|(\|f(\cdot)+g(\cdot)\|_X - \|g(\cdot)\|_X)\|_E \geqq \theta\|f\|/16.$$

Since E is uniformly rotund,

$$\|g + \tfrac{1}{2}f\| \leqq 1 - \delta_E(\theta\|f\|/16).$$

If $\|f\chi_{B\backslash B_1}\| \geqq \|f\|/8$, it follows that

$$(1-\theta)\max\{\|f(t)+g(t)\|, \|g(t)\|\} \leqq \min\{\|f(t)+g(t)\|, \|g(t)\|\}$$

if $t \in B\backslash B_1$. Since X is uniformly rotund in every direction and, for $t \in B$,

$$\left\| \frac{(f(t)+g(t))-g(t)}{\max\{\|f(t)+g(t)\|, \|g(t)\|\}} \right\| \geqq \frac{\|f\|}{4},$$

it follows that whenever $t \in B\backslash B_1$,

$$\begin{aligned} &\left\| g(t) + \frac{f(t)}{2} \right\| \\ &\leqq \frac{1-\delta_X(\rightarrow f(t), \|f\|/4)}{1-\theta} \min\{\|f(t)+g(t)\|, \|g(t)\|\} \\ &\leqq \frac{1-1/n_0}{1-\theta} \frac{\|f(t)+g(t)\| + \|g(t)\|}{2} \\ &= (1-\alpha)\frac{\|f(t)+g(t)\| + \|g(t)\|}{2} \end{aligned}$$

where $\alpha > 0$ is defined to be $1-(1-1/n_0)/(1-\theta)$. Set

$$u = \|g(\cdot) + \tfrac{1}{2}f(\cdot)\|_X \quad \text{and} \quad v = \tfrac{1}{2}(\|f(\cdot)+g(\cdot)\|_X + \|g(\cdot)\|_X).$$

Then u and v are in the unit ball of the Köthe space E and

$$v(t) - u(t) \geqq \alpha v(t) \quad \text{for } t \in B\backslash B_1.$$

Thus,

$$\|v-u\| \geqq \alpha\|f\|/16.$$

Since E is uniformly rotund,

$$\|g + \tfrac{1}{2}f\|_E \leqq \left\|\frac{u+v}{2}\right\|_E \leqq 1 - \delta_E(\alpha\|f\|/16).$$

Combining the two cases above completes the proof of the theorem.

As a corollary to Theorem 6, we obtain the analgous result in the context of Lebesgue-Bochner function spaces given in [**20**]. Theorem 6 is not however sharp enough to obtain the analogous results for Orlicz-Bochner spaces. In fact, it is shown in [**10**] that a Musielak-Orlicz space of Bochner type $L_\Phi(X)$ is uniformly rotund in every direction exactly when both L_Φ and X are. Moreover, for Musielak-Orlicz spaces, the notions of uniform rotundity in every direction and rotundity coincide and are weaker than uniform rotundity [**9**]. Also, if E is

$(L^\infty, |||\cdot|||)$ defined prior to the statement of Theorem 6, then $E(X)$ is uniformly rotund in every direction when X is. Since this particular E is not uniformly rotund (it is not even reflexive), Theorem 6 is clearly not the best possible.

One geometric property of Banach spaces which lies in strength between uniform rotundity and uniform rotundity in every direction is weak uniform rotundity. A Banach space X is *weakly uniformly rotund* if, for each $\epsilon > 0$ and $x^* \in X^*$ with $\|x^*\| = 1$, there exists $\delta(x^*, \epsilon) > 0$ such that if x and y are in X with $\|x\| = \|y\| = 1$ and $x^*(x - y) \geqq \epsilon$, then

$$\|\tfrac{1}{2}(x+y)\| \leqq 1 - \delta(x^*, \epsilon).$$

Weakly uniformly rotund Banach spaces need not be reflexive. In fact, c_0, or more generally, any Banach space with a separable dual can be renormed to be weakly uniformly rotund [**22**]. The situation is different for weakly uniformly rotund KB-spaces.

LEMMA 7. *Weakly uniformly rotund KB-spaces are reflexive.*

Proof. Let E be a weakly uniformly rotund KB-space. If E is not reflexive, then, from our introductory remarks, E^* is not a KB-space and hence E contains an isomorphic copy of l^1 [**15**]. But l^1 can not be renormed to be weakly uniformly rotund (see e.g. [**11**] or [**23**]). This contradiction completes the proof of the lemma.

The next theorem provides a characterization of some weakly uniformly rotund $E(X)$ spaces.

THEOREM 8. *Let the Köthe space E be a KB-space. The space $E(X)$ is weakly uniformly rotund if and only if both E and X are weakly uniformly rotund.*

Proof. Assume that E and X are weakly uniformly rotund. Let f and g be norm-one elements of $E(X)$ and let h be a norm-one element in $E(X)^*$ such that

$$\int \langle f - g, h\rangle d\mu \geqq \epsilon.$$

(Note by Lemma 4 that, since E is reflexive and hence has an order continuous norm, $E(X)^* = E_w^*(X^*)$. Also note that the measurability of f and g allows us to assume, without loss of generality, that X is separable.) Let (T_n) be an increasing sequence of measurable sets with $\mu(T_n) < \infty$ and $\mu(T \setminus \cup T_n) = 0$ such that $E|_{T_n}$ injects into $L^1|_{T_n}$. By the order continuity of the norm of E^*,

$$\|h\chi_{T\setminus T_n}\| \downarrow 0;$$

so there exists $n_0 \in \mathbf{N}$ such that

$$\int \langle f - g, h\chi_{T_{n_0}}\rangle d\mu \geqq \epsilon/2.$$

Let $K > 0$ satisfy

$$\|u\chi_{T_{n_0}}\|_1 \leqq K\|u\chi_{T_{n_0}}\|$$

and choose $\theta > 0$ such that

$$\epsilon_0 \equiv \tfrac{1}{4}\epsilon - 2\theta K > 0.$$

Setting

$$A_n = \{t \in \operatorname{supp} h : \delta_X(h(t), \theta) \geqq 1/n\},$$

we have that A_n is measurable, by the separability of X, and

$$\|h\chi_{T_{n_0}\cap(T\setminus A_n)}\| \downarrow 0.$$

Thus, for some $m_0 \in \mathbf{N}$ and with $T_0 \equiv A_{m_0} \cap T_{n_0}$, it follows that

$$(*) \qquad \int \langle f - g, h\chi_{T_0}\rangle d\mu \geqq \epsilon/4.$$

Defining

$$A = \{t \in T_0 : \langle f(t) - g(t), h(t)\rangle \geqq \theta m2x\{\|f(t)\|, \|g(t)\|\} > 0,$$

we have

$$\int_{T_0\setminus A} \langle f - g, h\rangle d\mu \leqq \theta \left(\int_{T_{n_0}} \|f(\cdot) d\mu + \int_{T_{n_0}} \|g(\cdot)\| d\mu \right)$$
$$\leqq 2\theta K.$$

This combines with (*) to yield

$$\int_A \langle f - g, h\rangle d\mu \geqq \epsilon_0.$$

Fixing θ_1 in $(0, 1/m_0)$ and setting

$$B = \{t \in A : |\ \|f(t)\| - \|g(t)\|\ | \geqq \theta_1 \max\{\|f(t)\|, \|g(t)\|\}\},$$

either

$$\int_B \langle f - g, h\rangle d\mu \geqq \epsilon_0/2 \quad \text{or} \quad \int_{A\setminus B} \langle f - g, h\rangle d\mu \geqq \epsilon_0/2.$$

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If $\int_B \langle f - g, h\rangle d\mu \geqq \epsilon_0/2$ occurs, then

$$\tfrac{1}{2}\epsilon_0 \leqq \int_B 2\max\{\|f(\cdot)\|, \|g(\cdot)\|\}\|h(\cdot)\|d\mu$$
$$\leqq \frac{2}{\theta_1}\int_T |\ \|f(\cdot)\| - \|g(\cdot)\|\ |\ \|h(\cdot)\|d\mu.$$

Thus

$$\int_T |\ \|f(\cdot)\| - \|g(\cdot)\|\ |\ \|h(t)\|d\mu \geqq \theta_1\epsilon_0/4.$$

Letting

$$\sigma(\cdot) \equiv \operatorname{sign}(\|f(\cdot)\| - \|g(\cdot)\|)$$

yields that

$$\langle \sigma(\cdot)\|f(\cdot)\| - \sigma(\cdot)\|g(\cdot)\|, \|h(\cdot)\|\rangle \geqq \theta_1\epsilon_0/4.$$

The weak uniform rotundity of E then implies that

$$\|\tfrac{1}{2}(f+g)\|_{E(X)} \leqq \|\tfrac{1}{2}(\|f(\cdot)\|_X + \|g(\cdot)\|_X)\|_E$$
$$= \|\tfrac{1}{2}(\sigma(\cdot)\|f(\cdot)\|_X + \sigma(\cdot)\|g(\cdot)\|_X)\|_E$$
$$\leqq 1 - \delta_E(\|h(\cdot)\|, \theta_1\epsilon_0/4).$$

Now consider the case

$$\int_{A\backslash B} \langle f - g, h\rangle d\mu \geqq \epsilon_0/2.$$

If $t \in A\backslash B$, then

$$(1-\theta_1)\max\{\|f(t)\|, \|g(t)\|\} \leqq \min\{\|f(t)\|, \|g(t)\|\}.$$

Thus the weak uniform rotundity of X implies, for $t \in A\backslash B$, that

$$(**)\qquad \|\tfrac{1}{2}(f(t)+g(t))\| \leqq (1 - \delta_X(h(t), \theta))\max\{\|f(t)\|, \|g(t)\|\}$$
$$\leqq \frac{1 - 1/m_0}{1-\theta_1}\min\{\|f(t)\|, \|g(t)\|\}$$
$$\leqq (1-\alpha)\frac{\|f(t)\| + \|g(t)\|}{2}$$

where

$$\alpha = \frac{1/m_0 - \theta_1}{1-\theta_1} > 0$$

by the choice of θ_1. Setting

$$u(\cdot) = \tfrac{1}{2}\|f(\cdot) + g(\cdot)\| \quad \text{and} \quad v(\cdot) = \tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|),$$

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we have

$$\int_T |u(\cdot) - v(\cdot)| \, \|h(\cdot)\| \, \|d\mu \geqq \frac{\alpha}{2} \int_{A\setminus B} (\|f(\cdot)\| + \|g(\cdot)\|)\|h(\cdot)\|d\mu$$
$$\geqq \frac{\alpha}{2} \int_{A\setminus B} \|f(\cdot) - g(\cdot)\| \, \|h(\cdot)\|d\mu$$
$$\geqq \frac{\alpha}{2} \int_{A\setminus B} \langle f - g, h\rangle d\mu$$
$$\geqq \frac{1}{4}\alpha\epsilon_0.$$

Using the sign function as in the previous case, it follows that

$$\|\tfrac{1}{2}(f + g)\|_{E(X)} \leqq \|\tfrac{1}{2}(u + v)\|_E$$
$$\leqq 1 - \delta_E(\|h(\cdot)\|, \alpha\epsilon_0/4).$$

Combining the two cases finishes the proof of Theorem 8.

The analogous theorems in the setting of Bochner L^p-spaces ([**6**], [**20**]) or in the setting of Orlicz-Bochner spaces [**11**] can be obtained as corollaries of Theorem 8.

The final generalization of uniform rotundity considered here is another notion of weak uniform rotundity due to Cudia [**3**]. A Banach space X is *weakly uniformly rotund in the sense of Cudia* if, for every $\epsilon > 0$ and $x^* \in X$, there exists $\delta^c(x^*, \epsilon) > 0$ such that if x and y are in X with $\|x\| = \|y\| = 1$ and $\|x - y\| \geqq \epsilon$, then

$$|x^*(\tfrac{1}{2}(x + y))| \leqq 1 - \delta^c(x^*, \epsilon).$$

The following theorem summarizes what is known connecting this property with other geometric and approximation theoretic properties.

THEOREM 9. ([**7**], [**11**], [**19**]). *The following statements are equivalent.*

(a) *X is weakly uniformly rotund in the sense of Cudia.*

(b) *X is an E-space (in the sense of Fan and Glicksberg).*

(c) *X is reflexive and every point in the unit sphere of X is strongly exposed.*

(d) *X is reflexive and rotund and has the Kadec-Klee property.*

One should note that if, in the definition of weak uniform rotundity in the sense of Cudia, x and y are allowed to be in the ball of X and not just on the unit sphere, an equivalent definition is obtained.

Before characterizing the $E(X)$-spaces which are weakly uniformly rotund in the sense of Cudia, we need a lemma.

LEMMA 10. *If a Köthe space E is weakly uniformly rotund in the sense of Cudia, then, for every norm-one positive functional $F \in E^*$ and for every $\epsilon > 0$, there exists $\gamma(F, \epsilon) > 0$ such that, for all $u \in E$ and $A \in \Sigma$, $\|u\| \leqq 1$ and $\|u\chi_A\| \geqq \epsilon$ implies that*

$$F(u\chi_{T\setminus A}) \leqq 1 - \gamma(F, \epsilon).$$

PROOF. In order to obtain a contradiction, suppose that there exist a norm-one positive functional $F \in E^*$, $\epsilon > 0$, and sequences (u_n) and (A_n) with

$$0 \leqq u_n, \|u_n\| \leqq 1, \quad \|u_n \chi_{A_n}\| \geqq \epsilon, \quad \text{and}$$
$$F(u_n \chi_{T \setminus A_n}) \geqq 1 - \frac{1}{n}.$$

Setting $v_n = u_n \chi_{T \setminus A_n}$ yields

$$\|u_n - v_n\| \geqq \epsilon \quad \text{and}$$
$$F(\tfrac{1}{2}(u_n + v_n)) \geqq F(u_n \chi_{T \setminus A_n}) \geqq 1 - \frac{1}{n}.$$

This contradicts that E is weakly uniformly rotund in the sense of Cudia and the proof of Lemma 10 is complete.

Our final result is:

THEOREM 11. *The space $E(X)$ is weakly uniformly rotund in the sense of Cudia if and only if both E and X are weakly uniformly rotund in the sense of Cudia.*

Proof. Let E and X be weakly uniformly rotund in the sense of Cudia. Again, there is no loss of generality in assuming that X is separable. Let $\epsilon > 0$ be given and let f and g be norm-one elements in $E(X)$ with $\|f - g\| \geqq \epsilon$. Since E is reflexive, choose

$$H \in E(X)^* = E_w^*(X^*) \quad \text{with } \|H\| = 1.$$

Define $\tilde{H} \in E^*$ by

$$\tilde{H}(u) = \int u(\cdot)\|H(\cdot)\|_{X^*} d\mu;$$

$\tilde{H}$ is a positive norm-one linear functional on E.

Choose $\theta > 0$ such that $\epsilon_0 \equiv \epsilon - 2\theta > 0$. Setting

$$A_n = \{t \in \operatorname{supp} H : \delta_X^c(H(t)/\|H(t)\|, \theta) \geqq 1/n\},$$

there exists $n_0 \in \mathbf{N}$ such that

$$\|H \chi_{T \setminus A_{n_0}}\| \leqq \tfrac{1}{2}\gamma(\tilde{H}, \epsilon_0/6)$$

where $\gamma(\tilde{H}, \epsilon_0/6)$ is defined in the preceding lemma. Choose $\theta_1 > 0$ such that

$$(1 - 1/n_0)/(1 - \theta_1) < 1$$

and define the sets

$$A = \{t \in T : \|f(t) - g(t)\| \geqq \theta \max\{\|f(t)\|, \|g(t)\|\} \geqq 0\}$$
$$B = \{t \in T : |\ \|f(t)\| - \|g(t)\|\ | \geqq \theta_1 \max\{\|f(t)\|, \|g(t)\|\}\}.$$

Since

$$A = B \cup [(A \backslash B) \cap A_{n_0}] \cup [(A \backslash B) \backslash A_{N_0}] \quad \text{and} \quad \|(f-g)\chi_A\| \geqq \epsilon_0,$$

we consider three cases.

If $\|(f-g)\chi_B\| \geqq \epsilon_0/3$, then

$$\|(\|f(\cdot)\|_X - \|g(\cdot)\|_X)\|_E \geqq \theta_1 \epsilon_0/6.$$

Since E is weakly uniformly rotund in the sense of Cudia,

$$H(\tfrac{1}{2}(f+g)) \leqq \tilde{H}(\tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|)) \leqq 1 - \delta_E^c(\tilde{H}, \theta_1\epsilon_0/6).$$

If $\|(f-g)\chi_{(A\backslash B)\cap A_{n_0}}\| \geqq \epsilon_0/3$, the Cudia weak uniform rotundity of X yields, similar to the proof of (**) in Theorem 8, that

$$|\langle f(t)+g(t), H(t)\rangle| \leqq (1-\alpha)\|H(t)\|(\|f(t)\| + \|g(t)\|)/2$$

for $t \in (A\backslash B) \cap A_{n_0}$ where

$$\alpha = (1/n_0 - \theta_1)/(1-\theta_1).$$

Now set

$$u(\cdot) = |\langle \tfrac{1}{2}(f(\cdot)+g(\cdot)), H(\cdot)/\|H(\cdot)\|\rangle|\chi_{\operatorname{supp} H}(\cdot) \quad \text{and}$$
$$v(\cdot) = \tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|).$$

Then

$$\|u - v\| \geqq \alpha\epsilon_0/6 \quad \text{and}$$

$$|H(\tfrac{1}{2}(f+g))| \leqq \tilde{H}(\tfrac{1}{2}(u+v)) \leqq 1 - \delta_E^c(\tilde{H}, \alpha\epsilon_0/6).$$

Finally, if

$$\|(f-g)\chi_{(A\backslash B)\backslash A_{n_0}}\| \geqq \epsilon_0/3,$$

then

$$\|\tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|)\chi_{T\backslash A_{n_0}}\| \geqq \epsilon_0/6.$$

So, by Lemma 10,

$$\tilde{H}(\tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|)\chi_{A_{n_0}}) \leqq 1 - \gamma(\tilde{H}, \epsilon_0/6).$$

Then, by the choice of n_0,

$$|H(\tfrac{1}{2}(f+g))| \leqq \tilde{H}(\tfrac{1}{2}(\|f(\cdot)\| + \|g(\cdot)\|)\chi_{A_{n_0}}) + \|H\chi_{T\setminus A_{n_0}}\| \\ \leqq 1 - \tfrac{1}{2}\gamma(\tilde{H}, \epsilon_0/6).$$

Thus, no matter which case occurs, $|H(\frac{1}{2}(f+g))|$ is bounded below 1. This completes the proof of Theorem 11.

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