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Abstract

Bershadsky, Cecotti, Ooguri and Vafa constructed a real-valued invariant for Calabi–Yau manifolds, which is now called the BCOV invariant. In this paper, we extend the BCOV invariant to such pairs (X, D), where X is a compact Kähler manifold and D is a pluricanonical divisor on X with simple normal crossing support. We also study the behavior of the extended BCOV invariant under blow-ups. The results in this paper lead to a joint work with Fu proving that birational Calabi–Yau manifolds have the same BCOV invariant.

Contents

Introduction

In this paper, we consider a real-valued invariant for Calabi–Yau manifolds equipped with Ricci flat metrics, which is now called the BCOV torsion. The BCOV torsion was introduced by

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Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] as the stringy genus-one partition function of $N = 2$ superconformal field theory. Their work extended the mirror symmetry conjecture of Candelas, de la Ossa, Green and Parkes [COGP91]. Fang and Lu [FL05] used BCOV torsion to study the moduli space of Calabi–Yau manifolds.

The BCOV torsion is an invariant on the B-side. Its mirror on the A-side is conjecturally the genus-one Gromov–Witten invariant. Though genus $\geqslant 2$ Gromov–Witten invariants have been intensively studied recently, there is no rigorously defined genus $\geqslant 2$ invariant on the B-side.

The BCOV invariant is a real-valued invariant for Calabi–Yau manifolds, which could be viewed as a normalization of the BCOV torsion. Fang, Lu and Yoshikawa [FLY08] constructed the BCOV invariant for Calabi–Yau threefolds and established the asymptotics of the BCOV invariant (of Calabi–Yau threefolds) for one-parameter normal crossings degenerations. They also confirmed the (B-side) genus-one mirror symmetry conjecture of Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] for quintic threefolds.

Eriksson, Freixas i Montplet and Mourougane [EFM21] constructed the BCOV invariant for Calabi–Yau manifolds of arbitrary dimension and established the asymptotics of the BCOV invariant for one-parameter normal crossings degenerations. In another paper [EFM22], they confirmed the (B-side) genus-one mirror symmetry conjecture of Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] for Calabi–Yau hypersurfaces of arbitrary dimension, which is compatible with the results of Zinger [Zin08, Zin09] on the A-side.

For a Calabi–Yau manifold X, we denote by $\tau(X)$ the logarithm of the BCOV invariant of X defined in [EFM21].

Yoshikawa [Yos06, Conjecture 2.1] conjectured that for a pair of birational projective Calabi–Yau threefolds (X, X'), we have $\tau(X') = \tau(X)$. Eriksson, Freixas i Montplet and Mourougane [EFM21, Conjecture B] conjectured the following higher-dimensional analogue.

Conjecture 0.1. For a pair of birational projective Calabi–Yau manifolds (X, X'), we have

$$\tau(X') = \tau(X). \tag{0.1}$$

Let X and X' be projective Calabi–Yau threefolds defined over a field L. Let T be a finite set of embeddings $L \hookrightarrow \mathbb{C}$. For $\sigma \in T$, we denote by X_σ (respectively, X'_σ) the base change of X (respectively, X') to $\mathbb{C}$ via the embedding σ. We denote by $D^b(X_\sigma)$ (respectively, $D^b(X'_\sigma)$) the bounded derived category of coherent sheaves on X_σ (respectively, X'_σ). Maillot and Rössler [MR12, Theorem 1.1] showed that if one of the following conditions holds:

(a) there exists $\sigma \in T$ such that X_σ and X'_σ are birational;
(b) there exists $\sigma \in T$ such that $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent;

then there exist a positive integer n and a non-zero element $\alpha \in L$ such that

$$\tau(X'_\sigma) - \tau(X_\sigma) = \frac{1}{n} \log |\sigma(\alpha)| \quad \text{for all } \sigma \in T. \tag{0.2}$$

Although a result of Bridgeland [Bri02, Theorem 1.1] showed that condition (a) implies condition (b), Maillot and Rössler gave separate proofs for conditions (a) and (b).

Let X be a Calabi–Yau threefold. Let $Z \hookrightarrow X$ be a $(-1,-1)$-curve. Let X' be the Atiyah flop of X along Z, which is also a Calabi–Yau threefold. We assume that both X and X' are compact and Kähler. The current author [Zha22, Corollary 0.5] showed that

$$\tau(X') = \tau(X). \tag{0.3}$$

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In other words, Conjecture 0.1 holds for three-dimensional Atiyah flops. The proof of (0.3) consists of two key ingredients:

(i) we extend the BCOV invariant from Calabi–Yau manifolds to certain 'Calabi–Yau pairs', more precisely, we consider manifolds equipped with smooth reduced canonical divisors;
(ii) we study the behavior of the extended BCOV invariant under blow-ups.

To fully confirm Conjecture 0.1 following this strategy, it is necessary to further extend the BCOV invariant as well as the blow-up formula. This is exactly the purpose of this paper. We consider pairs consisting of a compact Kähler manifold and a canonical divisor with rational coefficients on the manifold with simple normal crossing support and without component of multiplicity $\leqslant -1$. We construct the BCOV invariant of such pairs and establish a blow-up formula for our BCOV invariant.

In the joint work with Fu [FZ20], we use the results in this paper together with a factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [AKMW02, Theorem 0.3.1] to confirm Conjecture 0.1 in full generality.

Let us now give more detail about the matter of this paper.

BCOV torsion. We use the notation in (0.23) and (0.24). Let X be an n-dimensional compact Kähler manifold. Let $H^\bullet_{\mathrm{dR}}(X)$ be the de Rham cohomology of X. Let $H^k_{\mathrm{dR}}(X)=\bigoplus_{p+q=k}H^{p,q}(X)$ be the Hodge decomposition. Set

$$\begin{aligned}\lambda_p(X)&=\det H^{p,\bullet}(X)=\bigotimes_{q=0}^{n}\big(\det H^{p,q}(X)\big)^{(-1)^q}\quad\text{for } p=0,\dots,n,\\ \lambda_{\mathrm{tot}}(X)&=\bigotimes_{k=1}^{2n}\big(\det H^k_{\mathrm{dR}}(X)\big)^{(-1)^k k}=\bigotimes_{p=1}^{n}\big(\lambda_p(X)\otimes\overline{\lambda_p(X)}\big)^{(-1)^p p}.\end{aligned}\tag{0.4}$$

Let $H^\bullet_{\mathrm{Sing}}(X,\mathbb{C})$ be the singular cohomology of X with coefficients in $\mathbb{C}$. We identify $H^k_{\mathrm{dR}}(X)$ with $H^k_{\mathrm{Sing}}(X,\mathbb{C})$ (see (1.121)). For $k=0,\dots,2n$, let

$$\sigma_{k,1},\dots,\sigma_{k,b_k}\in\mathrm{Im}\big(H^k_{\mathrm{Sing}}(X,\mathbb{Z})\to H^k_{\mathrm{Sing}}(X,\mathbb{R})\big)\subseteq H^k_{\mathrm{dR}}(X)\tag{0.5}$$

be a basis of the lattice. Set

$$\sigma_X=\bigotimes_{k=1}^{2n}(\sigma_{k,1}\wedge\dots\wedge\sigma_{k,b_k})^{(-1)^k k}\in\lambda_{\mathrm{tot}}(X),\tag{0.6}$$

which is well-defined up to ±1.

Let ω be a Kähler form on X. Let $\|\cdot\|_{\lambda_p(X),\omega}$ be the Quillen metric (see § 1.4) on $\lambda_p(X)$ associated with ω. Let $\|\cdot\|_{\lambda_{\mathrm{tot}}(X),\omega}$ be the metric on $\lambda_{\mathrm{tot}}(X)$ induced by $\|\cdot\|_{\lambda_p(X),\omega}$ via (0.4). Set

$$\tau_{\mathrm{BCOV}}(X,\omega)=\log\|\sigma_X\|_{\lambda_{\mathrm{tot}}(X),\omega},\tag{0.7}$$

which we call the unnormalized BCOV invariant of (X,ω).

BCOV invariant. For a compact complex manifold X and a divisor D on X, we denote

$$D=\sum_{j=1}^{l}m_jD_j,\tag{0.8}$$

where $m_j\in\mathbb{Z}\backslash\{0\}$, $D_1,\dots,D_l\subseteq X$ are mutually distinct and irreducible. We call D a divisor with simple normal crossing support if $D_1,\dots,D_l$ are smooth and transversally intersect. Let d

be a non-zero integer. We assume that D is of simple normal crossing support and $m_j \neq -d$ for $j = 1, \dots, l$. For $J \subseteq \{1, \dots, l\}$, we denote

$$\begin{aligned} w_d^J &= \prod_{j\in J} \frac{-m_j}{m_j + d}, \quad D_J = X \cap \bigcap_{j\in J} D_J, \\ w_d^{\emptyset} &= 1, \quad D_{\emptyset} = X. \end{aligned} \tag{0.9}$$

See [FZ20, § 4] for an interpretation of this construction.

Now let X be a compact Kähler manifold. Let K_X be the canonical line bundle over X. Let K_X^d be the dth tensor power of K_X. Let $\gamma \in \mathscr{M}(X, K_X^d)$ be an invertible element.

Definition 0.2. We call (X, γ) a d-Calabi–Yau pair if:

(i) $\mathrm{div}(\gamma) = \sum_{j=1}^{l} m_j D_j$ is of simple normal crossing support;
(ii) $m_j \neq -d$ for $j = 1, \dots, l$.

Here are some examples of d-Calabi–Yau pairs.

(a) If X is a compact Kähler Calabi–Yau manifold and $0 \neq \gamma \in H^0(X, K_X^d)$, then (X, γ) is a d-Calabi–Yau pair.
(b) If (X, γ) is a d-Calabi–Yau pair with $d > 0$ and $Y \subseteq X$ transversally intersects with $\mathrm{div}(\gamma)$ in the sense of Definition 1.1, then $(\mathrm{Bl}_Y X, f^*\gamma)$ is a d-Calabi–Yau pair, where $f : \mathrm{Bl}_Y X \to X$ is the blow-up along Y.

Now we assume that (X, γ) is a d-Calabi–Yau pair. Let w_d^J and D_J be as in (0.9). Let ω be a Kähler form on X. Recall that $\tau_{\mathrm{BCOV}}(\cdot, \cdot)$ was constructed in (0.7). The BCOV invariant of (X, γ) is defined as

$$\tau_d(X, \gamma) = \sum_{J\subseteq\{1,\dots,l\}} w_d^J \tau_{\mathrm{BCOV}}(D_J, \omega|_{D_J}) + \text{correction terms}, \tag{0.10}$$

where the correction terms are Bott–Chern-type integrations (see Definition 3.2 and (3.10)). We construct $\tau_d(X, \gamma)$ and show that it is independent of ω.

We can further extend our construction to canonical divisors with rational coefficients. We consider a pair (X, D), where X is an n-dimensional compact Kähler manifold, D is a canonical divisor with rational coefficients on X such that:

(i) D is of simple normal crossing support;
(ii) each component of D is of multiplicity > -1.

Definition 0.3. Let d be a positive integer such that dD is a divisor with integer coefficients. Let γ be a meromorphic section of K_X^d such that $\mathrm{div}(\gamma) = dD$. We define

$$\tau(X, D) = \tau_d(X, \gamma) + \frac{\chi_d(X, dD)}{12} \log \left((2\pi)^{-n} \int_{X\backslash|D|} |\gamma\bar{\gamma}|^{1/d} \right), \tag{0.11}$$

where $\chi_d(\cdot, \cdot)$ is defined in Definition 1.3, $|D|$ is defined in (0.25), $|\gamma\bar{\gamma}|^{1/d}$ is the unique positive volume form on $X\backslash|D|$ whose dth tensor power equals $i^{n^2 d}\gamma\bar{\gamma}$. By Propositions 3.3, 3.4, the BCOV invariant $\tau(X, D)$ is well-defined, i.e. independent of d and γ.

Our BCOV invariant differs from the one defined in [EFM21] by a topological invariant. More precisely, if X is a Calabi–Yau manifold, the logarithm of the BCOV invariant of X

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defined in [EFM21] is equal to

$$\tau(X,\emptyset)+\frac{\log(2\pi)}{2}\sum_{k=0}^{2n}(-1)^k k(n-k)b_k(X), \tag{0.12}$$

where $b_k(X)$ is the kth Betti number of X. The sum of Betti numbers in (0.12) comes from our choice of the L^2-metric (see (1.70)) and the identification between singular cohomology and de Rham cohomology (see (1.121)).

Curvature formula. Let $\pi:\mathscr{X}\to S$ be a holomorphic submersion. We assume that π is locally Kähler in the sense of [BGS88b, Definition 1.25], i.e. for any $s\in S$, there exists an open subset $s\in U\subseteq S$ such that $\pi^{-1}(U)$ is Kähler. For $s\in S$, we denote $X_s=\pi^{-1}(s)$. Let

$$\big(\gamma_s\in\mathscr{M}(X_s,K^d_{X_s})\big)_{s\in S} \tag{0.13}$$

be a holomorphic family. We assume that (X_s,γ_s) is a d-Calabi–Yau pair for any $s\in S$. We assume that there exist $l\in\mathbb{N}$, $m_1,\dots,m_l\in\mathbb{Z}\backslash\{0,-d\}$ and $(D_{j,s}\subseteq X_s)_{j\in\{1,\dots,l\},s\in S}$ such that

$$\operatorname{div}(\gamma_s)=\sum_{j=1}^{l}m_jD_{j,s}\quad\text{for } s\in S. \tag{0.14}$$

For $J\subseteq\{1,\dots,l\}$ and $s\in S$, let $D_{J,s}\subseteq X_s$ be as in (0.9) with X replaced by X_s and D_j replaced by $D_{j,s}$. We assume that $(D_{J,s})_{s\in S}$ is a smooth holomorphic family for each J.

Let $\tau_d(X,\gamma)$ be the function $s\mapsto\tau_d(X_s,\gamma_s)$ on S. Let w_d^J be as in (0.9). Let $H^\bullet(D_J)$ be the variation of Hodge structure associated with $(D_{J,s})_{s\in S}$. Let $\omega_{H^\bullet(D_J)}\in\Omega^{1,1}(S)$ be its Hodge form (see [Zha22, § 1.2]).

Theorem 0.4. *The following identity holds:*

$$\frac{\bar\partial\partial}{2\pi i}\tau_d(X,\gamma)=\sum_{J\subseteq\{1,\dots,l\}}w_d^J\omega_{H^\bullet(D_J)}. \tag{0.15}$$

Blow-up formula. Let (X,γ) be a d-Calabi–Yau pair in the sense of Definition 0.2 with $d>0$.

Let $Y\subseteq X$ be a connected complex submanifold such that $Y,D_1,\dots,D_l$ transversally intersect (in the sense of Definition 1.1). We assume that $m_j>0$ for j satisfying $Y\subseteq D_j$. Let r be the codimension of $Y\subseteq X$. Let q be the number of D_j containing Y. Then we have $q\leqslant r$. Without loss of generality, we assume that

$$Y\subseteq D_j\quad\text{for } j=1,\dots,q;\quad Y\nsubseteq D_j\quad\text{for } j=q+1,\dots,l. \tag{0.16}$$

Let $f:X'\to X$ be the blow-up along Y. Let $D_j'\subseteq X'$ be the strict transformation of $D_j\subseteq X$. Set $E=f^{-1}(Y)$. Let $f^*\gamma\in\mathscr{M}(X',K_{X'})$ be the pull-back of γ. We denote $D'=\operatorname{div}(f^*\gamma)$. We denote

$$m_0=m_1+\cdots+m_q+rd-d. \tag{0.17}$$

We have (cf. [MM07, Proposition 2.1.11])

$$D'=m_0E+\sum_{j=1}^{l}m_jD_j'. \tag{0.18}$$

Hence, $(X',f^*\gamma)$ is a d-Calabi–Yau pair.

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Set

$$D_Y = \sum_{j=q+1}^{l} m_j (D_j \cap Y), \quad D_E = \sum_{j=1}^{l} m_j (D'_j \cap E). \tag{0.19}$$

Then D_Y (respectively, D_E) is a divisor on Y (respectively, E) with simple normal crossing support.

We identify $\mathbb{C}\mathrm{P}^r$ with $\mathbb{C}^r \cup \mathbb{C}\mathrm{P}^{r-1}$. Let $(z_1, \ldots, z_r) \in \mathbb{C}^r$ be the coordinates. Let $\gamma_{r,m_1,\ldots,m_q} \in \mathscr{M}(\mathbb{C}\mathrm{P}^r, K^d_{\mathbb{C}\mathrm{P}^r})$ be such that

$$\gamma_{r,m_1,\ldots,m_q}|_{\mathbb{C}^r} = (dz_1 \wedge \cdots \wedge dz_r)^d \prod_{j=1}^{q} z_j^{m_j}. \tag{0.20}$$

Let $H_k \subseteq \mathbb{C}\mathrm{P}^r$ be the closure of $\{z_k = 0\} \subseteq \mathbb{C}^r$. Let $H_\infty = \mathbb{C}\mathrm{P}^{r-1} \subseteq \mathbb{C}\mathrm{P}^r$. We have

$$\mathrm{div}(\gamma_{r,m_1,\ldots,m_q}) = -(m_1 + \cdots + m_q + rd + d) H_\infty + \sum_{j=1}^{q} m_j H_j. \tag{0.21}$$

Thus, $(\mathbb{C}\mathrm{P}^r, \gamma_{r,m_1,\ldots,m_q})$ is a d-Calabi–Yau pair.

Theorem 0.5. *The following identities hold:*

$$\begin{aligned} \chi_d(X', f^*\gamma) - \chi_d(X, \gamma) &= 0, \\ \tau_d(X', f^*\gamma) - \tau_d(X, \gamma) &= \chi_d(E, D_E)\tau_d(\mathbb{C}\mathrm{P}^1, \gamma_{1,m_0}) \\ &\quad - \chi_d(Y, D_Y)\tau_d(\mathbb{C}\mathrm{P}^r, \gamma_{r,m_1,\ldots,m_q}), \end{aligned} \tag{0.22}$$

where $\chi_d(\cdot,\cdot)$ is given by Definition 1.3.

The proof of Theorem 0.5 is based on:

(i) the deformation to the normal cone introduced by Baum, Fulton and MacPherson [BFM75, § 1.5];
(ii) the immersion formula for Quillen metrics due to Bismut and Lebeau [BL91];
(iii) the submersion formula for Quillen metrics due to Berthomieu and Bismut [BB94];
(iv) the blow-up formula for Quillen metrics due to Bismut [Bis97];
(v) the relation between the holomorphic torsion and the de Rham torsion established by Bismut [Bis04].

We remark that the Quillen metric can be extended to orbifolds, and the immersion formula and the submersion formula still hold (see [Ma05, Ma21]).

Notation. For a complex vector space V, we denote

$$\det V = \Lambda^{\dim V} V, \tag{0.23}$$

which is a complex line. For a complex line λ, we denote by λ^{-1} the dual of λ. For a graded complex vector space $V^\bullet = \bigoplus_{k=0}^m V^k$, we denote

$$\det V^\bullet = \bigotimes_{k=0}^{m} (\det V^k)^{(-1)^k}. \tag{0.24}$$

For a complex manifold X and a divisor $D = m_1 D_1 + \cdots + m_l D_l$ on X, where $m_1, \ldots, m_l \in \mathbb{Z}\backslash\{0\}$, $D_1, \ldots, D_l$ are mutually distinct and irreducible, we denote

$$|D| = D_1 \cup \cdots \cup D_l \subseteq X, \tag{0.25}$$

which we call the support of D.

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For a complex manifold X, we denote by $\Omega^{p,q}(X)$ the vector space of (p,q)-forms on X. We denote by $\mathscr{O}_X$ the analytic coherent sheaf of holomorphic functions on X. We denote by Ω^p_X the analytic coherent sheaf of holomorphic p-forms on X. For a complex vector bundle E over X, we denote by $\Omega^{p,q}(X,E)$ the vector space of (p,q)-forms on X with values in E. We denote by $\mathscr{M}(X,E)$ the vector space of meromorphic sections of E. We denote by $\mathscr{O}_X(E)$ the analytic coherent sheaf of holomorphic sections of E. For an analytic coherent sheaf $\mathscr{F}$ on X, we denote by $H^q(X,\mathscr{F})$ the qth cohomology of $\mathscr{F}$. We denote $H^q(X,E) = H^q(X,\mathscr{O}_X(E))$. We denote $H^{p,q}(X) = H^q(X,\Omega^p_X)$. We denote by $H^k_{\mathrm{dR}}(X)$ the kth de Rham cohomology of X with coefficients in $\mathbb{C}$. If X is a compact Kähler manifold, we identify $H^{p,q}(X)$ with a vector subspace of $H^{p+q}_{\mathrm{dR}}(X)$ via the Hodge decomposition.

1. Preliminaries

1.1 Divisor with simple normal crossing support

For $I \subseteq \{1,\ldots,n\}$, we denote

$$\mathbb{C}^n_I = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : z_i = 0 \text{ for } i \in I\} \subseteq \mathbb{C}^n. \tag{1.1}$$

Let X be an n-dimensional complex manifold.

DEFINITION 1.1. For closed complex submanifolds $Y_1,\ldots,Y_l \subseteq X$, we say that $Y_1,\ldots,Y_l$ transversally intersect if for any $x \in X$, there exists a holomorphic local chart $\mathbb{C}^n \supseteq U \xrightarrow{\varphi} X$ such that:

(i) $0 \in U$ and $\varphi(0) = x$;
(ii) for each k, either $\varphi^{-1}(Y_k) = \emptyset$ or $\varphi^{-1}(Y_k) = U \cap \mathbb{C}^n_{I_k}$ for certain $I_k \subseteq \{1,\ldots,n\}$.

Let D be a divisor on X. We denote

$$D = \sum_{j=1}^{l} m_j D_j, \tag{1.2}$$

where $m_j \in \mathbb{Z}\backslash\{0\}$, $D_1,\ldots,D_l \subseteq X$ are mutually distinct and irreducible.

DEFINITION 1.2. We call D a divisor with simple normal crossing support if $D_1,\ldots,D_l$ are smooth and transversally intersect.

For $J \subseteq \{1,\ldots,l\}$, let w^J_d and D_J be as in (0.9), let $\chi(D_J)$ be the topological Euler characteristic of D_J.

DEFINITION 1.3. If D is a divisor with simple normal crossing support, we define

$$\chi_d(X,D) = \sum_{J \subseteq \{1,\ldots,l\}} w^J_d \chi(D_J). \tag{1.3}$$

Moreover, if there is a meromorphic section γ of a holomorphic line bundle over X such that $\mathrm{div}(\gamma) = D$, we define

$$\chi_d(X,\gamma) = \chi_d(X,D). \tag{1.4}$$

Now we assume that D is a divisor with simple normal crossing support. Let L be a holomorphic line bundle over X together with $\gamma \in \mathscr{M}(X,L)$ such that

$$\mathrm{div}(\gamma) = D. \tag{1.5}$$

Let $\gamma^{-1} \in \mathscr{M}(X,L^{-1})$ be the inverse of γ.

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We denote by $(T^*X \oplus \overline{T^*X})^{\otimes k}$ the kth tensor power of $T^*X \oplus \overline{T^*X}$. We denote

$$E_k^{\pm} = (T^*X \oplus \overline{T^*X})^{\otimes k} \otimes L^{\pm 1}. \tag{1.6}$$

In particular, we have $E_0^{\pm} = L^{\pm 1}$. Let $\nabla^{E_k^{\pm}}$ be a connection on $E_k^{\pm}$.

Let L_j be the normal line bundle of $D_j \hookrightarrow X$.

Definition 1.4. We define $\mathrm{Res}_{D_j}(\gamma) \in \mathscr{M}(D_j, L \otimes L_j^{-m_j})$ as follows:

$$\mathrm{Res}_{D_j}(\gamma) = \begin{cases} \dfrac{1}{m_j!}\Big(\nabla^{E^+_{m_j-1}} \cdots \nabla^{E_0^+}\gamma\Big)\Big|_{D_j} & \text{if } m_j > 0, \\ \dfrac{1}{|m_j|!}\Big(\Big(\nabla^{E^-_{|m_j|-1}} \cdots \nabla^{E_0^-}\gamma^{-1}\Big)\Big|_{D_j}\Big)^{-1} & \text{if } m_j < 0. \end{cases} \tag{1.7}$$

Here $\mathrm{Res}_{D_j}(\gamma)$ is independent of $(\nabla^{E_k^{\pm}})_{k \in \mathbb{N}}$.

For $j \in \{1, \dots, l\}$, we have

$$\mathrm{div}\big(\mathrm{Res}_{D_j}(\gamma)\big) = \sum_{k \in \{1,\dots,l\} \backslash \{j\}} m_k (D_j \cap D_k). \tag{1.8}$$

For distinct $j, k \in \{1, \dots, l\}$, we have

$$\begin{aligned} &\mathrm{Res}_{D_j \cap D_k}\big(\mathrm{Res}_{D_j}(\gamma)\big) = \mathrm{Res}_{D_j \cap D_k}\big(\mathrm{Res}_{D_k}(\gamma)\big) \\ &\in \mathscr{M}(D_j \cap D_k, L \otimes L_j^{-m_j} \otimes L_k^{-m_k}). \end{aligned} \tag{1.9}$$

1.2 Some characteristic classes

For an $(m \times m)$-matrix A, we define

$$\mathrm{ch}(A) = \mathrm{Tr}[e^A], \quad \mathrm{Td}(A) = \det\left(\frac{A}{\mathrm{Id} - e^{-A}}\right), \quad c(A) = \det(\mathrm{Id} + A). \tag{1.10}$$

We have

$$c(tA) = 1 + \sum_{k=1}^{m} t^k c_k(A), \tag{1.11}$$

where $c_k(A)$ is the kth elementary symmetric polynomial of the eigenvalues of A.

Let V be an m-dimensional complex vector space. Let $R \in \mathrm{End}(V)$. Let V^* be the dual of V. Let $R^* \in \mathrm{End}(V^*)$ be the dual of R. For $r = 1, \dots, m$, we construct $R_r \in \mathrm{End}(\Lambda^r V^*)$ by induction,

$$R_1 = -R^*, \quad R_r = R_1 \wedge \mathrm{Id}_{\Lambda^{r-1}V^*} + \mathrm{Id}_{V^*} \wedge R_{r-1}. \tag{1.12}$$

We use the convention $\Lambda^0 V^* = \mathbb{C}$ and $R_0 = 0$.

Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of R. For $p \in \mathbb{N}$ and F a polynomial of $\lambda_1, \dots, \lambda_m$, we denote by $\{F\}^{[p]}$ the component of F of degree p.

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PROPOSITION 1.5. *The following identities hold:*

$$\begin{aligned} \mathrm{Td}(R)\Big(\sum_{r=0}^{m}(-1)^r \mathrm{ch}(R_r)\Big) &= c_m(R), \\ \Big\{\mathrm{Td}(R)\Big(\sum_{r=1}^{m}(-1)^r r\,\mathrm{ch}(R_r)\Big)\Big\}^{[\leqslant m]} &= -c_{m-1}(R) + \frac{m}{2}c_m(R), \\ \Big\{\mathrm{Td}(R)\Big(\sum_{r=2}^{m}(-1)^r r(r-1)\mathrm{ch}(R_r)\Big)\Big\}^{[m]} &= \frac{1}{6}(c_1c_{m-1})(R) + \frac{m(3m-5)}{12}c_m(R). \end{aligned} \tag{1.13}$$

Proof. Note that the eigenvalues of R_r are given by $((-1)^r\lambda_{j_1}\cdots\lambda_{j_r})_{1\leqslant j_1<\cdots<j_r\leqslant m}$, we have

$$\mathrm{Td}(R) = \prod_{j=1}^{m}\frac{\lambda_j}{1-e^{-\lambda_j}}, \quad \sum_{r=0}^{m}(-1)^r t^r \mathrm{ch}(R_r) = \prod_{j=1}^{m}(1-te^{-\lambda_j}). \tag{1.14}$$

Taking $t=1$ in (1.14), we obtain the first identity in (1.13).

Taking the derivative of the second identity in (1.14) at $t=1$, we obtain

$$\sum_{r=0}^{m}(-1)^r r\,\mathrm{ch}(R_r) = -\Big(\sum_{j=1}^{m}\frac{e^{-\lambda_j}}{1-e^{-\lambda_j}}\Big)\prod_{j=1}^{m}(1-e^{-\lambda_j}). \tag{1.15}$$

From the first identity in (1.14), (1.15) and the identity

$$\frac{e^{-\lambda_j}}{1-e^{-\lambda_j}} = \lambda_j^{-1} - \frac{1}{2} + \frac{1}{12}\lambda_j + \cdots, \tag{1.16}$$

we obtain the second identity in (1.13).

Taking the second derivative of the second identity in (1.14) at $t=1$, we obtain

$$\sum_{r=0}^{m}(-1)^r r(r-1)\mathrm{ch}(R_r) = \Big(\Big(\sum_{j=1}^{m}\frac{e^{-\lambda_j}}{1-e^{-\lambda_j}}\Big)^2 - \sum_{j=1}^{m}\Big(\frac{e^{-\lambda_j}}{1-e^{-\lambda_j}}\Big)^2\Big)\prod_{j=1}^{m}(1-e^{-\lambda_j}). \tag{1.17}$$

From the first identity in (1.14), (1.16) and (1.17), we obtain the third identity in (1.13). This completes the proof. □

For an $(m\times m)$-matrix A, we define

$$\mathrm{Td}'(A) = \frac{\partial}{\partial t}\mathrm{Td}(A + t\,\mathrm{Id})\Big|_{t=0}. \tag{1.18}$$

PROPOSITION 1.6. *We have*

$$\begin{aligned} \Big\{\mathrm{Td}'(R)\Big(\sum_{r=0}^{m}(-1)^r \mathrm{ch}(R_r)\Big)\Big\}^{[m]} &= \frac{m}{2}c_m(R), \\ \Big\{\mathrm{Td}'(R)\Big(\sum_{r=0}^{m}(-1)^r r\,\mathrm{ch}(R_r)\Big)\Big\}^{[m]} &= \frac{1}{12}(c_1c_{m-1})(R) + \frac{m^2}{4}c_m(R). \end{aligned} \tag{1.19}$$

Proof. Let c'_k be as in (1.18) with Td replaced by c_k. We have

$$c'_1(R) = m, \quad c'_2(R) = (m-1)c_1(R). \tag{1.20}$$

On the other hand, we have

$$\{\mathrm{Td}(R)\}^{[\leqslant 2]} = 1 + \tfrac{1}{2}c_1(R) + \tfrac{1}{12}\big(c_1^2(R) + c_2(R)\big). \tag{1.21}$$

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By (1.20) and (1.21), we have

$$\left\{\frac{\mathrm{Td}'(R)}{\mathrm{Td}(R)}\right\}^{[\leqslant 1]} = \frac{m}{2} - \frac{1}{12}c_1(R). \tag{1.22}$$

From (1.13) and (1.22), we obtain (1.19). This completes the proof. □

1.3 Chern form and Bott–Chern form

Let S be a compact Kähler manifold. We denote

$$\begin{aligned} Q^S &= \bigoplus_{p=0}^{\dim S} \Omega^{p,p}(S), \\ Q^{S,0} &= \bigoplus_{p=1}^{\dim S} \big(\partial\Omega^{p-1,p}(S) + \bar\partial\Omega^{p,p-1}(S)\big) \subseteq Q^S. \end{aligned} \tag{1.23}$$

Let E be a holomorphic vector bundle over S. Let g^E be a Hermitian metric on E. Let $R^E \in \Omega^{1,1}(S, \mathrm{End}(E))$ be the curvature of the Chern connection on (E, g^E). Recall that $c(\cdot)$ was defined in (1.10). The total Chern form of (E, g^E) is defined by

$$c(E, g^E) = c\Big(-\frac{R^E}{2\pi i}\Big) \in Q^S. \tag{1.24}$$

The total Chern class of E is defined by

$$c(E) = \big[c(E, g^E)\big] \in H^{\mathrm{even}}_{\mathrm{dR}}(S), \tag{1.25}$$

which is independent of g^E.

Let $E' \subseteq E$ be a holomorphic subbundle. Let $E'' = E/E'$. We have a short exact sequence of holomorphic vector bundles over S,

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0, \tag{1.26}$$

where α (respectively, β) is the canonical embedding (respectively, projection). We have

$$c(E) = c(E')c(E''). \tag{1.27}$$

Let $g^{E'}$ be a Hermitian metric on E'. Let $g^{E''}$ be a Hermitian metric on E''. The Bott–Chern form [BGS88a, § 1f)]

$$\tilde{c}(g^{E'}, g^E, g^{E''}) \in Q^S/Q^{S,0} \tag{1.28}$$

is such that

$$\begin{aligned} \frac{\bar\partial\partial}{2\pi i}\tilde{c}(g^{E'}, g^E, g^{E''}) &= c(E, g^E) - c(E' \oplus E'', g^{E'} \oplus g^{E''}) \\ &= c(E, g^E) - c(E', g^{E'})c(E'', g^{E''}). \end{aligned} \tag{1.29}$$

Let $\alpha^* g^E$ be the Hermitian metric on E' induced by g^E via the embedding $\alpha : E' \to E$. Let $\beta_* g^E$ be the quotient Hermitian metric on E'' induced by g^E via the surjection $\beta : E \to E''$. We denote

$$\tilde{c}(E', E, g^E) = \tilde{c}(\alpha^* g^E, g^E, \beta_* g^E). \tag{1.30}$$

Let $\beta^* g^{E''}$ be the Hermitian pseudometric on E induced by $g^{E''}$ via the surjection $\beta : E \to E''$. For $\varepsilon > 0$, set

$$g^E_\varepsilon = g^E + \frac{1}{\varepsilon}\beta^* g^{E''}. \tag{1.31}$$

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We equip $Q^S \subseteq \Omega^{\bullet,\bullet}(S)$ with the compact-open topology. We equip $Q^S/Q^{S,0}$ with the quotient topology.

Proposition 1.7. *As* $\varepsilon \to 0$,

$$c(E, g^E_\varepsilon) \to c(E', \alpha^* g^E) c(E'', g^{E''}), \quad \tilde{c}(E', E, g^E_\varepsilon) \to 0. \tag{1.32}$$

Proof. We follow the proof of [BGS88a, Theorem 1.29].

Let $\mathrm{pr} : S \times \mathbb{C} \to S$ be the canonical projection. Let

$$\tilde{\alpha} : \mathrm{pr}^* E' \to \mathrm{pr}^* E \tag{1.33}$$

be the pull-back of $\alpha : E' \to E$. Let $(s, z) \in S \times \mathbb{C}$ be coordinates. Let $\sigma \in H^0(S \times \mathbb{C}, \mathbb{C})$ be the holomorphic function $\sigma(s, z) = z$. Let

$$\tilde{\sigma} : \mathrm{pr}^* E' \to \mathrm{pr}^* E' \tag{1.34}$$

be the multiplication by σ. Set

$$\mathcal{E}' = \mathrm{pr}^* E', \quad \mathcal{E} = \mathrm{Coker}(\tilde{\alpha} \oplus \tilde{\sigma} : \mathrm{pr}^* E' \to \mathrm{pr}^* E \oplus \mathrm{pr}^* E'). \tag{1.35}$$

We get a short exact sequence of holomorphic vector bundles over $S \times \mathbb{C}$,

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0, \tag{1.36}$$

where $\mathcal{E}' \to \mathcal{E}$ is induced by the embedding $0 \oplus \mathrm{Id}_{\mathrm{pr}^* E'} : \mathrm{pr}^* E' \hookrightarrow \mathrm{pr}^* E \oplus \mathrm{pr}^* E'$, and $\mathcal{E} \to \mathcal{E}'' := \mathrm{Coker}(\mathcal{E}' \to \mathcal{E})$ is the canonical projection. For $z \in \mathbb{C}$, let

$$0 \to \mathcal{E}'_z \to \mathcal{E}_z \to \mathcal{E}''_z \to 0 \tag{1.37}$$

be the restriction of (1.36) to $S \times \{z\}$. For $z \neq 0$, let

$$\phi_z : E \to \mathcal{E}_z = \mathrm{Coker}(\alpha \oplus z\,\mathrm{Id}_{E'} : E' \to E \oplus E') \tag{1.38}$$

be the isomorphism induced by the embedding $\mathrm{Id}_E \oplus 0 : E \hookrightarrow E \oplus E'$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}'_z & \longrightarrow & \mathcal{E}_z & \longrightarrow & \mathcal{E}''_z & \longrightarrow & 0 \end{array} \tag{1.39}$$

where the vertical maps are induced by ϕ_z. Let

$$\phi_0 : E' \oplus E'' \to \mathcal{E}_0 = \mathrm{Coker}(\alpha \oplus 0 : E' \to E \oplus E') = E'' \oplus E' \tag{1.40}$$

be the obvious isomorphism. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E' \oplus E'' & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}'_0 & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{E}''_0 & \longrightarrow & 0 \end{array} \tag{1.41}$$

where the vertical maps are induced by ϕ_0.

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We can construct a Hermitian metric $g^{\mathcal{E}}$ on $\mathcal{E}$ such that

$$\phi_z^* g^{\mathcal{E}} = |z|^2 g^E + \beta^* g^{E''} \quad \text{for } z \neq 0, \quad \phi_0^* g^{\mathcal{E}} = \alpha^* g^E \oplus g^{E''}. \tag{1.42}$$

To show that $g^{\mathcal{E}}$ is a smooth metric, we consider the metric $g^{\mathrm{pr}^*E\oplus \mathrm{pr}^*E'}$ on $\mathrm{pr}^*E \oplus \mathrm{pr}^*E'$ defined by

$$g^{\mathrm{pr}^*E\oplus \mathrm{pr}^*E'}|_{S\times\{z\}} = (1+|z|^2)(g^E \oplus \alpha^* g^E). \tag{1.43}$$

We can directly verify that $g^{\mathcal{E}}$ is the quotient metric induced by $g^{\mathrm{pr}^*E\oplus \mathrm{pr}^*E'}$ via the canonical projection $\mathrm{pr}^*E \oplus \mathrm{pr}^*E' \to \mathcal{E}$.

By (1.39) and (1.42), for $\varepsilon = |z|^2 > 0$, we have

$$c(\mathcal{E}_z, g^{\mathcal{E}_z}) = c(E, g^E_\varepsilon), \quad \tilde{c}(\mathcal{E}'_z, \mathcal{E}_z, g^{\mathcal{E}_z}) = \tilde{c}(E', E, g^E_\varepsilon). \tag{1.44}$$

By [BGS88a, Theorem 1.29 iii)], (1.41) and (1.42), we have

$$c(\mathcal{E}_0, g^{\mathcal{E}_0}) = c(E', \alpha^* g^E) c(E'', g^{E''}), \quad \tilde{c}(\mathcal{E}'_0, \mathcal{E}_0, g^{\mathcal{E}_0}) = 0. \tag{1.45}$$

On the other hand, by [BGS88a, Theorem 1.29 ii)], we have

$$\lim_{z\to 0} c(\mathcal{E}_z, g^{\mathcal{E}_z}) = c(\mathcal{E}_0, g^{\mathcal{E}_0}), \quad \lim_{z\to 0} \tilde{c}(\mathcal{E}'_z, \mathcal{E}_z, g^{\mathcal{E}_z}) = \tilde{c}(\mathcal{E}'_0, \mathcal{E}_0, g^{\mathcal{E}_0}). \tag{1.46}$$

From (1.44)–(1.46), we obtain (1.32). This completes the proof. □

Remark 1.8. We can also prove Proposition 1.7 by applying the arguments in [BB94, (4.67)–(4.70) and (4.75)–(4.81)], which show that the connection of E converges to a triangular 2×2 matrix with diagonal elements given by the connections of E' and E'' as $\varepsilon \to 0$. Though [BB94, (4.67)–(4.70) and (4.75)–(4.81)] work with tangent bundles, the argument equally holds in our case (because the connections under consideration are Chern connections).

Let $F \subseteq E$ be a holomorphic subbundle. Set $F' = \alpha^{-1}(F) \subseteq E'$, $F'' = \beta(F) \subseteq E''$.

Proposition 1.9. *If $F' = E'$, as $\varepsilon \to 0$,*

$$\tilde{c}(F, E, g^E_\varepsilon) \to c(E', \alpha^* g^E)\tilde{c}(F'', E'', g^{E''}). \tag{1.47}$$

If $F'' = E''$, as $\varepsilon \to 0$,

$$\tilde{c}(F, E, g^E_\varepsilon) \to c(E'', g^{E''})\tilde{c}(F', E', \alpha^* g^E). \tag{1.48}$$

Proof. We use the notation from the proof of Proposition 1.7. Set

$$\mathcal{F} = \mathrm{Coker}(\tilde{\alpha} \oplus \tilde{\sigma}|_{\mathrm{pr}^*F'} : \mathrm{pr}^*F' \to \mathrm{pr}^*F \oplus \mathrm{pr}^*F') \subseteq \mathcal{E}. \tag{1.49}$$

For $z \in \mathbb{C}$, let $\mathcal{F}_z$ be the restriction of $\mathcal{F}$ to $S \times \{z\}$.

For $z \neq 0$, we have $\phi_z(F) = \mathcal{F}_z \subseteq \mathcal{E}_z$. By (1.42), for $\varepsilon = |z|^2 > 0$, we have

$$\tilde{c}(\mathcal{F}_z, \mathcal{E}_z, g^{\mathcal{E}_z}) = \tilde{c}(F, E, g^E_\varepsilon). \tag{1.50}$$

We have $\phi_0(F) = F' \oplus F'' \subseteq E' \oplus E'' = \mathcal{E}_0$. By (1.42), we have

$$\tilde{c}(\mathcal{F}_0, \mathcal{E}_0, g^{\mathcal{E}_0}) = \tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}). \tag{1.51}$$

By [BGS88a, Theorem 1.29], we have

$$\begin{aligned} \tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}) &= c(E', \alpha^* g^E)\tilde{c}(F'', E'', g^{E''}) \quad \text{if } F' = E', \\ \tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}) &= c(E'', g^{E''})\tilde{c}(F', E', \alpha^* g^E) \quad \text{if } F'' = E''. \end{aligned} \tag{1.52}$$

On the other hand, by [BGS88a, Theorem 1.29 ii)], we have

$$\lim_{z\to 0} \tilde{c}(\mathcal{F}_z, \mathcal{E}_z, g^{\mathcal{E}_z}) = \tilde{c}(\mathcal{F}_0, \mathcal{E}_0, g^{\mathcal{E}_0}). \tag{1.53}$$

From (1.50)–(1.53), we obtain (1.47) and (1.48). This completes the proof. □

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Recall that $\mathrm{Td}(\cdot)$ was defined in (1.10). The Bott–Chern form [BGS88a, § 1f)]

$$\widetilde{\mathrm{Td}}(g^{E'}, g^{E}, g^{E''}) \in Q^S/Q^{S,0} \tag{1.54}$$

is such that

$$\frac{\bar{\partial}\partial}{2\pi i}\widetilde{\mathrm{Td}}(g^{E'}, g^{E}, g^{E''}) = \mathrm{Td}(E, g^E) - \mathrm{Td}(E', g^{E'})\mathrm{Td}(E'', g^{E''}). \tag{1.55}$$

Proposition 1.10. *Propositions 1.7 and 1.9 hold with $c(\cdot)$ replaced by $\mathrm{Td}(\cdot)$.*

Recall that $\mathrm{ch}(\cdot)$ was defined in (1.10). The Bott–Chern form [BGS88a, § 1f)]

$$\widetilde{\mathrm{ch}}(g^{E'}, g^{E}, g^{E''}) \in Q^S/Q^{S,0} \tag{1.56}$$

is such that

$$\frac{\bar{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}(g^{E'}, g^{E}, g^{E''}) = \mathrm{ch}(E', g^{E'}) - \mathrm{ch}(E, g^E) + \mathrm{ch}(E'', g^{E''}). \tag{1.57}$$

For another Hermitian metric $\hat{g}^E$ on E, let

$$\widetilde{\mathrm{ch}}(\hat{g}^E, g^E) \in Q^S/Q^{S,0} \tag{1.58}$$

be the Bott–Chern form [BGS88a, § 1f)] such that

$$\frac{\bar{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}(\hat{g}^E, g^E) = \mathrm{ch}(E, \hat{g}^E) - \mathrm{ch}(E, g^E). \tag{1.59}$$

The following proposition is a direct consequence of the construction of the Bott–Chern form [BGS88a, § 1f)].

Proposition 1.11. *For another Hermitian metric $\hat{g}^E$ (respectively, $\hat{g}^{E'}$, $\hat{g}^{E''}$) on E (respectively, E', E''), we have*

$$\widetilde{\mathrm{ch}}(\hat{g}^{E'}, \hat{g}^{E}, \hat{g}^{E''}) = \widetilde{\mathrm{ch}}(g^{E'}, g^{E}, g^{E''}) + \widetilde{\mathrm{ch}}(\hat{g}^{E'}, g^{E'}) - \widetilde{\mathrm{ch}}(\hat{g}^{E}, g^{E}) + \widetilde{\mathrm{ch}}(\hat{g}^{E''}, g^{E''}). \tag{1.60}$$

For $a, b > 0$, we have

$$\widetilde{\mathrm{ch}}(a g^E, b g^E) = \mathrm{ch}(E, g^E)(\log b - \log a). \tag{1.61}$$

For $(g_t^E)_{t\in\mathbb{R}}$ a smooth family of Hermitian metrics on E, the map $t \mapsto \widetilde{\mathrm{ch}}(g_t^E, g_0^E)$ is continuous. In particular, we have

$$\widetilde{\mathrm{ch}}(g_t^E, g_0^E) \to 0 \quad \text{as } t \to 0. \tag{1.62}$$

Let E^* be the dual of E. Following [BB94, § 1a)], for $p = 0, \dots, \dim E$ and $s = 0, \dots, p-1$, set

$$I_s^p = \big\{u \in \Lambda^p E^* : u(v_1, \dots, v_p) = 0 \text{ for any } v_1, \dots, v_{s+1} \in E', v_{s+2}, \dots, v_p \in E\big\}. \tag{1.63}$$

For convenience, we denote $I_p^p = \Lambda^p E^*$ and $I_{-1}^p = 0$. We obtain a filtration

$$\Lambda^p E^* = I_p^p \hookleftarrow I_{p-1}^p \hookleftarrow \cdots \hookleftarrow I_{-1}^p = 0. \tag{1.64}$$

For $r = 0, \dots, \dim E''$ and $s = 0, \dots, \dim E'$, we denote $E_{r,s} = \Lambda^s E'^* \otimes \Lambda^r E''^*$. We have a short exact sequence of holomorphic vector bundles over S,

$$0 \to I_{s-1}^{r+s} \to I_s^{r+s} \to E_{r,s} \to 0. \tag{1.65}$$

Recall that g_ε^E was defined in (1.31). Let $g_\varepsilon^{\Lambda^p E^*}$ be the Hermitian metric on $\Lambda^p E^*$ induced by g_ε^E. Let $g_\varepsilon^{I_s^{r+s}}$ be the restriction of $g_\varepsilon^{\Lambda^p E^*}$ to I_s^{r+s}. Let $g_\varepsilon^{E_{r,s}}$ be the quotient metric on $E_{r,s}$ induced by $g_\varepsilon^{I_s^{r+s}}$ via the surjection $I_s^{r+s} \to E_{r,s}$.

Similarly to Proposition 1.7, we have the following proposition.

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Proposition 1.12. *As* $\varepsilon \to 0$,

$$\widetilde{\mathrm{ch}}\Big(g_\varepsilon^{I_{s-1}^{r+s}}, g_\varepsilon^{I_s^{r+s}}, g_\varepsilon^{E_{r,s}}\Big) \to 0. \tag{1.66}$$

Proof. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be as in (1.36). Let $\mathcal{I}_s^p \subseteq \Lambda^p \mathcal{E}^*$ be as in (1.63) with E replaced by $\mathcal{E}$ and E' replaced by $\mathcal{E}'$. We denote $\mathcal{E}_{r,s} = \Lambda^s \mathcal{E}'^* \otimes \Lambda^r \mathcal{E}''^*$. We have a short exact sequence of holomorphic vector bundles over $S \times \mathbb{C}$,

$$0 \to \mathcal{I}_{s-1}^{r+s} \to \mathcal{I}_s^{r+s} \to \mathcal{E}_{r,s} \to 0. \tag{1.67}$$

Proceeding in the same way as in the proof of Proposition 1.7 with (1.36) replaced by (1.67), we obtain (1.66). This completes the proof. □

1.4 Quillen metric

Let X be an n-dimensional compact Kähler manifold. Let E be a holomorphic vector bundle over X. Let $\bar\partial^E$ be the Dolbeault operator on

$$\Omega^{0,\bullet}(X,E) = \mathscr{C}^\infty\big(X, \Lambda^\bullet(\overline{T^*X}) \otimes E\big). \tag{1.68}$$

For $q = 0, \dots, n$, we have $H^q(X,E) = H^q(\Omega^{0,\bullet}(X,E), \bar\partial^E)$. Set

$$\lambda(E) = \det H^\bullet(X,E) := \bigotimes_{q=0}^{n} \big(\det H^q(X,E)\big)^{(-1)^q}. \tag{1.69}$$

Let g^{TX} be a Kähler metric on TX. Let g^E be a Hermitian metric on E. Let $\langle\cdot,\cdot\rangle_{\Lambda^\bullet(\overline{T^*X})\otimes E}$ be the Hermitian product on $\Lambda^\bullet(\overline{T^*X}) \otimes E$ induced by g^{TX} and g^E. Let dv_X be the Riemannian volume form on X induced by g^{TX}. For $s_1, s_2 \in \Omega^{0,\bullet}(X,E)$, set

$$\langle s_1, s_2\rangle = (2\pi)^{-n} \int_X \langle s_1, s_2\rangle_{\Lambda^\bullet(\overline{T^*X})\otimes E}\, dv_X, \tag{1.70}$$

which we call the L^2-product.

Let $\bar\partial^{E,*}$ be the formal adjoint of $\bar\partial^E$ with respect to the Hermitian product (1.70). The Kodaira Laplacian on $\Omega^{0,\bullet}(X,E)$ is defined by

$$\square^E = \bar\partial^E \bar\partial^{E,*} + \bar\partial^{E,*}\bar\partial^E. \tag{1.71}$$

Let $\square_q^E$ be the restriction of $\square^E$ to $\Omega^{0,q}(X,E)$.

By the Hodge theorem, we have

$$\mathrm{Ker}(\square_q^E) = \{s \in \Omega^{0,q}(X,E) : \bar\partial^E s = 0, \bar\partial^{E,*} s = 0\}. \tag{1.72}$$

Still by the Hodge theorem, the following map is bijective:

$$\begin{aligned} \mathrm{Ker}(\square_q^E) &\to H^q(X,E) \\ s &\mapsto [s]. \end{aligned} \tag{1.73}$$

Let $|\cdot|_{\lambda(E)}$ be the L^2-metric on $\lambda(E)$ induced by the metric (1.70) via (1.69) and (1.73).

Let $\mathrm{Sp}(\square_q^E)$ be the spectrum of $\square_q^E$, which is a multiset.[1] For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > n$, set

$$\theta(z) = \sum_{q=1}^{n} (-1)^{q+1} q \sum_{\lambda \in \mathrm{Sp}(\square_q^E), \lambda \neq 0} \lambda^{-z}. \tag{1.74}$$

[1] A multiset allows for multiple instances for each of its elements.

By [See67], the function $\theta(z)$ extends to a meromorphic function of $z \in \mathbb{C}$, which is holomorphic at $z = 0$.

The following definition is due to Quillen [Qui85] and Bismut, Gillet and Soulé [BGS88b, § 1d)].

Definition 1.13. The Quillen metric on $\lambda(E)$ is defined by

$$\|\cdot\|_{\lambda(E)} = \exp\left(\tfrac{1}{2}\theta'(0)\right)|\cdot|_{\lambda(E)}. \tag{1.75}$$

Remark 1.14. Denote $\chi(X,E) = \sum_{q=0}^{n}(-1)^q \dim H^q(X,E)$. For $a > 0$, if we replace g^E by ag^E, then $\|\cdot\|_{\lambda(E)}$ is replaced by $a^{\chi(X,E)/2}\|\cdot\|_{\lambda(E)}$.

1.5 Analytic torsion form

Let $\pi : X \to Y$ be a holomorphic submersion between Kähler manifolds with compact fiber Z.

Let E be a holomorphic vector bundle over X. Let $R^\bullet\pi_*E$ be the derived direct image of E, which is a graded analytic coherent sheaf on Y. We assume that $R^\bullet\pi_*E$ is a graded holomorphic vector bundle. Let $H^\bullet(Z,E)$ be the fiberwise cohomology. More precisely, its fiber at $y \in Y$ is given by $H^\bullet(Z_y, E|_{Z_y})$. We have a canonical identification $R^\bullet\pi_*E = H^\bullet(Z,E)$. We have the Grothendieck–Riemann–Roch formula,

$$\mathrm{ch}(H^\bullet(Z,E)) := \sum_j (-1)^j \mathrm{ch}(H^j(Z,E)) = \int_Z \mathrm{Td}(TZ)\mathrm{ch}(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(Y). \tag{1.76}$$

Let $\omega \in \Omega^{1,1}(X)$ be a Kähler form. Let g^{TZ} be the Hermitian metric on TZ associated with ω. Let g^E be a Hermitian metric on E. Let $g^{H^\bullet(Z,E)}$ be the L^2-metric on $H^\bullet(Z,E)$ associated with g^{TZ} and g^E via (1.73).

We use the notation in (1.23). Let $\mathrm{ch}(H^\bullet(Z,E), g^{H^\bullet(Z,E)}) \in Q^Y$ be the Chern character form of $(H^\bullet(Z,E), g^{H^\bullet(Z,E)})$. We introduce $\mathrm{Td}(TZ, g^{TZ}) \in Q^X$ and $\mathrm{ch}(E, g^E) \in Q^X$ in the same way.

Bismut and Köhler [BK92, Definition 3.8] defined the analytic torsion forms. The analytic torsion form associated with $(\pi : X \to Y, \omega, E, g^E)$ is a differential form on Y, which we denote by $T(\omega, g^E)$. Moreover, we have

$$T(\omega, g^E) \in Q^Y. \tag{1.77}$$

We sometimes view $T(\omega, g^E)$ as an element in $Q^Y/Q^{Y,0}$. By [BK92, Theorem 3.9], we have

$$\frac{\bar\partial\partial}{2\pi i} T(\omega, g^E) = \mathrm{ch}\big(H^\bullet(Z,E), g^{H^\bullet(Z,E)}\big) - \int_Z \mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^E). \tag{1.78}$$

The identity (1.78) is a refinement of the Grothendieck–Riemann–Roch formula (1.76).

For $y \in Y$, let $\theta_y(z)$ be as in (1.74) with (X, g^{TX}, E, g^E) replaced by $(Z_y, g^{TZ_y}, E|_{Z_y}, g^E|_{Z_y})$. Let $\theta'(0)$ be the function $y \mapsto \theta'_y(0)$ on Y. By the construction of the analytic torsion forms, we have

$$\big\{T(\omega, g^E)\big\}^{(0,0)} = \theta'(0) \in \mathscr{C}^\infty(Y), \tag{1.79}$$

where $\{\cdot\}^{(0,0)}$ means the component of degree $(0,0)$.

Let F be a holomorphic vector bundle over Y. Let π^*F be its pull-back via π, which is a holomorphic vector bundle over X. Let g^F be a Hermitian metric on F. Let $g^{E\otimes\pi^*F}$ be the Hermitian metric on $E \otimes \pi^*F$ induced by g^E and g^F. Let

$$T(\omega, g^{E\otimes\pi^*F}) \in Q^Y \tag{1.80}$$

be the analytic torsion form associated with $(\pi : X \to Y, \omega, E \otimes \pi^*F, g^{E\otimes\pi^*F})$.

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The following proposition is a direct consequence of the construction of the analytic torsion forms.

Proposition 1.15. *The following identity holds in $Q^Y/Q^{Y,0}$:*

$$T(\omega, g^{E\otimes\pi^*F}) = \mathrm{ch}(F, g^F)T(\omega, g^E). \tag{1.81}$$

For $p = 0, \ldots, \dim Z$, let $g^{\Lambda^p(T^*Z)}$ be the metric on $\Lambda^p(T^*Z)$ induced by g^{TZ}. Let

$$T(\omega, g^{\Lambda^p(T^*Z)}) \in Q^Y \tag{1.82}$$

be the analytic torsion form associated with $(\pi : X \to Y, \omega, \Lambda^p(T^*Z), g^{\Lambda^p(T^*Z)})$.

Theorem 1.16 (Bismut [Bis04, Theorem 4.15]). *The following identity holds in $Q^Y/Q^{Y,0}$,*

$$\sum_{p=0}^{\dim Z} (-1)^p T(\omega, g^{\Lambda^p(T^*Z)}) = 0. \tag{1.83}$$

1.6 Properties of the Quillen metric

In this subsection, we state several results describing the behavior of the Quillen metric under submersion, resolution, immersion and blow-up.

Submersion. Let $\pi : X \to Y$, Z, E and $H^\bullet(Z, E)$ be as in § 1.5. We assume that X and Y are compact. We further assume that the Leray spectral sequence for E and π degenerates at E_2, i.e.

$$H^q(X, E) \simeq \bigoplus_{j+k=q} H^j(Y, H^k(Z, E)) \quad \text{for } q = 0, \ldots, \dim X. \tag{1.84}$$

We denote

$$\begin{aligned} \det H^\bullet(Y, H^\bullet(Z, E)) &= \bigotimes_{k=0}^{\dim Z} \big(\det H^\bullet\big(Y, H^k(Z, E)\big)\big)^{(-1)^k} \\ &= \bigotimes_{j=0}^{\dim Y} \bigotimes_{k=0}^{\dim Z} \big(\det H^j\big(Y, H^k(Z, E)\big)\big)^{(-1)^{j+k}}. \end{aligned} \tag{1.85}$$

Let

$$\sigma \in \det H^\bullet(X, E) \otimes \big(\det H^\bullet\big(Y, H^\bullet(Z, E)\big)\big)^{-1} \tag{1.86}$$

be the canonical section induced by (1.84).

Let $\omega_X \in \Omega^{1,1}(X)$ and $\omega_Y \in \Omega^{1,1}(Y)$ be Kähler forms. For $\varepsilon > 0$, set

$$\omega_\varepsilon = \omega_X + \frac{1}{\varepsilon}\pi^*\omega_Y. \tag{1.87}$$

Let g^E be a Hermitian metric on E.

Let g^{TX}_ε be the metric on TX associated with ω_ε. Let

$$\|\cdot\|_{\det H^\bullet(X,E),\varepsilon} \tag{1.88}$$

be the Quillen metric on $\det H^\bullet(X, E)$ associated with g^{TX}_ε and g^E. Let g^{TY} be the metric on TY associated with ω_Y. Let g^{TZ} be the metric on TZ associated with $\omega_X|_Z$. Let $g^{H^\bullet(Z,E)}$ be the

L^2-metric on $H^\bullet(Z,E)$ associated with g^{TZ} and g^E. For $k=0,\dots,\dim Z$, let

$$\|\cdot\|_{\det H^\bullet(Y,H^k(Z,E))} \tag{1.89}$$

be the Quillen metric on $\det H^\bullet(Y,H^k(Z,E))$ associated with g^{TY} and $g^{H^k(Z,E)}$. Let

$$\|\cdot\|_{\det H^\bullet(Y,H^\bullet(Z,E))} \tag{1.90}$$

be the metric on $\det H^\bullet(Y,H^\bullet(Z,E))$ induced by the Quillen metrics (1.89) via (1.85). Let $\|\sigma\|_\varepsilon$ be the norm of σ with respect to the metrics (1.88) and (1.90).

We use the notation in (1.23). Let $\mathrm{Td}(TY,g^{TY})\in Q^Y$ be the Todd form of (TY,g^{TY}). Let

$$T(\omega,g^E)\in Q^Y \tag{1.91}$$

be the analytic torsion form (see § 1.5) associated with $(\pi:X\to Y,\omega_X,E,g^E)$.

Recall that $\mathrm{Td}'(\cdot)$ was defined by (1.18).

Theorem 1.17 (Berthomieu and Bismut [BB94, Theorem 3.2]). *As $\varepsilon\to 0$,*

$$\log\|\sigma\|^2_\varepsilon+\int_Y \mathrm{Td}'(TY)\int_Z \mathrm{Td}(TZ)\mathrm{ch}(E)\log\varepsilon\to\int_Y \mathrm{Td}(TY,g^{TY})T(\omega,g^E). \tag{1.92}$$

Resolution. Let X be a compact Kähler manifold. Let

$$0\to E^0\to E^1\to E^2\to 0 \tag{1.93}$$

be a short exact sequence of holomorphic vector bundles over X. Let

$$\sigma\in\bigotimes_{k=0}^{2}\big(\det H^\bullet(X,E^k)\big)^{(-1)^{k+1}} \tag{1.94}$$

be the canonical section induced by the long exact sequence induced by (1.93).

Let g^{TX} be a Kähler metric on TX. For $k=0,1,2$, let g^{E^k} be a Hermitian metric on E^k. Let

$$\|\cdot\|_{\det H^\bullet(X,E^k)} \tag{1.95}$$

be the Quillen metric on $\det H^\bullet(X,E^k)$ associated with g^{TX} and g^{E^k}. Let $\|\sigma\|$ be the norm of σ with respect to the metrics (1.95).

We use the notation in (1.23). Let $\mathrm{Td}(TX,g^{TX})\in Q^X$ be the Todd form of (TX,g^{TX}). Let $\mathrm{ch}(E^k,g^{E^k})\in Q^X$ be the Chern character form of (E^k,g^{E^k}). Let

$$\widetilde{\mathrm{ch}}(g^{E^\bullet})\in Q^X/Q^{X,0} \tag{1.96}$$

be the Bott–Chern form [BGS88a, § 1f)] such that

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}(g^{E^\bullet})=\sum_{k=0}^{2}(-1)^k\mathrm{ch}(E^k,g^{E^k}). \tag{1.97}$$

Theorem 1.18 (Bismut, Gillet and Soulé [BGS88b, Theorem 1.23]). *The following identity holds:*

$$\log\|\sigma\|^2=\int_X \mathrm{Td}(TX,g^{TX})\widetilde{\mathrm{ch}}(g^{E^\bullet}). \tag{1.98}$$

Immersion. Let X be a compact Kähler manifold. Let $Y\subseteq X$ be a complex submanifold of codimension one. Let $i:Y\hookrightarrow X$ be the canonical embedding. Let F be a holomorphic vector bundle over Y. Let $v:E_1\to E_0$ be a map between holomorphic vector bundles over X which,

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together with a restriction map $r : E_0|_Y \to F$, provides a resolution of $i_*\mathscr{O}_Y(F)$. More precisely, we have an exact sequence of analytic coherent sheaves on X,

$$0 \to \mathscr{O}_X(E_1) \xrightarrow{v} \mathscr{O}_X(E_0) \xrightarrow{r} i_*\mathscr{O}_Y(F) \to 0. \tag{1.99}$$

Let

$$\sigma \in \big(\det H^\bullet(X, E_1)\big)^{-1} \otimes \det H^\bullet(X, E_0) \otimes \big(\det H^\bullet(Y, F)\big)^{-1} \tag{1.100}$$

be the canonical section induced by the long exact sequence induced by (1.99).

Let $\omega \in \Omega^{1,1}(X)$ be a Kähler form. For $k = 0, 1$, let g^{E_k} be a Hermitian metric on E_k. Let g^F be a Hermitian metric on F. Assume that there is an open neighborhood $Y \subseteq U \subseteq X$ such that $v|_{X\backslash U}$ is isometric, i.e.

$$g^{E_1}|_{X\backslash U} = v^* g^{E_0}|_{X\backslash U}. \tag{1.101}$$

Let g^{TX} be the metric on TX associated with ω. For $k = 0, 1$, let

$$\|\cdot\|_{\det H^\bullet(X,E_k)} \tag{1.102}$$

be the Quillen metric on $\det H^\bullet(X, E_k)$ associated with g^{TX} and g^{E_k}. Let g^{TY} be the metric on TY associated with $\omega|_Y$. Let

$$\|\cdot\|_{\det H^\bullet(Y,F)} \tag{1.103}$$

be the Quillen metric on $\det H^\bullet(Y, F)$ associated with g^{TY} and g^F. Let $\|\sigma\|$ be the norm of σ with respect to the metrics (1.102) and (1.103).

The following theorem is a direct consequence of the immersion formula due to Bismut and Lebeau [BL91, Theorem 0.1] and the anomaly formula due to Bismut, Gillet and Soulé [BGS88b, Theorem 1.23].

Theorem 1.19. *We have*

$$\log\|\sigma\|^2 = \alpha(U, \omega|_U, v|_U, g^{E_\bullet}|_U, r, g^F), \tag{1.104}$$

where $\alpha(U, \omega|_U, v|_U, r|_U, g^{E_\bullet}, g^F)$ is a real number determined by

$$U, \quad \omega|_U, \quad v|_U : E_1|_U \to E_0|_U, \quad g^{E_\bullet}|_U, \quad r : E_0|_Y \to F, \quad g^F. \tag{1.105}$$

More precisely, given

$$\tilde{Y} \subseteq \tilde{U} \subseteq \tilde{X}, \quad \tilde{\omega}, \quad \tilde{v} : \tilde{E}_1 \to \tilde{E}_0, \quad \tilde{r} : \tilde{E}_0|_{\tilde{Y}} \to \tilde{F}, \quad g^{\tilde{E}_\bullet}, \quad g^{\tilde{F}} \tag{1.106}$$

satisfying the same properties that

$$Y \subseteq U \subseteq X, \quad \omega, \quad v : E_1 \to E_0, \quad r : E_0|_Y \to F, \quad g^{E_\bullet}, \quad g^F \tag{1.107}$$

satisfy, if there is a biholomorphic map $U \to \tilde{U}$ inducing an isomorphism between the restrictions of the data above to U and $\tilde{U}$, then

$$\log\|\sigma\|^2 = \log\|\tilde{\sigma}\|^2, \tag{1.108}$$

where

$$\tilde{\sigma} \in \big(\det H^\bullet(\tilde{X}, \tilde{E}_1)\big)^{-1} \otimes \det H^\bullet(\tilde{X}, \tilde{E}_0) \otimes \big(\det H^\bullet(\tilde{Y}, \tilde{F})\big)^{-1} \tag{1.109}$$

is the canonical section, and $\|\tilde{\sigma}\|$ is its norm with respect to the Quillen metrics.

Remark 1.20. The real number $\alpha(U, \omega|_U, v|_U, r|_U, g^{E_\bullet}, g^F)$ depends continuously on the input data.

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Blow-up. Let X be a compact Kähler manifold. Let $Y \subseteq X$ be a complex submanifold of codimension $r \geqslant 2$. Let $f : X' \to X$ be the blow-up along Y. Let E be a holomorphic vector bundle over X. Let f^*E be the pull-back of E via f, which is a holomorphic vector bundle over X'. Applying spectral sequence, we obtain a canonical identification

$$H^\bullet(X', f^*E) = H^\bullet(X, E). \tag{1.110}$$

Let

$$\sigma \in \big(\det H^\bullet(X,E)\big)^{-1} \otimes \det H^\bullet(X', f^*E) \tag{1.111}$$

be the canonical section induced by (1.110).

Let $\omega \in \Omega^{1,1}(X)$ and $\omega' \in \Omega^{1,1}(X')$ be Kähler forms. Assume that there are open neighborhoods $Y \subseteq U \subseteq X$ and $f^{-1}(Y) \subseteq U' \subseteq X'$ such that

$$f^{-1}(U) = U', \quad f^*(\omega|_{X\backslash U}) = \omega'|_{X'\backslash U'}. \tag{1.112}$$

For the existence of such ω and ω', see the proof of [Voi02, Proposition 3.24]. Let g^E be a Hermitian metric on E.

Let g^{TX} be the metric on TX associated with ω. Let

$$\|\cdot\|_{\det H^\bullet(X,E)} \tag{1.113}$$

be the Quillen metric on $\det H^\bullet(X,E)$ associated with g^{TX} and g^E. Let $g^{TX'}$ be the metric on TX' associated with ω'. Let

$$\|\cdot\|_{\det H^\bullet(X',f^*E)} \tag{1.114}$$

be the Quillen metric on $\det H^\bullet(X', f^*E)$ associated with $g^{TX'}$ and f^*g^E. Let $\|\sigma\|$ be the norm of σ with respect to the metrics (1.113) and (1.114).

The following theorem is a direct consequence of the blow-up formula due to Bismut [Bis97, Theorem 8.10].

Theorem 1.21. *We have*

$$\log\|\sigma\|^2 = \alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U), \tag{1.115}$$

where $\alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U)$ is a real number determined by

$$U, \quad \omega|_U, \quad U', \quad \omega'|_{U'}, \quad E|_U, \quad g^E|_U. \tag{1.116}$$

Remark 1.22. The real number $\alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U)$ depends continuously on the input data.

1.7 Topological torsion and BCOV torsion

Let X be an n-dimensional compact Kähler manifold. For $p = 0, \dots, n$, set

$$\lambda_p(X) = \det H^{p,\bullet}(X) := \bigotimes_{q=0}^{n} (\det H^{p,q}(X))^{(-1)^q}. \tag{1.117}$$

Set

$$\begin{aligned}\eta(X) = \det H^\bullet_{\mathrm{dR}}(X) &:= \bigotimes_{k=0}^{2n} \big(\det H^k_{\mathrm{dR}}(X)\big)^{(-1)^k} \\ &= \bigotimes_{p=0}^{n} \big(\lambda_p(X)\big)^{(-1)^p}.\end{aligned} \tag{1.118}$$

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Set

$$\begin{aligned}\lambda(X) &= \bigotimes_{0\leqslant p,q\leqslant n} \big(\det H^{p,q}(X)\big)^{(-1)^{p+q}p} = \bigotimes_{p=1}^{n} \big(\lambda_p(X)\big)^{(-1)^p p},\\ \lambda_{\mathrm{tot}}(X) &= \bigotimes_{k=1}^{2n} \big(\det H^k_{\mathrm{dR}}(X)\big)^{(-1)^k k} = \lambda(X)\otimes\overline{\lambda(X)}.\end{aligned} \tag{1.119}$$

The identities in (1.119) appeared in [Kat14]. They were applied to the theory of BCOV invariant by Eriksson, Freixas i Montplet and Mourougane [EFM21].

For $\mathbb{A}=\mathbb{Z},\mathbb{R},\mathbb{C}$, we denote by $H^\bullet_{\mathrm{Sing}}(X,\mathbb{A})$ the singular cohomology of X with coefficients in $\mathbb{A}$. For $k=0,\dots,2n$, let

$$\sigma_{k,1},\dots,\sigma_{k,b_k}\in \mathrm{Im}\big(H^k_{\mathrm{Sing}}(X,\mathbb{Z})\to H^k_{\mathrm{Sing}}(X,\mathbb{R})\big) \tag{1.120}$$

be a basis of the lattice. We fix a square root of i. In what follows, the choice of square root is irrelevant. We identify $H^k_{\mathrm{dR}}(X)$ with $H^k_{\mathrm{Sing}}(X,\mathbb{C})$ as follows:

$$\begin{aligned} H^k_{\mathrm{dR}}(X) &\to H^k_{\mathrm{Sing}}(X,\mathbb{C})\\ [\alpha] &\mapsto \bigg[\mathfrak{a}\mapsto (2\pi i)^{-k/2}\int_{\mathfrak{a}}\alpha\bigg],\end{aligned} \tag{1.121}$$

where α is a closed k-form on X and $\mathfrak{a}$ is a k-chain in X. Then $\sigma_{k,1},\dots,\sigma_{k,b_k}$ form a basis of $H^k_{\mathrm{dR}}(X)$. Set

$$\begin{aligned}&\sigma_k = \sigma_{k,1}\wedge\cdots\wedge\sigma_{k,b_k}\in \det H^k_{\mathrm{dR}}(X),\\ &\epsilon_X = \bigotimes_{k=0}^{2n}\sigma_k^{(-1)^k}\in\eta(X),\quad \sigma_X=\bigotimes_{k=1}^{2n}\sigma_k^{(-1)^k k}\in\lambda_{\mathrm{tot}}(X),\end{aligned} \tag{1.122}$$

which are well-defined up to ±1.

Let ω be a Kähler form on X. Let $\|\cdot\|_{\lambda_p(X),\omega}$ be the Quillen metric on $\lambda_p(X)$ associated with ω. Let $\|\cdot\|_{\eta(X)}$ be the metric on $\eta(X)$ induced by $\|\cdot\|_{\lambda_p(X),\omega}$ via (1.118). The same calculation as in [Zha22, Theorem 2.1] together with the first identity in Proposition 1.5 shows that $\|\cdot\|_{\eta(X)}$ is independent of ω.

Definition 1.23. We define

$$\tau_{\mathrm{top}}(X)=\log\|\epsilon_X\|_{\eta(X)}. \tag{1.123}$$

Indeed $\|\cdot\|_{\eta(X)}$ is the classical Ray–Singer metric up to a normalization. Later, we use this fact to show that $\tau_{\mathrm{top}}(X)=0$.

Let $\|\cdot\|_{\lambda(X),\omega}$ be the metric on $\lambda(X)$ induced by $\|\cdot\|_{\lambda_p(X),\omega}$ via the first identity in (1.119). Let $\|\cdot\|_{\lambda_{\mathrm{tot}}(X),\omega}$ be the metric on $\lambda_{\mathrm{tot}}(X)$ induced by $\|\cdot\|_{\lambda(X),\omega}$ via the second identity in (1.119).

Definition 1.24. We define

$$\tau_{\mathrm{BCOV}}(X,\omega)=\log\|\sigma_X\|_{\lambda_{\mathrm{tot}}(X),\omega}. \tag{1.124}$$

For $p=0,\dots,n$, let $g^{\Lambda^p(T^*X)}_\omega$ be the metric on $\Lambda^p(T^*X)$ induced by ω. Let $g^{\Omega^{p,q}(X)}_\omega$ be the L^2-metric on $\Omega^{p,q}(X)$. More precisely, $g^{\Omega^{p,q}(X)}_\omega$ is defined by (1.70) with (E,g^E) replaced by $(\Lambda^p(T^*X),g^{\Lambda^p(T^*X)}_\omega)$. Let $g^{H^{p,q}(X)}_\omega$ be the L^2-metric on $H^{p,q}(X)$. More precisely, $g^{H^{p,q}(X)}_\omega$ is induced by $g^{\Omega^{p,q}(X)}_\omega$ via the Hodge theorem. Let $|\cdot|_{\eta(X),\omega}$ be the metric on $\eta(X)$ induced by $(g^{H^{p,q}(X)}_\omega)_{0\leqslant p,q\leqslant n}$ via (1.117) and (1.118).

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Proposition 1.25. *The following identity holds,*

$$\tau_{\mathrm{top}}(X) = \log|\epsilon_X|_{\eta(X),\omega} = 0. \tag{1.125}$$

Proof. Let $\Box_p$ be as in (1.71) with $(\Omega^{0,\bullet}(X,E), \bar{\partial}^E, g^E)$ replaced by $(\Omega^{p,\bullet}(X), \bar{\partial}, g_\omega^{\Lambda^p(T^*X)})$. Let $\Box_{p,q}$ be the restriction of $\Box_p$ to $\Omega^{p,q}(X)$. Let $\theta_p(z)$ be as in (1.74) with $\Box_q^E$ replaced by $\Box_{p,q}$. By Definition 1.13, 1.23, the first equality in (1.125) is equivalent to

$$\sum_{p=0}^{n}(-1)^p\theta_p'(0) = 0, \tag{1.126}$$

which was indicated in [Bis04, p. 1304].

Denote by $\mathrm{covol}(H^k_{\mathrm{Sing}}(X,\mathbb{Z}),\omega)$ the covolume of $\mathrm{Im}(H^k_{\mathrm{Sing}}(X,\mathbb{Z}) \to H^k_{\mathrm{Sing}}(X,\mathbb{R}))$ with respect to the metric induced by $\bigoplus_{p+q=k} g_\omega^{H^{p,q}(X)}$ via (1.121). We have

$$|\epsilon_X|_{\eta(X),\omega} = \prod_{k=0}^{2n}\left(\mathrm{covol}(H^k_{\mathrm{Sing}}(X,\mathbb{Z}),\omega)\right)^{(-1)^k}. \tag{1.127}$$

On the other hand, by [EFM21, Remark 5.5(ii)], we have

$$\mathrm{covol}\big(H^k_{\mathrm{Sing}}(X,\mathbb{Z}),\omega\big)\mathrm{covol}\big(H^{2n-k}_{\mathrm{Sing}}(X,\mathbb{Z}),\omega\big) = 1. \tag{1.128}$$

Here we remark that, due to the normalization in (1.70) and (1.121), the covolume in the sense of [EFM21, Remark 5.5(ii)] equals $(2\pi)^{(n-k)b_k/2}\mathrm{covol}(H^k_{\mathrm{Sing}}(X,\mathbb{Z}),\omega)$, where b_k is the kth Betti number of X. From (1.127) and (1.128), we obtain $|\epsilon_X|_{\eta(X),\omega} = 1$, which is equivalent to the second equality in (1.125). This completes the proof. ☐

2. Several properties of the BCOV torsion

2.1 Kähler metric on projective bundle

For a complex vector space V, we denote by $\mathbb{P}(V)$ the set of complex lines in V. Then $\mathbb{P}(V)$ is complex manifold.

Let Y be an m-dimensional compact Kähler manifold. Let N be a holomorphic vector bundle over Y of rank n. Let $\mathbb{1}$ be the trivial line bundle over Y. Set

$$X = \mathbb{P}(N \oplus \mathbb{1}). \tag{2.1}$$

Let $\pi : X \to Y$ be the canonical projection. For $y \in Y$, we denote $Z_y = \pi^{-1}(y)$, which is isomorphic to $\mathbb{CP}^n$. Let $\omega_{\mathbb{CP}^n}$ be the Kähler form on $\mathbb{CP}^n$ associated with the Fubini–Study metric. More precisely, $-i\omega_{\mathbb{CP}^n}$ is equal to the curvature of the tautological line bundle over $\mathbb{CP}^n$ equipped with the standard metric.

Lemma 2.1. *There exists a Kähler form ω on X such that for any $y \in Y$, there exists an isomorphism $\phi_y : \mathbb{CP}^n \to Z_y$ such that $\phi_y^*(\omega|_{Z_y}) = \omega_{\mathbb{CP}^n}$.*

Here $(\phi_y)_{y\in Y}$ is merely a set of maps parameterized by $y \in Y$. It is not even required to depend continuously on y.

Proof. We refer the reader to the proof of [Voi02, Proposition 3.18]. ☐

Let $s \in \{1,\dots,n\}$. We assume that there are holomorphic line bundles $L_1,\dots,L_s$ over Y together with a surjection between holomorphic vector bundles,

$$N \to L_1 \oplus \cdots \oplus L_s. \tag{2.2}$$

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For $k = 1, \ldots, s$, let $N \to L_k$ be the composition of (2.2) and the canonical projection $L_1 \oplus \cdots \oplus L_s \to L_k$. Set

$$N_k = \mathrm{Ker}(N \to L_k) \subseteq N, \quad X_k = \mathbb{P}(N_k \oplus \mathbb{1}) \subseteq X, \quad X_0 = \mathbb{P}(N) \subseteq X. \tag{2.3}$$

Let $[\xi_0 : \cdots : \xi_n]$ be homogenous coordinates on $\mathbb{C}\mathrm{P}^n$. For $k = 0, \ldots, n$, we denote $H_k = \{\xi_k = 0\} \subseteq \mathbb{C}\mathrm{P}^n$.

LEMMA 2.2. *There exists a Kähler form ω on X such that for any $y \in Y$, there exists an isomorphism $\phi_y : \mathbb{C}\mathrm{P}^n \to Z_y$ such that $\phi_y^*(\omega|_{Z_y}) = \omega_{\mathbb{C}\mathrm{P}^n}$ and $\phi_y^{-1}(X_k \cap Z_y) = H_k$ for $k = 0, \ldots, s$.*

Proof. Let N^* be the dual of N. We have $L_1^{-1} \oplus \cdots \oplus L_s^{-1} \hookrightarrow N^*$. Let g^{N^*} be a Hermitian metric on N^* such that $L_1^{-1}, \ldots, L_s^{-1} \subseteq N^*$ are mutually orthogonal. Let g^N be the dual metric on N. Now, proceeding in the same way as in the proof of [Voi02, Proposition 3.18], we obtain ω satisfying the desired properties. This completes the proof. □

2.2 Behavior under adiabatic limit

We use the notation in § 2.1. By Lemma 2.1, there exists a Kähler form ω_X on X such that for any $y \in Y$, there exists an isomorphism $\phi_y : \mathbb{C}\mathrm{P}^n \to Z_y$ such that

$$\phi_y^*(\omega_X|_{Z_y}) = \omega_{\mathbb{C}\mathrm{P}^n}. \tag{2.4}$$

Let $\omega_{Z_y} = \omega_X|_{Z_y}$. Note that $(Z_y, \omega_{Z_y})_{y \in Y}$ are mutually isometric, we omit the index y as long as there is no confusion. Let ω_Y be a Kähler form on Y. For $\varepsilon > 0$, set

$$\omega_\varepsilon = \omega_X + \frac{1}{\varepsilon}\pi^*\omega_Y. \tag{2.5}$$

We denote

$$(c_1 c_{m-1})(Y) = \int_Y c_1(TY) c_{m-1}(TY). \tag{2.6}$$

Let $\chi(\cdot)$ be the topological Euler characteristic. Recall that $\tau_{\mathrm{BCOV}}(\cdot, \cdot)$ was defined in Definition 1.24.

THEOREM 2.3. *As $\varepsilon \to 0$,*

$$\begin{aligned} &\tau_{\mathrm{BCOV}}(X, \omega_\varepsilon) - \tfrac{1}{12}\chi(Z)\big(m\chi(Y) + (c_1 c_{m-1})(Y)\big) \log \varepsilon \\ &\quad \to \chi(Z)\tau_{\mathrm{BCOV}}(Y, \omega_Y) + \chi(Y)\tau_{\mathrm{BCOV}}(Z, \omega_Z). \end{aligned} \tag{2.7}$$

Proof. The proof consists of several steps.

Recall that $\eta(\cdot)$ was constructed in (1.118) and $\lambda_{\mathrm{tot}}(\cdot)$ was constructed in (1.119).

Step 1. We construct two canonical sections of

$$\lambda_{\mathrm{tot}}(X) \otimes \big(\lambda_{\mathrm{tot}}(Y)\big)^{-\chi(Z)} \otimes \big(\eta(Y)\big)^{-n\chi(Z)}. \tag{2.8}$$

For $p = 0, \ldots, m+n$ and $s = 0, \ldots, p-1$, set

$$I_s^p = \big\{u \in \Lambda^p(T^*X) : u(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in TZ, v_{s+2}, \ldots, v_p \in TX\big\}. \tag{2.9}$$

For convenience, we denote $I_p^p = \Lambda^p(T^*X)$ and $I_{-1}^p = 0$. We obtain a filtration

$$\Lambda^p(T^*X) = I_p^p \hookleftarrow I_{p-1}^p \hookleftarrow \cdots \hookleftarrow I_{-1}^p = 0. \tag{2.10}$$

For $r = 0, \ldots, m$ and $s = 0, \ldots, n$, we denote

$$E_{r,s} = \Lambda^s(T^*Z) \otimes \pi^*\Lambda^r(T^*Y). \tag{2.11}$$

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We have a short exact sequence of holomorphic vector bundles over X,

$$0 \to I_{s-1}^{r+s} \to I_s^{r+s} \to E_{r,s} \to 0. \tag{2.12}$$

Let

$$\alpha_{r,s} \in \big(\det H^\bullet(X, I_{s-1}^{r+s})\big)^{-1} \otimes \det H^\bullet(X, I_s^{r+s}) \otimes \big(\det H^\bullet(X, E_{r,s})\big)^{-1}. \tag{2.13}$$

be the canonical section induced by the long exact sequence induced by (2.12).

Let $H^{\bullet,\bullet}(Z)$ be the fiberwise cohomology. As $Z \simeq \mathbb{C}\mathrm{P}^n$, we have

$$H^{p,p}(Z) = \mathbb{C} \quad \text{for } p = 0, \ldots, n, \quad H^{p,q}(Z) = 0 \quad \text{for } p \neq q. \tag{2.14}$$

Applying spectral sequence while using (2.11) and (2.14), we obtain

$$H^q(X, E_{r,s}) \simeq H^{r,q-s}\big(Y, H^{s,s}(Z)\big) := H^{q-s}\big(Y, \Lambda^r(T^*Y) \otimes H^{s,s}(Z)\big). \tag{2.15}$$

Let

$$\beta_{r,s} \in \det H^\bullet(X, E_{r,s}) \otimes \big(\det H^{r,\bullet}\big(Y, H^{s,s}(Z)\big)\big)^{-(-1)^s} \tag{2.16}$$

be the canonical section induced by (2.15).

We have a generator of lattice,

$$\delta_s \in H^{2s}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n, \mathbb{Z}) \subseteq H^{2s}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n, \mathbb{R}) \subseteq H^{2s}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n, \mathbb{C}). \tag{2.17}$$

We identify $H^{2s}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n, \mathbb{C})$ with $H^{2s}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) = H^{s,s}(\mathbb{C}\mathrm{P}^n)$ (see (1.121)). Since $H^{s,s}(Z) = H^{s,s}(\mathbb{C}\mathrm{P}^n) = H^{2s}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n, \mathbb{C})$ is a trivial line bundle over Y, we have an isomorphism (cf. [GH94, p. 607])

$$\begin{aligned} H^{r,\bullet}(Y) &\to H^{r,\bullet}\big(Y, H^{s,s}(Z)\big) = H^{r,\bullet}(Y) \otimes H^{s,s}(\mathbb{C}\mathrm{P}^n) \\ u &\mapsto u \otimes \delta_s. \end{aligned} \tag{2.18}$$

Let

$$\gamma_{r,s} \in \big(\det H^{r,\bullet}\big(Y, H^{s,s}(Z)\big)\big)^{(-1)^s} \otimes \big(\det H^{r,\bullet}(Y)\big)^{-(-1)^s} \tag{2.19}$$

be the canonical section induced by (2.18). By (2.13), (2.16) and (2.19), we have

$$\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s} \in \big(\det H^\bullet(X, I_{s-1}^{r+s})\big)^{-1} \otimes \det H^\bullet(X, I_s^{r+s}) \otimes \big(\det H^{r,\bullet}(Y)\big)^{-(-1)^s}. \tag{2.20}$$

Recall that $\lambda(\cdot)$ was defined in (1.119). By (1.119) and (2.10), we have

$$\begin{aligned} \lambda(X) &= \bigotimes_{p=1}^{m+n} \big(\det H^\bullet\big(X, \Lambda^p(T^*X)\big)\big)^{(-1)^p p} \\ &= \bigotimes_{p=1}^{m+n} \big(\det H^\bullet(X, I_p^p)\big)^{(-1)^p p} \\ &= \bigotimes_{r=0}^{m} \bigotimes_{s=0}^{n} \Big(\big(\det H^\bullet(X, I_{s-1}^{r+s})\big)^{-1} \otimes \det H^\bullet(X, I_s^{r+s})\Big)^{(-1)^{r+s}(r+s)}. \end{aligned} \tag{2.21}$$

On the other hand, by (1.118), (1.119) and the identities

$$n + 1 = \chi(Z), \quad \sum_{s=0}^{n} s = \frac{n(n+1)}{2} = \frac{n}{2}\chi(Z), \tag{2.22}$$

we have

$$\bigotimes_{r=0}^{m} \bigotimes_{s=0}^{n} \big(\det H^{r,\bullet}(Y)\big)^{(-1)^r (r+s)} = \big(\lambda(Y)\big)^{\chi(Z)} \otimes \big(\eta(Y)\big)^{n\chi(Z)/2}. \tag{2.23}$$

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By (2.20), (2.21) and (2.23), we have

$$\prod_{r=0}^{m}\prod_{s=0}^{n}(\alpha_{r,s}\otimes\beta_{r,s}\otimes\gamma_{r,s})^{(-1)^{r+s}(r+s)}\in\lambda(X)\otimes\big(\lambda(Y)\big)^{-\chi(Z)}\otimes\big(\eta(Y)\big)^{-n\chi(Z)/2}. \tag{2.24}$$

By (1.119) and (2.24), we have

$$\prod_{r=0}^{m}\prod_{s=0}^{n}(\alpha_{r,s}\otimes\beta_{r,s}\otimes\gamma_{r,s})^{(-1)^{r+s}(r+s)}\otimes\overline{\prod_{r=0}^{m}\prod_{s=0}^{n}(\alpha_{r,s}\otimes\beta_{r,s}\otimes\gamma_{r,s})^{(-1)^{r+s}(r+s)}}$$
$$\in\lambda_{\mathrm{tot}}(X)\otimes\big(\lambda_{\mathrm{tot}}(Y)\big)^{-\chi(Z)}\otimes\big(\eta(Y)\big)^{-n\chi(Z)}, \tag{2.25}$$

where $\bar{\cdot}$ is the conjugation.

Let $\sigma_X\in\lambda_{\mathrm{tot}}(X)$, $\sigma_Y\in\lambda_{\mathrm{tot}}(Y)$ and $\epsilon_Y\in\eta(Y)$ be as in (1.122). Obviously, we have

$$\sigma_X\otimes\sigma_Y^{-\chi(Z)}\otimes\epsilon_Y^{-n\chi(Z)}\in\lambda_{\mathrm{tot}}(X)\otimes\big(\lambda_{\mathrm{tot}}(Y)\big)^{-\chi(Z)}\otimes\big(\eta(Y)\big)^{-n\chi(Z)}. \tag{2.26}$$

Step 2. We show that

$$\prod_{r=0}^{m}\prod_{s=0}^{n}(\alpha_{r,s}\otimes\beta_{r,s}\otimes\gamma_{r,s})^{(-1)^{r+s}(r+s)}\otimes\overline{\prod_{r=0}^{m}\prod_{s=0}^{n}(\alpha_{r,s}\otimes\beta_{r,s}\otimes\gamma_{r,s})^{(-1)^{r+s}(r+s)}}$$
$$=\pm\sigma_X\otimes\sigma_Y^{-\chi(Z)}\otimes\epsilon_Y^{-n\chi(Z)}. \tag{2.27}$$

Let $\mathbb{Z}(-1)$ be the inverse of the Tate twist, which is a Hodge structure of pure weight two. For $j\in\mathbb{N}$, we denote by $\mathbb{Z}(-j)$ its jth tensor power. We have canonical identifications of Hodge structures,

$$\begin{aligned}H^{2j}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n,\mathbb{Z})&=\mathbb{Z}(-j)\quad\text{for } j=0,\dots,n,\\ H^{k}_{\mathrm{Sing}}(X,\mathbb{Z})&=\bigoplus_{j=0}^{n}H^{k-2j}_{\mathrm{Sing}}(Y,\mathbb{Z})\otimes H^{2j}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^n,\mathbb{Z})\\ &=\bigoplus_{j=0}^{n}H^{k-2j}_{\mathrm{Sing}}(Y,\mathbb{Z})\otimes\mathbb{Z}(-j).\end{aligned} \tag{2.28}$$

Complexifying (2.28) and applying Hodge decomposition, we obtain

$$\begin{aligned}&H^{j,j}(\mathbb{C}\mathrm{P}^n)=\mathbb{C}\quad\text{for } j=0,\dots,n,\\ &H^{p,q}(X)=\bigoplus_{j=0}^{n}H^{p-j,q-j}(Y)\otimes H^{j,j}(\mathbb{C}\mathrm{P}^n)=\bigoplus_{j=0}^{n}H^{p-j,q-j}(Y).\end{aligned} \tag{2.29}$$

We use the identifications in (2.28) and (2.29) until the end of Step 2.

Claim. For complex vector spaces A and B, the canonical identification $\det A\otimes\det B\otimes(\det(A\oplus B))^{-1}=\mathbb{C}$ is such that the canonical section of $\det A\otimes\det B\otimes(\det(A\oplus B))^{-1}$ is identified with $1\in\mathbb{C}$.

Recall that I^{r+s}_s was defined in (2.9) and $E_{r,s}$ was defined in (2.11). We have

$$H^q(X,I^{r+s}_s)=\bigoplus_{j=0}^{s}H^{r+s-j,q-j}(Y),\quad H^q(X,E_{r,s})=H^{r,q-s}(Y). \tag{2.30}$$

By (2.30), we have

$$H^\bullet(X,I^{r+s}_s)=H^\bullet(X,I^{r+s}_{s-1})\oplus H^\bullet(X,E_{r,s}). \tag{2.31}$$

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Applying the claim in the last paragraph to (2.31), we obtain

$$\big(\det H^\bullet(X, I_{s-1}^{r+s})\big)^{-1} \otimes \det H^\bullet(X, I_s^{r+s}) \otimes \big(\det H^\bullet(X, E_{r,s})\big)^{-1} = \mathbb{C}, \quad \alpha_{r,s} = 1. \tag{2.32}$$

A similar argument shows that

$$\begin{aligned} \det H^\bullet(X, E_{r,s}) \otimes \big(\det H^{r,\bullet}\big(Y, H^{s,s}(Z)\big)\big)^{-(-1)^s} &= \mathbb{C}, \quad \beta_{r,s} = 1, \\ \big(\det H^{r,\bullet}\big(Y, H^{s,s}(Z)\big)\big)^{(-1)^s} \otimes \big(\det H^{r,\bullet}(Y)\big)^{-(-1)^s} &= \mathbb{C}, \quad \gamma_{r,s} = 1. \end{aligned} \tag{2.33}$$

Using (1.119), (1.121) and (2.28), we can show that

$$\begin{aligned} \lambda_{\mathrm{tot}}(X) \otimes \big(\lambda_{\mathrm{tot}}(Y)\big)^{-\chi(Z)} \otimes \big(\eta(Y)\big)^{-n\chi(Z)} &= \mathbb{C}, \\ \sigma_X \otimes \sigma_Y^{-\chi(Z)} \otimes \epsilon_Y^{-n\chi(Z)} &= \pm 1. \end{aligned} \tag{2.34}$$

From (2.32)–(2.34), we obtain (2.27).

Step 3. We introduce several Quillen metrics.

- Let g_ε^{TX} be the metric on TX induced by ω_ε.
- Let $g_\varepsilon^{\Lambda^p(T^*X)}$ be the metric on $\Lambda^p(T^*X)$ induced by g_ε^{TX}.
- Let $g_\varepsilon^{I_s^p}$ be the metric on I_s^p induced by $g_\varepsilon^{\Lambda^p(T^*X)}$ via (2.10).
- Let g^{TY} be the metric on TY induced by ω_Y.
- Let $g^{\Lambda^r(T^*Y)}$ be the metric on $\Lambda^r(T^*Y)$ induced by g^{TY}.
- Let g^{TZ} be the metric on TZ induced by $\omega_Z = \omega_\varepsilon|_Z$.
- Let $g^{\Lambda^s(T^*Z)}$ be the metric on $\Lambda^s(T^*Z)$ induced by g^{TZ}.
- Let $g^{E_{r,s}}$ be the metric on $E_{r,s}$ induced by $g^{\Lambda^r(T^*Y)}$ and $g^{\Lambda^s(T^*Z)}$ via (2.11).

Let

$$\|\cdot\|_{\det H^\bullet(X, I_s^p), \varepsilon} \tag{2.35}$$

be the Quillen metric on $\det H^\bullet(X, I_s^p)$ associated with g_ε^{TX} and $g_\varepsilon^{I_s^p}$. Let

$$\|\cdot\|_{\det H^\bullet(X, E_{r,s}), \varepsilon} \tag{2.36}$$

be the Quillen metric on $\det H^\bullet(X, E_{r,s})$ associated with g_ε^{TX} and $g^{E_{r,s}}$. Recall that $\alpha_{r,s}$ was defined by (2.13). Let $\|\alpha_{r,s}\|_\varepsilon$ be the norm of $\alpha_{r,s}$ with respect to the metrics (2.35) and (2.36).

- Let $g^{\Omega^{s,s}(Z)}$ be the L^2-metric on $\Omega^{s,s}(Z)$ induced by g^{TZ} (see (1.70)).
- Let $g^{H^{s,s}(Z)}$ be the metric on $H^{s,s}(Z)$ induced by $g^{\Omega^{s,s}(Z)}$ via the Hodge theorem.

Let

$$\|\cdot\|_{\det H^{r,\bullet}(Y, H^{s,s}(Z))} \tag{2.37}$$

be the Quillen metric on $\det H^{r,\bullet}(Y, H^{s,s}(Z)) = \det H^\bullet(Y, \Lambda^r(T^*Y) \otimes H^{s,s}(Z))$ associated with g^{TY} and $g^{\Lambda^r(T^*Y)} \otimes g^{H^{s,s}(Z)}$. Recall that $\beta_{r,s}$ was defined by (2.16). Let $\|\beta_{r,s}\|_\varepsilon$ be the norm of $\beta_{r,s}$ with respect to the metrics (2.36) and (2.37). Let

$$\|\cdot\|_{\det H^{r,\bullet}(Y)} \tag{2.38}$$

be the Quillen metric on $\det H^{r,\bullet}(Y) = \det H^\bullet(Y, \Lambda^r(T^*Y))$ associated with g^{TY} and $g^{\Lambda^r(T^*Y)}$. Recall that $\gamma_{r,s}$ was defined by (2.19). Let $\|\gamma_{r,s}\|$ be the norm of $\gamma_{r,s}$ with respect to the metrics (2.37) and (2.38).

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By (1.119) and (2.10), we have

$$\sigma_X \in \lambda_{\mathrm{tot}}(X) = \bigotimes_{p=1}^{m+n} \big(\det H^\bullet(X, I_p^p)\big)^{(-1)^p p} \otimes \overline{\bigotimes_{p=1}^{m+n} \big(\det H^\bullet(X, I_p^p)\big)^{(-1)^p p}}. \tag{2.39}$$

Let $\|\sigma_X\|_\varepsilon$ be the norm of σ_X with respect to the metrics (2.35) with $s = p$. By (1.118) and (1.119), we have

$$\begin{aligned} \epsilon_Y \in \eta(Y) &= \bigotimes_{r=0}^{m} \big(\det H^{r,\bullet}(Y)\big)^{(-1)^r}, \\ \sigma_Y \in \lambda_{\mathrm{tot}}(Y) &= \bigotimes_{r=1}^{m} \big(\det H^{r,\bullet}(Y)\big)^{(-1)^r r} \otimes \overline{\bigotimes_{r=1}^{m} \big(\det H^{r,\bullet}(Y)\big)^{(-1)^r r}}. \end{aligned} \tag{2.40}$$

Let $\|\epsilon_Y\|$ be the norm of ϵ_Y with respect to the metrics (2.38). Let $\|\sigma_Y\|$ be the norm of σ_Y with respect to the metrics (2.38). By (2.27), we have

$$\begin{aligned} &\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}(r+s)\big(\log\|\alpha_{r,s}\|_\varepsilon^2 + \log\|\beta_{r,s}\|_\varepsilon^2 + \log\|\gamma_{r,s}\|^2\big) \\ &\qquad = \log\|\sigma_X\|_\varepsilon - \chi(Z)\log\|\sigma_Y\| - n\chi(Z)\log\|\epsilon_Y\|. \end{aligned} \tag{2.41}$$

On the other hand, by Definition 1.23 and Proposition 1.25, we have

$$\log\|\epsilon_Y\| = 0. \tag{2.42}$$

By Definition 1.24, (2.41) and (2.42), we have

$$\begin{aligned} \tau_{\mathrm{BCOV}}(X,\omega_\varepsilon) &= \chi(Z)\tau_{\mathrm{BCOV}}(Y,\omega_Y) \\ &\quad + \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}(r+s)\big(\log\|\alpha_{r,s}\|_\varepsilon^2 + \log\|\beta_{r,s}\|_\varepsilon^2 + \log\|\gamma_{r,s}\|^2\big). \end{aligned} \tag{2.43}$$

Step 4. We estimate $\log\|\alpha_{r,s}\|_\varepsilon^2$.

Recall that I_s^{r+s} was defined in (2.9), $E_{r,s}$ was defined in (2.11), $g_\varepsilon^{I_s^{r+s}}$ and $g^{E_{r,s}}$ were defined at the beginning of Step 3. Let $g_\varepsilon^{E_{r,s}}$ be quotient metric on $E_{r,s}$ induced by $g_\varepsilon^{I_s^{r+s}}$ via the surjection $I_s^{r+s} \to E_{r,s}$ in (2.12). Note that $g_\varepsilon^{I_s^{r+s}}$ is induced by ω_ε. By (2.5), as $\varepsilon \to 0$,

$$\varepsilon^{-r} g_\varepsilon^{E_{r,s}} \to g^{E_{r,s}}. \tag{2.44}$$

We use the notation from (1.23). Let

$$\tilde{T}_{r,s,\varepsilon} = \widetilde{\mathrm{ch}}\Big(g_\varepsilon^{I_{s-1}^{r+s}}, g_\varepsilon^{I_s^{r+s}}, g_\varepsilon^{E_{r,s}}\Big) \in Q^X/Q^{X,0} \tag{2.45}$$

be the Bott–Chern form (1.56) with $0 \to E' \to E \to E'' \to 0$ replaced by (2.12) and $(g^{E'}, g^E, g^{E''})$ replaced by $(g_\varepsilon^{I_{s-1}^{r+s}}, g_\varepsilon^{I_s^{r+s}}, g_\varepsilon^{E_{r,s}})$. Let

$$T_{r,s,\varepsilon} = \widetilde{\mathrm{ch}}\Big(g_\varepsilon^{I_{s-1}^{r+s}}, g_\varepsilon^{I_s^{r+s}}, g^{E_{r,s}}\Big) \in Q^X/Q^{X,0} \tag{2.46}$$

be the Bott–Chern form (1.56) with $0 \to E' \to E \to E'' \to 0$ replaced by (2.12) and $(g^{E'}, g^E, g^{E''})$ replaced by $(g_\varepsilon^{I_{s-1}^{r+s}}, g_\varepsilon^{I_s^{r+s}}, g^{E_{r,s}})$. By Proposition 1.11 and (2.44), as $\varepsilon \to 0$,

$$T_{r,s,\varepsilon} - \tilde{T}_{r,s,\varepsilon} - \mathrm{ch}(E_{r,s}, g^{E_{r,s}})r\log\varepsilon = \widetilde{\mathrm{ch}}(g^{E_{r,s}}, g_\varepsilon^{E_{r,s}}) - \mathrm{ch}(E_{r,s}, g^{E_{r,s}})r\log\varepsilon \to 0. \tag{2.47}$$

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On the other hand, by Proposition 1.12, as $\varepsilon \to 0$,

$$\tilde{T}_{r,s,\varepsilon} \to 0. \tag{2.48}$$

By (2.47) and (2.48), as $\varepsilon \to 0$,

$$T_{r,s,\varepsilon} - \mathrm{ch}(E_{r,s}, g^{E_{r,s}}) r \log \varepsilon \to 0. \tag{2.49}$$

Applying Theorem 1.18 to the short exact sequence (2.12), we obtain

$$\log\|\alpha_{r,s}\|^2_\varepsilon = \int_X \mathrm{Td}(TX, g^{TX}_\varepsilon) T_{r,s,\varepsilon}. \tag{2.50}$$

By Proposition 1.10, as $\varepsilon \to 0$,

$$\mathrm{Td}(TX, g^{TX}_\varepsilon) \to \pi^* \mathrm{Td}(TY, g^{TY}) \mathrm{Td}(TZ, g^{TZ}). \tag{2.51}$$

On the other hand, by the Grothendieck–Riemann–Roch formula (1.76), (2.11) and (2.14), we have

$$\begin{aligned}
&\int_X \pi^* \mathrm{Td}(TY, g^{TY}) \mathrm{Td}(TZ, g^{TZ}) \mathrm{ch}(E_{r,s}, g^{E_{r,s}}) \\
&\quad = \int_Y \mathrm{Td}(TY) \mathrm{ch}\big(H^\bullet(Z, E_{r,s})\big) \\
&\quad = \int_Y \mathrm{Td}(TY) \mathrm{ch}\big(\Lambda^r(T^*Y)\big) \mathrm{ch}\big(H^{s,\bullet}(Z)\big) \\
&\quad = (-1)^s \int_Y \mathrm{Td}(TY) \mathrm{ch}\big(\Lambda^r(T^*Y)\big).
\end{aligned} \tag{2.52}$$

By (2.49)–(2.52), as $\varepsilon \to 0$,

$$\log\|\alpha_{r,s}\|^2_\varepsilon - (-1)^s r \int_Y \mathrm{Td}(TY) \mathrm{ch}\big(\Lambda^r(T^*Y)\big) \log \varepsilon \to 0. \tag{2.53}$$

By Proposition 1.5, (2.22) and (2.53), as $\varepsilon \to 0$,

$$\begin{aligned}
&\sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} (r+s) \log\|\alpha_{r,s}\|^2_\varepsilon \\
&\quad - \left(\frac{m(3m+3n+1)}{12} \chi(Y) + \frac{1}{6} (c_1 c_{m-1})(Y) \right) \chi(Z) \log \varepsilon \to 0.
\end{aligned} \tag{2.54}$$

Step 5. We estimate $\log\|\beta_{r,s}\|^2_\varepsilon$.

Let

$$T_{r,s} \in Q^Y \tag{2.55}$$

be the Bismut–Köhler analytic torsion form (see § 1.5) associated with $(\pi : X \to Y, \omega_X, E_{r,s}, g^{E_{r,s}})$. Applying Theorem 1.17 with $E = E_{r,s}$, as $\varepsilon \to 0$,

$$\log\|\beta_{r,s}\|^2_\varepsilon + \int_Y \mathrm{Td}'(TY) \int_Z \mathrm{Td}(TZ) \mathrm{ch}(E_{r,s}) \log \varepsilon \to \int_Y \mathrm{Td}(TY, g^{TY}) T_{r,s}. \tag{2.56}$$

Similarly to (2.52), we have

$$\int_Y \mathrm{Td}'(TY) \int_Z \mathrm{Td}(TZ) \mathrm{ch}(E_{r,s}) = (-1)^s \int_Y \mathrm{Td}'(TY) \mathrm{ch}\big(\Lambda^r(T^*Y)\big). \tag{2.57}$$

Applying Proposition 1.15 with $E = E_{0,s}$ and $F = \Lambda^r(T^*Y)$, we obtain

$$T_{r,s} = \mathrm{ch}\big(\Lambda^r(T^*Y), g^{\Lambda^r(T^*Y)}\big) T_{0,s} \text{ modulo } Q^{Y,0}. \tag{2.58}$$

By (2.56)–(2.58), as $\varepsilon \to 0$,

$$\log\|\beta_{r,s}\|_{\varepsilon}^{2} + (-1)^{s}\int_{Y}\mathrm{Td}'(TY)\mathrm{ch}\big(\Lambda^{r}(T^{*}Y)\big)\log\varepsilon \\ \to \int_{Y}\mathrm{Td}(TY,g^{TY})\mathrm{ch}\big(\Lambda^{r}(T^{*}Y),g^{\Lambda^{r}(T^{*}Y)}\big)T_{0,s}. \tag{2.59}$$

On the other hand, by Theorem 1.16, we have

$$\sum_{s=0}^{n}(-1)^{s}T_{0,s} = 0 \text{ modulo } Q^{Y,0}. \tag{2.60}$$

By Propositions 1.5, 1.6, (2.22), (2.59) and (2.60), as $\varepsilon \to 0$,

$$\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}(r+s)\log\|\beta_{r,s}\|_{\varepsilon}^{2} + \Big(\frac{m(m+n)}{4}\chi(Y) + \frac{1}{12}(c_{1}c_{m-1})(Y)\Big)\chi(Z)\log\varepsilon \\ \to \int_{Y}c_{m}(TY,g^{TY})\sum_{s=0}^{n}(-1)^{s}sT_{0,s} \\ = \int_{Y}c_{m}(TY,g^{TY})\sum_{s=0}^{n}(-1)^{s}s\{T_{0,s}\}^{(0,0)}, \tag{2.61}$$

where $\{\cdot\}^{(0,0)}$ means the component of degree $(0,0)$.

Step 6. We calculate $\log\|\gamma_{r,s}\|^{2}$.

Recall that $H^{s,s}(Z)$ is a trivial line bundle over Y. Recall that $g^{H^{s,s}(Z)}$ was constructed in the paragraph above (2.37). By our assumption (2.4), $g^{H^{s,s}(Z)}$ is a constant metric. Recall that $\delta_{s} \in H^{s,s}(Z)$ was constructed in (2.17). Let $|\delta_{s}|$ be the norm of δ_{s} with respect to $g^{H^{s,s}(Z)}$, which is a constant function on Y. In the following, we do not distinguish between a constant function and its value. We denote $\chi_{r}(Y) = \sum_{q=0}^{m}(-1)^{q}\dim H^{r,q}(Y)$. By Remark 1.14, we have

$$\log\|\gamma_{r,s}\|^{2} = (-1)^{s}\chi_{r}(Y)\log|\delta_{s}|^{2}. \tag{2.62}$$

Let $\epsilon_{Z} \in \eta(Z)$ be as in (1.122). We have

$$\epsilon_{Z} = \pm\bigotimes_{s=0}^{n}\delta_{s}. \tag{2.63}$$

Let $|\epsilon_{Z}|$ be the norm of ϵ_{Z} with respect to the metrics $g^{H^{s,s}(Z)}$. By Proposition 1.25 and (2.63), we have

$$\sum_{s=0}^{n}\log|\delta_{s}|^{2} = \log|\epsilon_{Z}|^{2} = 0. \tag{2.64}$$

Let $\sigma_{Z} \in \lambda_{\mathrm{tot}}(Z)$ be as in (1.122). We have

$$\sigma_{Z} = \pm\bigotimes_{s=1}^{n}\delta_{s}^{2s}. \tag{2.65}$$

Let $|\sigma_{Z}|$ be the norm of σ_{Z} with respect to the metrics $g^{H^{s,s}(Z)}$. By (2.65), we have

$$\sum_{s=0}^{n}s\log|\delta_{s}|^{2} = \log|\sigma_{Z}|. \tag{2.66}$$

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By (2.62), (2.64), (2.66) and the identity $\sum_{r=0}^{m}(-1)^r\chi_r(Y)=\chi(Y)$, we have

$$\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}(r+s)\log\|\gamma_{r,s}\|^2=\chi(Y)\log|\sigma_Z|. \tag{2.67}$$

Step 7. We conclude.

By (2.43), (2.54), (2.61) and (2.67), as $\varepsilon\to 0$,

$$\begin{aligned}&\tau_{\mathrm{BCOV}}(X,\omega_\varepsilon)-\frac{1}{12}\chi(Z)\big(m\chi(Y)+(c_1c_{m-1})(Y)\big)\log\varepsilon\\&\to\chi(Z)\tau_{\mathrm{BCOV}}(Y,\omega_Y)+\chi(Y)\log|\sigma_Z|\\&\quad+\int_Y c_m(TY,g^{TY})\sum_{s=0}^{n}(-1)^s s\{T_{0,s}\}^{(0,0)}.\end{aligned} \tag{2.68}$$

Let $\theta_s(z)$ be as in (1.74) with (X,ω) replaced by (Z,ω_Z) and (E,g^E) replaced by $(\Lambda^s(T^*Z),g^{\Lambda^s(T^*Z)})$. By Definition 1.13, 1.24, we have

$$\tau_{\mathrm{BCOV}}(Z,\omega_Z)=\log|\sigma_Z|+\sum_{s=0}^{n}(-1)^s s\theta_s'(0). \tag{2.69}$$

By (2.4), all the terms in (2.69) are constant functions on Y. By (1.79), we have

$$\{T_{0,s}\}^{(0,0)}=\theta_s'(0). \tag{2.70}$$

From (2.68)–(2.70), we obtain (2.7). This completes the proof. □

Remark 2.4. The key ingredient in the proof of Theorem 2.3 is [BB94, Theorem 3.2], which is a consequence of [BB94, Theorem 3.1]. Of course, we can replace [BB94, Theorem 3.2] by [BB94, Theorem 3.1] in our proof to obtain a formula for $\tau_{\mathrm{BCOV}}(X,\omega_X)$. However, because [BB94, Theorem 3.1] involves a Bott–Chern form, the formula obtained will be far from clean.

2.3 Behavior under blow-ups

The following lemma is direct consequence of Bott formula [Bot57] (see also [OSS11, p. 5]).

Lemma 2.5. *Let L be the holomorphic line bundle of degree one over $\mathbb{C}\mathrm{P}^n$. For $k=1,\dots,n$ and $s=1,\dots,k$, we have*

$$H^\bullet\big(\mathbb{C}\mathrm{P}^n,\Lambda^k(T^*\mathbb{C}\mathrm{P}^n)\otimes L^s\big)=0. \tag{2.71}$$

Let X be an n-dimensional compact Kähler manifold. Let $Y\subseteq X$ be a closed complex submanifold. Let $f:X'\to X$ be the blow-up along Y. Let $Y\subseteq U\subseteq X$ be an open neighborhood of Y. Set $U'=f^{-1}(U)$. Let ω be a Kähler form on X. Let ω' be a Kähler form on X' such that

$$\omega'|_{X'\backslash U'}=f^*(\omega|_{X\backslash U}). \tag{2.72}$$

For the existence of such ω', see the proof of [Voi02, Proposition 3.24].

Theorem 2.6. *We have*

$$\tau_{\mathrm{BCOV}}(X',\omega')-\tau_{\mathrm{BCOV}}(X,\omega)=\alpha(U,U',\omega|_U,\omega'|_{U'}), \tag{2.73}$$

where $\alpha(U,U',\omega|_U,\omega'|_{U'})$ is a real number determined by U, U', $\omega|_U$ and $\omega'|_{U'}$.

Proof. The proof consists of several steps.

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Step 0. We introduce several pieces of notation.

We denote $D = f^{-1}(Y)$. Let $i : D \hookrightarrow X'$ be the canonical embedding. Let $\mathscr{I} \subseteq \mathscr{O}_{X'}$ be the ideal sheaf associated with D. More precisely, for open subset $U \subseteq X'$, we have

$$\mathscr{I}(U) = \{\theta \in \mathscr{O}_{X'}(U) : \theta|_{U\cap D} = 0\}. \tag{2.74}$$

For $p = 0, \ldots, n$, there exist holomorphic vector bundles over X' linked by holomorphic maps

$$f^*\Lambda^p(T^*X) = F^p_p \to F^p_{p-1} \to \cdots \to F^p_0 = \Lambda^p(T^*X') \tag{2.75}$$

such that for $s = 0, \ldots, p-1$,

- the induced map $\mathscr{O}_{X'}(F^p_{s+1}) \to \mathscr{O}_{X'}(F^p_s)$ is injective;
- we have $\mathscr{I} \otimes \mathscr{O}_{X'}(F^p_s) \hookrightarrow \mathscr{O}_{X'}(F^p_{s+1}) \hookrightarrow \mathscr{O}_{X'}(F^p_s)$.

Set

$$\mathscr{G}^p_s = \mathscr{O}_{X'}(F^p_s)/\mathscr{O}_{X'}(F^p_{s+1}). \tag{2.76}$$

Then we have a commutative diagram of analytic coherent sheaves on X',

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_{X'}(F^p_{s+1}) & \longrightarrow & \mathscr{O}_{X'}(F^p_s) & \longrightarrow & \mathscr{G}^p_s & \longrightarrow & 0 \\ &&&& & \searrow & \uparrow && \\ &&&&&& i_*\mathscr{O}_D(F^p_s|_D) && \end{array} \tag{2.77}$$

where the first row is exact. Now we briefly explain the existence of these F^p_s. We have

$$\mathscr{I}^{\otimes p} \otimes \mathscr{O}_{X'}\big(\Lambda^p(T^*X')\big) \hookrightarrow \mathscr{O}_{X'}\big(f^*\Lambda^p(T^*X)\big) \hookrightarrow \mathscr{O}_{X'}\big(\Lambda^p(T^*X')\big). \tag{2.78}$$

For $s = 0, \ldots, p$, let $\mathscr{F}^p_s$ be the sub-sheaf of $\mathscr{O}_{X'}(\Lambda^p(T^*X'))$ generated by $\mathscr{I}^{\otimes s} \otimes \mathscr{O}_{X'}(\Lambda^p(T^*X'))$ and $\mathscr{O}_{X'}(f^*\Lambda^p(T^*X))$. Then the desired properties hold with $\mathscr{O}_{X'}(F^p_s)$ replaced by $\mathscr{F}^p_s$. It remains to show that each $\mathscr{F}^p_s$ is given by a holomorphic vector bundle. Let r be the codimension of $Y \hookrightarrow X$. Let N_Y be the normal bundle of $Y \hookrightarrow X$. Let $\pi : D = \mathbb{P}(N_Y) \to Y$ be the canonical projection. Let $(y_0, y_1, \ldots, y_{n-r}, z_1, \ldots, z_{r-1}) \in \mathbb{C}^n$ be local coordinates on a neighborhood of $x \in D$ such that:

- $(y_1, \ldots, y_{n-r})$ are the coordinates on Y;
- $(z_1, \ldots, z_{r-1})$ are the coordinates on the fiber of $\pi : D \to Y$;
- $D \subseteq X'$ is given by the equation $y_0 = 0$.

Then the image of $\mathscr{O}_{X'}(f^*T^*X) \hookrightarrow \mathscr{O}_{X'}(T^*X')$ is generated by

$$dy_0, dy_1, \ldots, dy_{n-r}, y_0\, dz_1, \ldots, y_0\, dz_{r-1}. \tag{2.79}$$

As a consequence, the image of $\mathscr{F}^p_s \hookrightarrow \mathscr{O}_{X'}(\Lambda^p(T^*X'))$ is generated by

$$y_0^{\min\{s,|J|\}} \bigotimes_{i\in I} dy_i \otimes \bigotimes_{j\in J} dz_j \tag{2.80}$$

with $I \subseteq \{0, 1, \ldots, n-r\}$ and $J \subseteq \{1, \ldots, r-1\}$ satisfying $|I| + |J| = p$. Each term in (2.80) yields a holomorphic line bundle. Hence, $\mathscr{F}^p_s$ is given by a holomorphic vector bundle, which we denote by F^p_s.

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Let $TD \to \pi^* TY$ be the derivative of π. Set

$$T^V D = \mathrm{Ker}(TD \to \pi^* TY) \subseteq TD \subseteq TX'|_D. \tag{2.81}$$

Set

$$I_s^p = \big\{\alpha \in \Lambda^p(T^*X')|_D : \alpha(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in T^V D, v_{s+2}, \ldots, v_p \in TX'|_D\big\}. \tag{2.82}$$

We obtain a filtration of holomorphic vector bundles over D,

$$\Lambda^p(T^*X')|_D = I_p^p \supseteq I_{p-1}^p \supseteq \cdots \supseteq I_0^p. \tag{2.83}$$

Let N_D be the normal line bundle of $D \hookrightarrow X'$. From the calculation in local coordinates, we see that

$$\mathscr{G}_s^p = i_* \mathscr{O}_D\big(N_D^{-s} \otimes (I_p^p / I_s^p)\big) \quad \text{for } s = 0, \ldots, p-1. \tag{2.84}$$

For convenience, we denote

$$G_s^p = N_D^{-s} \otimes (I_p^p / I_s^p). \tag{2.85}$$

Then we obtain a short exact sequence

$$0 \to \mathscr{O}_{X'}(F_{s+1}^p) \to \mathscr{O}_{X'}(F_s^p) \to i_* \mathscr{O}_D(G_s^p) \to 0. \tag{2.86}$$

Step 1. We show that

$$\begin{aligned} H^q(D, G_0^p) &= \bigoplus_{k=1}^{r-1} H^{k,k}(\mathbb{CP}^{r-1}) \otimes H^{p-k,q-k}(Y), \\ H^q(D, G_s^p) &= 0 \quad \text{for } s = 1, \ldots, p-1. \end{aligned} \tag{2.87}$$

Set

$$J_s^p = \big\{\alpha \in \Lambda^p(T^*D) : \alpha(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in T^V D, v_{s+2}, \ldots, v_p \in TD\big\}. \tag{2.88}$$

Let $\phi : \Lambda^p(T^*X')|_D \to \Lambda^p(T^*D)$ be the canonical projection. By (2.82) and (2.88), we have

$$J_s^p = \phi(I_s^p) \subseteq \Lambda^p(T^*D). \tag{2.89}$$

By (2.83) and (2.89), we have a filtration of holomorphic vector bundles over D,

$$\Lambda^p(T^*D) = J_p^p \supseteq J_{p-1}^p \supseteq \cdots \supseteq J_0^p. \tag{2.90}$$

We also have

$$J_k^p / J_{k-1}^p = \pi^*\big(\Lambda^{p-k}(T^*Y)\big) \otimes \Lambda^k(T^{V,*}D), \tag{2.91}$$

and a short exact sequence of holomorphic vector bundles over D,

$$0 \to N_D^{-1} \otimes J_k^{p-1} \to I_k^p \to J_k^p \to 0. \tag{2.92}$$

Combining (2.91) and (2.92), we obtain a short exact sequence,

$$\begin{aligned} 0 \to N_D^{-1} \otimes \pi^*\big(\Lambda^{p-k-1}(T^*Y)\big) \otimes \Lambda^k(T^{V,*}D) \to I_k^p / I_{k-1}^p \\ \to \pi^*\big(\Lambda^{p-k}(T^*Y)\big) \otimes \Lambda^k(T^{V,*}D) \to 0. \end{aligned} \tag{2.93}$$

By (2.85) and (2.93), G_s^p admits a filtration with factors

$$\big(N_D^{-s-\epsilon} \otimes \pi^*\big(\Lambda^{p-k-\epsilon}(T^*Y)\big) \otimes \Lambda^k(T^{V,*}D)\big)_{\epsilon=0,1,k=s+1,\ldots,p}. \tag{2.94}$$

We remark that $\pi : D \to Y$ is a $\mathbb{CP}^{r-1}$-bundle and the restriction of N_D^{-1} to the fiber of $\pi : D \to Y$ is a holomorphic line bundle of degree one. Applying spectral sequence while using

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Lemma 2.5, we see that the cohomology of the holomorphic vector bundles in (2.94) vanishes unless $\epsilon = s = 0$. Hence, we obtain the second identity in (2.87). This argument also shows that

$$H^q(D, G_0^p) = H^q(D, I_p^p/I_0^p) = H^q(D, J_p^p/J_0^p). \tag{2.95}$$

Using spectral sequence and (2.91), we obtain

$$H^q(D, J_k^p/J_{k-1}^p) = H^{k,k}(\mathbb{C}\mathrm{P}^{r-1}) \otimes H^{p-k,q-k}(Y). \tag{2.96}$$

On the other hand, it is classical that

$$H^q(D, J_p^p) = H^q\big(D, \Lambda^p(T^*D)\big) = \bigoplus_{k=0}^{r-1} H^{k,k}(\mathbb{C}\mathrm{P}^{r-1}) \otimes H^{p-k,q-k}(Y). \tag{2.97}$$

From (2.95)–(2.97), we obtain the first identity in (2.87).

Set

$$\lambda(G_0^\bullet) = \bigotimes_{p=1}^{n} \big(\det H^\bullet(D, G_0^p)\big)^{(-1)^p p}, \quad \lambda_{\mathrm{tot}}(G_0^\bullet) = \lambda(G_0^\bullet) \otimes \overline{\lambda(G_0^\bullet)}. \tag{2.98}$$

Recall that $\lambda_{\mathrm{tot}}(X)$ was defined in (1.119).

Step 2. We construct two canonical sections of

$$\big(\lambda_{\mathrm{tot}}(X)\big)^{-1} \otimes \lambda_{\mathrm{tot}}(X') \otimes \big(\lambda_{\mathrm{tot}}(G_0^\bullet)\big)^{-1} \tag{2.99}$$

and show that they coincide up to ± 1.

Let

$$\mu_{p,s} \in \big(\det H^\bullet(X', F_{s+1}^p)\big)^{-1} \otimes \det H^\bullet(X', F_s^p) \otimes \big(\det H^\bullet(D, G_s^p)\big)^{-1} \tag{2.100}$$

be the canonical section induced by the long exact sequence induced by (2.86). Indeed, by (2.87), we have

$$\mu_{p,s} \in \big(\det H^\bullet(X', F_{s+1}^p)\big)^{-1} \otimes \det H^\bullet(X', F_s^p) \quad \text{for } s \neq 0. \tag{2.101}$$

Set

$$\begin{aligned}\mu_p = \bigotimes_{s=0}^{p-1} \mu_{p,s} &\in \big(\det H^\bullet(X', F_p^p)\big)^{-1} \otimes \det H^\bullet(X', F_0^p) \otimes \big(\det H^\bullet(D, G_0^p)\big)^{-1} \\ &= \big(\det H^\bullet(X', f^*\Lambda^p(T^*X))\big)^{-1} \otimes \det H^{p,\bullet}(X') \otimes \big(\det H^\bullet(D, G_0^p)\big)^{-1}.\end{aligned} \tag{2.102}$$

We remark that $f_*\mathscr{O}_{X'} = \mathscr{O}_X$ and $R^{>0}f_*\mathscr{O}_{X'} = 0$. Using spectral sequence, we obtain a canonical identification

$$H^{p,\bullet}(X) = H^\bullet\big(X', f^*\Lambda^p(T^*X)\big). \tag{2.103}$$

Let

$$\nu_p \in \big(\det H^{p,\bullet}(X)\big)^{-1} \otimes \det H^\bullet\big(X', f^*\Lambda^p(T^*X)\big) \tag{2.104}$$

be the canonical section induced by (2.103).

By (2.102) and (2.104), we have

$$\mu_p \otimes \nu_p \in \big(\det H^{p,\bullet}(X)\big)^{-1} \otimes \det H^{p,\bullet}(X') \otimes \big(\det H^\bullet(D, G_0^p)\big)^{-1}. \tag{2.105}$$

By (1.119), (2.98) and (2.105), we have

$$\bigotimes_{p=1}^{n} (\mu_p \otimes \nu_p)^{(-1)^p p} \in \big(\lambda(X)\big)^{-1} \otimes \lambda(X') \otimes \big(\lambda(G_0^\bullet)\big)^{-1}, \tag{2.106}$$

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and

$$\bigotimes_{p=1}^{n}(\mu_p\otimes\nu_p)^{(-1)^p p}\otimes\overline{\bigotimes_{p=1}^{n}(\mu_p\otimes\nu_p)^{(-1)^p p}}\in\big(\lambda_{\mathrm{tot}}(X)\big)^{-1}\otimes\lambda_{\mathrm{tot}}(X')\otimes\big(\lambda_{\mathrm{tot}}(G_0^\bullet)\big)^{-1}. \tag{2.107}$$

We have the Hodge decomposition

$$H^j_{\mathrm{dR}}(Y)=\bigoplus_{p+q=j}H^{p,q}(Y). \tag{2.108}$$

Let b_k be the kth Betti number of Y. By (2.87), (2.98) and (2.108), we have

$$\lambda_{\mathrm{tot}}(G_0^\bullet)=\bigotimes_{k=1}^{r-1}\bigotimes_{j=2k}^{2k+2n-2r}\Big(\big(\det H^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{r-1})\big)^{b_{j-2k}}\otimes\det H^{j-2k}_{\mathrm{dR}}(Y)\Big)^{(-1)^j j}. \tag{2.109}$$

Let

$$\delta_j\in H^j_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^{r-1},\mathbb{Z})\subseteq H^j_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^{r-1},\mathbb{C})=H^j_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{r-1}) \tag{2.110}$$

be a generator of $H^j_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^{r-1},\mathbb{Z})$. Let

$$\tau_{j,1},\dots,\tau_{j,b_j}\in\mathrm{Im}\big(H^j_{\mathrm{Sing}}(Y,\mathbb{Z})\to H^j_{\mathrm{Sing}}(Y,\mathbb{R})\big)\subseteq H^j_{\mathrm{dR}}(Y) \tag{2.111}$$

be a basis of the lattice. We denote $\tau_j=\tau_{j,1}\wedge\cdots\wedge\tau_{j,b_j}\in\det H^j_{\mathrm{dR}}(Y)$. Set

$$\sigma_{G_0^\bullet}=\bigotimes_{k=1}^{r-1}\bigotimes_{j=2k}^{2k+2n-2r}(\delta_{2k}^{b_{j-2k}}\otimes\tau_{j-2k})^{(-1)^j j}\in\lambda_{\mathrm{tot}}(G_0^\bullet). \tag{2.112}$$

Let $\sigma_X\in\lambda_{\mathrm{tot}}(X)$ and $\sigma_{X'}\in\lambda_{\mathrm{tot}}(X')$ be as in (1.122). Obviously, we have

$$\sigma_X^{-1}\otimes\sigma_{X'}\otimes\sigma_{G_0^\bullet}^{-1}\in\big(\lambda_{\mathrm{tot}}(X)\big)^{-1}\otimes\lambda_{\mathrm{tot}}(X')\otimes\big(\lambda_{\mathrm{tot}}(G_0^\bullet)\big)^{-1}. \tag{2.113}$$

We have a canonical identification (cf. [Voi02, Théorème 7.31])

$$H^j_{\mathrm{Sing}}(X',\mathbb{Z})=H^j_{\mathrm{Sing}}(X,\mathbb{Z})\oplus\bigoplus_{k=1}^{r-1}H^{2k}_{\mathrm{Sing}}(\mathbb{C}\mathrm{P}^{r-1},\mathbb{Z})\otimes H^{j-2k}_{\mathrm{Sing}}(Y,\mathbb{Z}), \tag{2.114}$$

which induces an isomorphism of Hodge structures. Similarly to Step 2 in the proof of Theorem 2.3, using (2.114), we can show that

$$\bigotimes_{p=1}^{n}(\mu_p\otimes\nu_p)^{(-1)^p p}\otimes\overline{\bigotimes_{p=1}^{n}(\mu_p\otimes\nu_p)^{(-1)^p p}}=\pm\sigma_X^{-1}\otimes\sigma_{X'}\otimes\sigma_{G_0^\bullet}^{-1}. \tag{2.115}$$

Step 3. We introduce Quillen metrics.

Let g^{TX} be the metric on TX induced by ω. Let $g^{\Lambda^p(T^*X)}$ be the metric on $\Lambda^p(T^*X)$ induced by g^{TX}. Let

$$\|\cdot\|_{\det H^{p,\bullet}(X)} \tag{2.116}$$

be the Quillen metric on $\det H^{p,\bullet}(X)=\det H^\bullet(X,\Lambda^p(T^*X))$ associated with g^{TX} and $g^{\Lambda^p(T^*X)}$.

Let $g^{TX'}$ be the metric on TX' induced by ω'. Let $g^{\Lambda^p(T^*X')}$ be the metric on $\Lambda^p(T^*X')$ induced by $g^{TX'}$. Let

$$\|\cdot\|_{\det H^{p,\bullet}(X')} \tag{2.117}$$

be the Quillen metric on $\det H^{p,\bullet}(X')=\det H^\bullet(X',\Lambda^p(T^*X'))$ associated with $g^{TX'}$ and $g^{\Lambda^p(T^*X')}$.

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Let

$$\|\cdot\|_{\det H^{\bullet}(X', f^*\Lambda^p(T^*X))} \tag{2.118}$$

be the Quillen metric on $\det H^{\bullet}(X', f^*\Lambda^p(T^*X))$ associated with $g^{TX'}$ and $f^*g^{\Lambda^p(T^*X)}$.

Let g^{TD} and g^{N_D} be the metrics on TD and N_D induced by $g^{TX'}$. Let $g^{I^p_s}$ be the metric on I^p_s induced by $g^{\Lambda^p(T^*X')}$ via (2.83). Let $g^{G^p_s}$ be the metric on G^p_s induced by g^{N_D} and $g^{I^p_s}$ via (2.85). Let

$$\|\cdot\|_{\det H^{\bullet}(D, G^p_s)} \tag{2.119}$$

be the Quillen metric on $\det H^{\bullet}(D, G^p_s)$ associated with g^{TD} and $g^{G^p_s}$. By the second identity in (2.87), we have a canonical identification $\det H^{\bullet}(D, G^p_s) = \mathbb{C}$ for $s \neq 0$. However, the metric (2.119) with $s \neq 0$ is not necessarily the standard metric on $\mathbb{C}$.

We remark that

$$\begin{aligned} \Lambda^p(T^*X')|_{X'\backslash U'} &= F^p_s|_{X'\backslash U'} \\ &= f^*\Lambda^p(T^*X)|_{X'\backslash U'} \quad \text{for } s = 0, \dots, p. \end{aligned} \tag{2.120}$$

We equip F^p_s with Hermitian metric $g^{F^p_s}$ such that

$$\begin{gathered} g^{F^p_0} = g^{\Lambda^p(T^*X')}, \quad g^{F^p_p} = f^*g^{\Lambda^p(T^*X)}, \\ g^{F^p_{s+1}}|_{X'\backslash U'} = g^{F^p_s}|_{X'\backslash U'} \quad \text{for } s = 0, \dots, p-1. \end{gathered} \tag{2.121}$$

Our assumption (2.72) implies $g^{\Lambda^p(T^*X')}|_{X'\backslash U'} = f^*(g^{\Lambda^p(T^*X)}|_{X\backslash U})$, which guarantees the existence of $g^{F^p_s}$ satisfying (2.121). Let

$$\|\cdot\|_{\det H^{\bullet}(X', F^p_s)} \tag{2.122}$$

be the Quillen metric on $\det H^{\bullet}(X', F^p_s)$ associated with $g^{TX'}$ and $g^{F^p_s}$. We remark that $H^{\bullet}(X', F^p_0) = H^{p,\bullet}(X')$ and

$$\|\cdot\|_{\det H^{\bullet}(X', F^p_0)} = \|\cdot\|_{\det H^{p,\bullet}(X')}. \tag{2.123}$$

Recall that $\mu_{p,s}$ was defined in (2.100). Let $\|\mu_{p,s}\|$ be the norm of $\mu_{p,s}$ with respect to the metrics (2.119) and (2.122).

Recall that ν_p was defined in (2.104). Let $\|\nu_p\|$ be the norm of ν_p with respect to the Quillen metrics (2.116) and (2.118).

Recall that $\sigma_{G^{\bullet}_0}$ was defined in (2.112). By (2.98) and the second identity in (2.87), we can and do view $\sigma_{G^{\bullet}_0}$ as the section of

$$\lambda_{\mathrm{tot}}(G^{\bullet}_{\bullet}) := \bigotimes_{p=1}^{n} \bigotimes_{s=0}^{p-1} \big(\det H^{\bullet}(D, G^p_s)\big)^{(-1)^p p} \otimes \overline{\bigotimes_{p=1}^{n} \bigotimes_{s=0}^{p-1} \big(\det H^{\bullet}(D, G^p_s)\big)^{(-1)^p p}}. \tag{2.124}$$

Let $\|\sigma_{G^{\bullet}_0}\|_{\lambda_{\mathrm{tot}}(G^{\bullet}_{\bullet})}$ be the norm of $\sigma_{G^{\bullet}_0} \in \lambda_{\mathrm{tot}}(G^{\bullet}_{\bullet})$ with respect to the metrics (2.119).

Let $\|\sigma_X\|_{\lambda_{\mathrm{tot}}(X)}$ be the norm of σ_X with respect to the metrics (2.116). Let $\|\sigma_{X'}\|_{\lambda_{\mathrm{tot}}(X')}$ be the norm of $\sigma_{X'}$ with respect to the metrics (2.117). By (2.102) and (2.115), we have

$$\begin{aligned} &\log\|\sigma_{X'}\|_{\lambda_{\mathrm{tot}}(X')} - \log\|\sigma_X\|_{\lambda_{\mathrm{tot}}(X)} - \log\|\sigma_{G^{\bullet}_0}\|_{\lambda_{\mathrm{tot}}(G^{\bullet}_{\bullet})} \\ &\quad = \sum_{p=1}^{n} (-1)^p p \bigg(\log\|\nu_p\|^2 + \sum_{s=0}^{p-1} \log\|\mu_{p,s}\|^2 \bigg). \end{aligned} \tag{2.125}$$

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By Definition 1.24 and (2.125), we have

$$\begin{aligned}&\tau_{\mathrm{BCOV}}(X',\omega')-\tau_{\mathrm{BCOV}}(X,\omega)\\&=\log\|\sigma_{G_0^\bullet}\|_{\lambda_{\mathrm{tot}}(G_\bullet^\bullet)}+\sum_{p=1}^{n}(-1)^p p\bigg(\log\|\nu_p\|^2+\sum_{s=0}^{p-1}\log\|\mu_{p,s}\|^2\bigg).\end{aligned}\tag{2.126}$$

Step 4. We conclude.

For ease of notation, we denote

$$\alpha_{p,s}=\log\|\mu_{p,s}\|^2.\tag{2.127}$$

Applying Theorem 1.19 to the short exact sequence (2.86) while using the second line in (2.121), we see that $\alpha_{p,s}$ is determined by $(U',\omega'|_{U'},g^{F^p_s}|_{U'},g^{F^p_{s+1}}|_{U'})$. We denote

$$\alpha_p=\sum_{s=0}^{p-1}\alpha_{p,s}.\tag{2.128}$$

We remark that for $s=1,\dots,p-1$, the contributions of the metric $\|\cdot\|_{\det H^\bullet(X',F^p_s)}$ (see (2.122)) to $\alpha_{p,s-1}$ and $\alpha_{p,s}$ cancel with each other. Thus, α_p is independent of $(g^{F^p_s})_{s=1,\dots,p-1}$. Hence, α_p is determined by $(U',\omega'|_{U'},g^{F^p_0}|_{U'},g^{F^p_p}|_{U'})$. Now, applying the first line in (2.121), we see that α_p is determined by $(U,U',\omega|_U,\omega'|_{U'})$.

For ease of notation, we denote

$$\beta_p=\log\|\nu_p\|^2,\tag{2.129}$$

Applying Theorem 1.21 with $E=\Lambda^p(T^*X)$ while using (2.72), we see that β_p is determined by $(U,U',\omega|_U,\omega'|_{U'})$.

By (2.126)–(2.129), we have

$$\tau_{\mathrm{BCOV}}(X',\omega')-\tau_{\mathrm{BCOV}}(X,\omega)=\log\|\sigma_{G_0^\bullet}\|_{\lambda_{\mathrm{tot}}(G_\bullet^\bullet)}+\sum_{p=1}^{n}(-1)^p p(\alpha_p+\beta_p).\tag{2.130}$$

Here:

- the section $\sigma_{G_0^\bullet}\in\lambda_{\mathrm{tot}}(G_\bullet^\bullet)$ is determined by $D\subseteq U'$ and its normal bundle;
- the Quillen metric $\|\cdot\|_{\lambda_{\mathrm{tot}}(G_\bullet^\bullet)}$ is determined by $\omega'|_{U'}$;
- the real number α_p is determined by $(U,U',\omega|_U,\omega'|_{U'})$;
- the real number β_p is determined by $(U,U',\omega|_U,\omega'|_{U'})$.

In conclusion, the right-hand side of (2.130) is determined by $(U,U',\omega|_U,\omega'|_{U'})$. This completes the proof. □

Let $\pi:\mathscr{U}\to\mathbb{C}$ be a holomorphic submersion between complex manifolds. Let $\mathscr{Y}\subseteq\mathscr{U}$ be a closed complex submanifold. We assume that $\pi|_{\mathscr{Y}}:\mathscr{Y}\to\mathbb{C}$ is a holomorphic submersion with compact fiber. For $z\in\mathbb{C}$, we denote $U_z=\pi^{-1}(z)$ and $Y_z=U_z\cap\mathscr{Y}$. Assume that for any $z\in\mathbb{C}$, U_z can be extended to a compact Kähler manifold. More precisely, there exist a compact Kähler manifold X_z and a holomorphic embedding $i_z:U_z\hookrightarrow X_z$ whose image is open. Here $\{X_z:z\in\mathbb{C}\}$ is just a set of complex manifolds parameterized by $\mathbb{C}$. The topology of X_z may vary as z varies. We identify U_z with $i_z(U_z)\subseteq X_z$. Let $f_z:X'_z\to X_z$ be the blow-up along Y_z.

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Set $U'_z = f_z^{-1}(U_z) \subseteq X'_z$. Let

$$\big(\omega_z \in \Omega^{1,1}(X_z)\big)_{z\in\mathbb{C}}, \quad \big(\omega'_z \in \Omega^{1,1}(X'_z)\big)_{z\in\mathbb{C}} \tag{2.131}$$

be Kähler forms. We assume that $(\omega_z|_{U_z})_{z\in\mathbb{C}}$ and $(\omega'_z|_{U'_z})_{z\in\mathbb{C}}$ are smooth families. We further assume that

$$\omega'_z|_{X'_z\backslash U'_z} = f_z^*(\omega_z|_{X_z\backslash U_z}) \quad \text{for } z \in \mathbb{C}. \tag{2.132}$$

Theorem 2.7. *The function* $z \mapsto \tau_{\mathrm{BCOV}}(X'_z, \omega'_z) - \tau_{\mathrm{BCOV}}(X_z, \omega_z)$ *is continuous.*

Proof. We proceed in the same way as in the proof of Theorem 2.6. Each object constructed becomes a function of $z \in \mathbb{C}$. In particular, the identity (2.130) becomes

$$\tau_{\mathrm{BCOV}}(X'_z, \omega'_z) - \tau_{\mathrm{BCOV}}(X_z, \omega_z) = \log\|\sigma_{G_0^\bullet}\|_{\lambda_{\mathrm{tot}}(G^\bullet_\bullet),z} + \sum_{p=1}^{n}(-1)^p p(\alpha_{p,z} + \beta_{p,z}). \tag{2.133}$$

From Remarks 1.20 and 1.22 and the last paragraph in the proof of Theorem 2.6, we see that each term on the right-hand side of (2.133) is a continuous function of z. This completes the proof. □

3. BCOV invariant

3.1 Several meromorphic sections

Let X be a compact complex manifold. Let K_X be the canonical line bundle of X. Let d be a non-zero integer. Let K_X^d be the dth tensor power of K_X. We assume that there is an invertible element $\gamma \in \mathscr{M}(X, K_X^d)$. We denote

$$\mathrm{div}(\gamma) = D = \sum_{j=1}^{l} m_j D_j, \tag{3.1}$$

where $m_j \in \mathbb{Z}\backslash\{0\}$, $D_1, \ldots, D_l \subseteq X$ are mutually distinct and irreducible. We assume that D is of simple normal crossing support (see Definition 1.2).

For $J \subseteq \{1, \ldots, l\}$, let $D_J \subseteq X$ be as in (0.9). For $j \in J \subseteq \{1, \ldots, l\}$, let $L_{J,j}$ be the normal line bundle of $D_J \hookrightarrow D_{J\backslash\{j\}}$. Set

$$K_J = K_X^d|_{D_J} \otimes \bigotimes_{j\in J} L_{J,j}^{-m_j} = K_{D_J}^d \otimes \bigotimes_{j\in J} L_{J,j}^{-m_j-d}, \tag{3.2}$$

which is a holomorphic line bundle over D_J. In particular, we have $K_\emptyset = K_X^d$.

Recall that $\mathrm{Res}_\cdot(\cdot)$ was defined in Definition 1.4. By (1.9), there exist

$$\big(\gamma_J \in \mathscr{M}(D_J, K_J)\big)_{J\subseteq\{1,\ldots,l\}} \tag{3.3}$$

such that

$$\gamma_\emptyset = \gamma, \quad \gamma_J = \mathrm{Res}_{D_J}(\gamma_{J\backslash\{j\}}) \quad \text{for } j \in J \subseteq \{1, \ldots, l\}. \tag{3.4}$$

By (1.8), we have

$$\mathrm{div}(\gamma_J) = \sum_{j\notin J} m_j D_{J\cup\{j\}}. \tag{3.5}$$

3.2 Construction of BCOV invariant

We use the notation from § 3.1. We further assume that X is Kähler and $m_j \neq -d$ for $j = 1, \ldots, l$. Then (X, γ) is a d-Calabi–Yau pair (see Definition 0.2).

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Let ω be a Kähler form on X. Let $|\cdot|_{K_{D_J},\omega}$ be the metric on K_{D_J} induced by ω. Let $|\cdot|_{L_{J,j},\omega}$ be the metric on $L_{J,j}$ induced by ω. Let $|\cdot|_{K_J,\omega}$ be the metric on K_J induced by $|\cdot|_{K_{D_J},\omega}$ and $|\cdot|_{L_{J,j},\omega}$ via (3.2).

We use the notation from (1.23). For $J \subseteq \{1,\ldots,l\}$, let $|J|$ be the number of elements in J, let $g^{TD_J}_\omega$ be the metric on TD_J induced by ω, let $c_k(TD_J, g^{TD_J}_\omega) \in Q^{D_J}$ be kth Chern form of $(TD_J, g^{TD_J}_\omega)$. Let $n = \dim X$. Set

$$a_J(\gamma,\omega) = \frac{1}{12}\int_{D_J} c_{n-|J|}\big(TD_J, g^{TD_J}_\omega\big)\log|\gamma_J|^{2/d}_{K_J,\omega}. \tag{3.6}$$

We consider the short exact sequence of holomorphic vector bundles over D_J,

$$0 \to TD_J \to TD_{J\backslash\{j\}}|_{D_J} \to L_{J,j} \to 0. \tag{3.7}$$

Let

$$\tilde{c}\Big(TD_J, TD_{J\backslash\{j\}}|_{D_J}, g^{TD_{J\backslash\{j\}}}_\omega|_{D_J}\Big) \in Q^{D_J}/Q^{D_J,0} \tag{3.8}$$

be the Bott–Chern form (1.30) with $0 \to E' \to E \to E''$ replaced by (3.7) and g^E replaced by $g^{TD_{J\backslash\{j\}}}_\omega|_{D_J}$. Set

$$b_{J,j}(\omega) = \frac{1}{12}\int_{D_J}\tilde{c}\Big(TD_J, TD_{J\backslash\{j\}}|_{D_J}, g^{TD_{J\backslash\{j\}}}_\omega|_{D_J}\Big). \tag{3.9}$$

Let w^J_d be as in (0.9). Recall that $\tau_{\mathrm{BCOV}}(\cdot,\cdot)$ was defined in Definition 1.24. For ease of notation, we denote $\tau_{\mathrm{BCOV}}(D_J,\omega) = \tau_{\mathrm{BCOV}}(D_J,\omega|_{D_J})$. We define

$$\tau_d(X,\gamma,\omega) = \sum_{J\subseteq\{1,\ldots,l\}} w^J_d\bigg(\tau_{\mathrm{BCOV}}(D_J,\omega) - a_J(\gamma,\omega) - \sum_{j\in J}\frac{m_j+d}{d}b_{J,j}(\omega)\bigg). \tag{3.10}$$

Theorem 3.1. *The real number $\tau_d(X,\gamma,\omega)$ is independent of ω.*

Proof. Let $(\omega_s)_{s\in\mathbb{CP}^1}$ be a smooth family of Kähler forms on X parameterized by $\mathbb{CP}^1$. It is sufficient to show that $\tau_d(X,\gamma,\omega_s)$ is independent of s.

We view the terms involved in (3.10) as smooth functions on $\mathbb{CP}^1$, i.e.

$$\begin{aligned} &\tau_d(X,\gamma,\omega) : s \mapsto \tau_d(X,\gamma,\omega_s),\\ &\tau_{\mathrm{BCOV}}(D_J,\omega) : s \mapsto \tau_{\mathrm{BCOV}}(D_J,\omega_s), \quad \text{etc.}\end{aligned} \tag{3.11}$$

We view TD_J and $L_{J,j}$ as holomorphic vector bundles over $D_J\times\mathbb{CP}^1$. Let $g^{TD_J}_\omega$ and $g^{L_{J,j}}_\omega$ be metrics on TD_J and $L_{J,j}$ induced by $(\omega_s)_{s\in\mathbb{CP}^1}$. More precisely, the restrictions $g^{TD_J}_\omega|_{D_J\times\{s\}}$ and $g^{L_{J,j}}_\omega|_{D_J\times\{s\}}$ are induced by ω_s. By [Zha22, Theorem 1.6], we have

$$\frac{\bar\partial\partial}{2\pi i}\tau_{\mathrm{BCOV}}(D_J,\omega) = \frac{1}{12}\int_{D_J} c_{n-|J|}(TD_J, g^{TD_J}_\omega)c_1(TD_J, g^{TD_J}_\omega). \tag{3.12}$$

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Similarly to [Zha22, (2.9)], by the Poincaré–Lelong formula, (3.2), (3.5) and (3.6), we have

$$\begin{aligned}\frac{\bar\partial\partial}{2\pi i}a_J(\gamma,\omega) &= \frac{1}{12d}\int_{D_J} c_{n-|J|}(TD_J, g^{TD_J}_\omega)\big(-c_1(K_J,|\cdot|_{K_J,\omega})+\delta_{\mathrm{div}(\gamma_J)}\big)\\ &= \frac{1}{12}\int_{D_J} c_{n-|J|}(TD_J,g^{TD_J}_\omega)c_1(TD_J,g^{TD_J}_\omega)\\ &\quad+\sum_{j\in J}\frac{m_j+d}{12d}\int_{D_J} c_{n-|J|}(TD_J,g^{TD_J}_\omega)c_1(L_{J,j},|\cdot|_{L_{J,j},\omega})\\ &\quad+\sum_{j\notin J}\frac{m_j}{12d}\int_{D_{J\cup\{j\}}} c_{n-|J|}(TD_J,g^{TD_J}_\omega).\end{aligned} \tag{3.13}$$

Similarly to [Zha22, (2.10)], by (1.29), (1.30) and (3.9), we have

$$\begin{aligned}\frac{\bar\partial\partial}{2\pi i}b_{J,j}(\omega) &= \frac{1}{12}\int_{D_J} c_{n-|J|+1}\Big(TD_{J\backslash\{j\}}, g_\omega^{TD_{J\backslash\{j\}}}\Big)\\ &\quad-\frac{1}{12}\int_{D_J} c_{n-|J|}\big(TD_J,g^{TD_J}_\omega\big)c_1\big(L_{J,j},g^{L_{J,j}}_\omega\big).\end{aligned} \tag{3.14}$$

By (3.12)–(3.14), we have

$$\begin{aligned}&\frac{\bar\partial\partial}{2\pi i}\Big(\tau_{\mathrm{BCOV}}(D_J,\omega)-a_J(\gamma,\omega)-\sum_{k\in J}\frac{m_j+d}{d}b_{J,j}(\omega)\Big)\\ &= -\sum_{j\in J}\frac{m_j+d}{12d}\int_{D_J} c_{n-|J|+1}\Big(TD_{J\backslash\{j\}},g_\omega^{TD_{J\backslash\{j\}}}\Big)-\sum_{j\notin J}\frac{m_j}{12d}\int_{D_{J\cup\{j\}}} c_{n-|J|}\big(TD_J,g^{TD_J}_\omega\big).\end{aligned} \tag{3.15}$$

From (0.9), (3.10) and (3.15), we obtain $\bar\partial\partial\tau_d(X,\gamma,\omega)=0$. Hence, $s\mapsto\tau_d(X,\gamma,\omega_s)$ is constant on $\mathbb{CP}^1$. This completes the proof. □

DEFINITION 3.2. The BCOV invariant of (X,γ) is defined by

$$\tau_d(X,\gamma)=\tau_d(X,\gamma,\omega). \tag{3.16}$$

By Theorem 3.1, $\tau_d(X,\gamma)$ is well-defined.

PROPOSITION 3.3. *For a non-zero integer r, let $\gamma^r\in\mathscr{M}(X,K_X^{rd})$ be the rth tensor power of γ. Then (X,γ^r) is a rd-Calabi–Yau pair and*

$$\tau_{rd}(X,\gamma^r)=\tau_d(X,\gamma). \tag{3.17}$$

Proof. Once we replace γ by γ^r, each γ_J is replaced by γ_J^r. We can directly verify that

$$\tau_{rd}(X,\gamma^r,\omega)=\tau_d(X,\gamma,\omega). \tag{3.18}$$

From Definition 3.2 and (3.18), we obtain (3.17). This completes the proof. □

Recall that $\chi_d(\cdot,\cdot)$ was defined in Definition 1.3.

PROPOSITION 3.4. *For $z\in\mathbb{C}^*$, we have*

$$\tau_d(X,z\gamma)=\tau_d(X,\gamma)-\frac{\chi_d(X,D)}{12}\log|z|^{2/d}. \tag{3.19}$$

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Proof. Once we replace γ by $z\gamma$, each γ_J is replaced by $z\gamma_J$. By (3.6), we have

$$a_J(z\gamma,\omega) - a_J(\gamma,\omega) = \frac{\chi(D_J)}{12}\log|z|^{2/d}. \tag{3.20}$$

By Definition 1.3, (3.10) and (3.20), we have

$$\tau_d(X,z\gamma,\omega) - \tau_d(X,\gamma,\omega) = -\frac{\chi_d(X,D)}{12}\log|z|^{2/d}. \tag{3.21}$$

From Definition 3.2 and (3.21), we obtain (3.19). This completes the proof. □

Proof of Theorem 0.4. As $\pi : \mathscr{X} \to S$ is locally Kähler, for any $s_0 \in S$, there exist an open subset $s_0 \in U \subseteq S$ and a Kähler form ω on $\pi^{-1}(U)$. For $s \in U$, we denote $\omega_s = \omega|_{X_s}$. Similarly to the proof of Theorem 3.1, we view the terms involved in (3.10) as smooth functions on U.

Though the fibration $\pi^{-1}(U) \to U$ is not necessarily trivial, the identities (3.13) and (3.14) still hold. On the other hand, by [Zha22, Theorem 1.6], we have

$$\frac{\bar\partial\partial}{2\pi i}\tau_{\mathrm{BCOV}}(D_J,\omega) = \omega_{H^\bullet(D_J)} + \frac{1}{12}\int_{D_J} c_{n-|J|}\big(TD_J, g_\omega^{TD_J}\big)c_1\big(TD_J, g_\omega^{TD_J}\big). \tag{3.22}$$

By (0.9), (3.10), (3.13), (3.14) and (3.22), we have

$$\frac{\bar\partial\partial}{2\pi i}\tau_d(X,\gamma,\omega)\Big|_U = \sum_{J\subseteq\{1,\dots,l\}} w_d^J\,\omega_{H^\bullet(D_J)}. \tag{3.23}$$

From Definition 3.2 and (3.23), we obtain (0.15). This completes the proof. □

3.3 BCOV invariant of projective bundle

Let Y be a compact Kähler manifold. Let N be a holomorphic vector bundle of rank $r \geqslant 2$ over Y. Let $\mathbb{1}$ be the trivial line bundle over Y. Set

$$X = \mathbb{P}(N \oplus \mathbb{1}). \tag{3.24}$$

Let $\pi : X \to Y$ be the canonical projection.

Let $q \in \{0,\dots,r\}$. Let $(L_k)_{k=1,\dots,q}$ be holomorphic line bundles over Y. We assume that there is a surjection between holomorphic vector bundles

$$N \to L_1 \oplus \cdots \oplus L_q. \tag{3.25}$$

Let N^* be the dual of N. Taking the dual of (3.25), we obtain

$$L_1^{-1} \oplus \cdots \oplus L_q^{-1} \hookrightarrow N^*. \tag{3.26}$$

Let $d, m_1, \dots, m_q$ be positive integers. Let

$$\gamma_Y \in \mathscr{M}\big(Y, (K_Y \otimes \det N^*)^d \otimes L_1^{-m_1} \otimes \cdots \otimes L_q^{-m_q}\big) \tag{3.27}$$

be an invertible element. We assume that

- $\mathrm{div}(\gamma_Y)$ is of simple normal crossing support;
- $\mathrm{div}(\gamma_Y)$ does not possess component of multiplicity $-d$.

Denote $m = m_1 + \cdots + m_q$. Let $S^m N^*$ be the mth symmetric tensor power of N^*. By (3.26) and (3.27), we have

$$\gamma_Y \in \mathscr{M}\big(Y, (K_Y \otimes \det N^*)^d \otimes S^m N^*\big). \tag{3.28}$$

Let $\mathcal{N}$ be the total space of N. We have

$$X = \mathcal{N} \cup \mathbb{P}(N), \quad K_X|_{\mathcal{N}} = \pi^*(K_Y \otimes \det N^*). \tag{3.29}$$

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We may view a section of $S^m N^*$ as a function on $\mathcal{N}$. By (3.28) and (3.29), γ_Y may be viewed as an element of $\mathscr{M}(\mathcal{N}, K_X^d)$. Let

$$\gamma_X \in \mathscr{M}(X, K_X^d) \tag{3.30}$$

be such that $\gamma_X|_{\mathcal{N}} = \gamma_Y$.

For $j = 1, \ldots, q$, let $N \to L_j$ be the composition of the map (3.25) and the canonical projection $L_1 \oplus \cdots \oplus L_q \to L_j$. Set

$$N_j = \mathrm{Ker}(N \to L_j), \quad X_j = \mathbb{P}(N_j \oplus \mathbb{K}) \subseteq X, \quad X_\infty = \mathbb{P}(N) \subseteq X. \tag{3.31}$$

We denote

$$\mathrm{div}(\gamma_Y) = \sum_{j=q+1}^{l} m_j Y_j, \tag{3.32}$$

where $Y_j \subseteq Y$ are mutually distinct and irreducible. For $j = q+1, \ldots, l$, set

$$X_j = \pi^{-1}(Y_j) \subseteq X. \tag{3.33}$$

Denote

$$m_\infty = -m_1 - \cdots - m_q - rd - d. \tag{3.34}$$

Note that:

- X is locally the product of an open subset of Y and $\mathbb{C}\mathrm{P}^r$;
- γ_X is locally the product of a d-canonical section on an open subset of Y and $\gamma_{r,m_1,\ldots,m_q}$ defined in (0.20);

we have

$$\mathrm{div}(\gamma_X) = \pi^* \mathrm{div}(\gamma_Y) + m_\infty X_\infty + \sum_{j=1}^{q} m_j X_j = m_\infty X_\infty + \sum_{j=1}^{l} m_j X_j, \tag{3.35}$$

which is of simple normal crossing support. Hence, (X, γ_X) is a d-Calabi–Yau pair.

For $y \in Y$, we denote $Z_y = \pi^{-1}(y)$. Let $K_{Y,y}$ be the fiber of K_Y at $y \in Y$. We have

$$K_X|_{Z_y} = K_{Z_y} \otimes \pi^* K_{Y,y}. \tag{3.36}$$

For $y \in Y \backslash \bigcup_{j=q+1}^{l} Y_j$, there exist $\gamma_{Z_y} \in \mathscr{M}(Z_y, K_{Z_y}^d)$ and $\eta_y \in K_{Y,y}^d$ such that

$$\gamma_X|_{Z_y} = \gamma_{Z_y} \otimes \pi^* \eta_y. \tag{3.37}$$

Then (Z_y, γ_{Z_y}) is a d-Calabi–Yau pair, which is independent of y up to isomorphism. We may omit the index y as long as there is no confusion. We remark that (Z, γ_Z) is isomorphic to $(\mathbb{C}\mathrm{P}^r, \gamma_{r,m_1,\ldots,m_q})$ constructed in the paragraph containing (0.20).

Recall that $\chi_d(\cdot,\cdot)$ was defined in Definition 1.3.

Lemma 3.5. *The following identity holds:*

$$\chi_d(Z, \gamma_Z) = 0. \tag{3.38}$$

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Proof. Set

$$f(t) = t^{r-q} \prod_{j\in\{1,\ldots,q,\infty\}} \Big(t - \frac{m_j}{m_j+d} \Big). \tag{3.39}$$

For $J \subseteq \{1,\ldots,q,\infty\}$, let w_d^J be as in (0.9). By (1.3), (1.4) and the fact that $\chi(\mathbb{C}\mathrm{P}^k) = k+1$, we have

$$\chi_d(Z,\gamma_Z) = \sum_{J\subseteq\{1,\ldots,q,\infty\}} w_d^J (r+1-|J|) = f'(1). \tag{3.40}$$

On the other hand, we have

$$\begin{aligned} \frac{f'(1)}{f(1)} &= r - q + \sum_{j\in\{1,\ldots,q,\infty\}} \Big(1 - \frac{m_j}{m_j+d} \Big)^{-1} \\ &= \frac{m_1+\cdots+m_q+m_\infty}{d} + r + 1. \end{aligned} \tag{3.41}$$

From (3.34), (3.40) and (3.41), we obtain (3.38). This completes the proof. □

Theorem 3.6. *The following identity holds:*

$$\tau_d(X,\gamma_X) = \chi_d(Y,\gamma_Y)\tau_d(Z,\gamma_Z). \tag{3.42}$$

Proof. The proof consists of several steps.

Step 0. We introduce several pieces of notation.

We denote $A = \{q+1,\ldots,l\}$ and $B = \{1,\ldots,q,\infty\}$. For $I \subseteq A$ and $J \subseteq B$, set

$$\begin{gathered} Y_I = Y \cap \bigcap_{j\in I} Y_j, \quad X_{I,J} = X \cap \bigcap_{j\in I\cup J} X_j, \\ X_I = X_{I,\emptyset}, \quad X_J = X_{\emptyset,J}. \end{gathered} \tag{3.43}$$

For $y \in Y$ and $J \subseteq B$, set

$$Z_{J,y} = Z_y \cap X_J. \tag{3.44}$$

Note that $Z_{J,y}$ is independent of y up to isomorphism, we may omit the index y as long as there is no confusion. We remark that $\pi|_{X_{I,J}} : X_{I,J} \to Y_I$ is a fibration with fiber Z_J.

Let ω_X be a Kähler form on X such that Lemma 2.2 holds. Let ω_Y be a Kähler form on Y. For $\varepsilon > 0$, set

$$\omega_\varepsilon = \omega_X + \frac{1}{\varepsilon}\pi^*\omega_Y. \tag{3.45}$$

For $I \subseteq A$, $J \subseteq B$ and $j \in (A\cup B)\backslash(I\cup J)$, let $a_{I,J}(\gamma_X,\omega_\varepsilon)$ and $b_{I,J,j}(\omega_\varepsilon)$ be as in (3.6) and (3.9) with (X,γ,ω) replaced by $(X,\gamma_X,\omega_\varepsilon)$ and J replaced by $I\cup J$. Let w_d^I be as in (0.9) with J replaced by I. By Definition 3.2, (0.9) and (3.10), we have

$$\begin{aligned} \tau_d(X,\gamma_X) &= \sum_{I\subseteq A}\sum_{J\subseteq B} w_d^I w_d^J \tau_{\mathrm{BCOV}}(X_{I,J},\omega_\varepsilon) \\ &\quad - \sum_{I\subseteq A}\sum_{J\subseteq B} w_d^I w_d^J a_{I,J}(\gamma_X,\omega_\varepsilon) \\ &\quad - \sum_{I\subseteq A}\sum_{J\subseteq B}\sum_{j\in I\cup J} w_d^I w_d^J \frac{m_j+d}{d} b_{I,J,j}(\omega_\varepsilon). \end{aligned} \tag{3.46}$$

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Step 1. We estimate $\tau_{\mathrm{BCOV}}(X_{I,J},\omega_\varepsilon)$.

For $y \in Y$, we denote $\omega_{Z_y} = \omega_X|_{Z_y}$. As ω_X satisfies Lemma 2.2, for any $J \subseteq B$, $(Z_{J,y}, \omega_{Z_y}|_{Z_{J,y}})_{y\in Y}$ are mutually isometric. We may omit the index y as long as there is no confusion. For ease of notation, we denote

$$\tau_{\mathrm{BCOV}}(Y_I,\omega_Y) = \tau_{\mathrm{BCOV}}(Y_I,\omega_Y|_{Y_I}), \quad \tau_{\mathrm{BCOV}}(Z_J,\omega_Z) = \tau_{\mathrm{BCOV}}(Z_J,\omega_Z|_{Z_J}). \tag{3.47}$$

For $I \subseteq A$ and $J \subseteq B$, by Theorem 2.3, as $\varepsilon \to 0$,

$$\begin{aligned}\tau_{\mathrm{BCOV}}(X_{I,J},\omega_\varepsilon) - \frac{\chi(Z_J)}{12}\big(\dim(Y_I)\chi(Y_I) + c_1 c_{\dim(Y_I)-1}(Y_I)\big)\log\varepsilon \\ \to \chi(Z_J)\tau_{\mathrm{BCOV}}(Y_I,\omega_Y) + \chi(Y_I)\tau_{\mathrm{BCOV}}(Z_J,\omega_Z).\end{aligned} \tag{3.48}$$

On the other hand, by Lemma 3.5, (1.3) and (1.4), we have

$$\sum_{I\subseteq A} w_d^I \chi(Y_I) = \chi_d(Y,\gamma_Y), \quad \sum_{J\subseteq B} w_d^J \chi(Z_J) = 0. \tag{3.49}$$

By (3.48) and (3.49), as $\varepsilon \to 0$,

$$\sum_{I\subseteq A}\sum_{J\subseteq B} w_d^I w_d^J \tau_{\mathrm{BCOV}}(X_{I,J},\omega_\varepsilon) \to \chi_d(Y,\gamma_Y)\sum_{J\subseteq B} w_d^J \tau_{\mathrm{BCOV}}(Z_J,\omega_Z). \tag{3.50}$$

Step 2. We estimate $a_{I,J}(\gamma_X,\omega_\varepsilon)$.

For $I \subseteq A$ and $J \subseteq B$, let $K_{I,J}$ be as in (3.2) with (X,γ) replaced by (X,γ_X) and J replaced by $I \cup J$. Then $K_{I,J}$ is a holomorphic line bundle over $X_{I,J}$. Let

$$\gamma_{I,J} \in \mathscr{M}(X_{I,J}, K_{I,J}) \tag{3.51}$$

be as in (3.4) with (X,γ) replaced by (X,γ_X) and J replaced by $I \cup J$.

Let $U \subseteq Y$ be a small open subset. Set $\mathcal{U} = \pi^{-1}(U)$. Recall that $\gamma_Z \in \mathscr{M}(Z, K_Z^d)$ was constructed in the paragraph containing (3.36). We fix an identification $\mathcal{U} = U \times Z$ such that there exists $\eta \in \mathscr{M}(U, K_Y^d)$ satisfying

$$\gamma_X|_{\mathcal{U}} = \mathrm{pr}_1^*\eta \otimes \mathrm{pr}_2^*\gamma_Z, \tag{3.52}$$

where $\mathrm{pr}_1 : U \times Z \to U$ and $\mathrm{pr}_2 : U \times Z \to Z$ are canonical projections.

For $I \subseteq A$, let K_I be as in (3.2) with (X,γ) replaced by (U,η). Then K_I is a holomorphic line bundle over $U \cap Y_I$. Let

$$\eta_I \in \mathscr{M}(U \cap Y_I, K_I) \tag{3.53}$$

be as in (3.4) with (X,γ) replaced by (U,η). For $J \subseteq B$, let K_J be as in (3.2) with (X,γ) replaced by (Z,γ_Z). Then K_J is a holomorphic line bundle over Z_J. Let

$$\gamma_J \in \mathscr{M}(Z_J, K_J) \tag{3.54}$$

be as in (3.4) with (X,γ) replaced by (Z,γ_Z). By the constructions of $K_{I,J}$ and $\gamma_{I,J}$ in the paragraph containing (3.51), we have

$$K_{I,J}|_{\mathcal{U}\cap X_{I,J}} = \mathrm{pr}_1^* K_I \otimes \mathrm{pr}_2^* K_J, \quad \gamma_{I,J}|_{\mathcal{U}\cap X_{I,J}} = \mathrm{pr}_1^*\eta_I \otimes \mathrm{pr}_2^*\gamma_J. \tag{3.55}$$

For $I \subseteq A$ and $J \subseteq B$, let $g_\varepsilon^{TX_{I,J}}$ (respectively, g^{TY_I}, g^{TZ_J}) be the metric on $TX_{I,J}$ (respectively, TY_I, TZ_J) induced by ω_ε (respectively, ω_Y, ω_Z), let $|\cdot|_{K_{I,J},\varepsilon}$ (respectively, $|\cdot|_{K_I}$, $|\cdot|_{K_J}$) be the norm on $K_{I,J}$ (respectively, K_I, K_J) induced by ω_ε (respectively, ω_Y, ω_Z) in the same

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way as in the paragraph above (3.6). We denote

$$a_{I,J}(\mathcal{U},\gamma_X,\omega_\varepsilon)=\frac{1}{12}\int_{\mathcal{U}\cap X_{I,J}}c(TX,g_\varepsilon^{TX})\log|\gamma_{I,J}|^{2/d}_{K_{I,J},\varepsilon}. \tag{3.56}$$

Recall that ω_ε was defined in (3.45). As $g_\varepsilon^{TX_{I,J}}$ is induced by ω_ε, by Proposition 1.7, as $\varepsilon\to 0$.

$$c\big(TX_{I,J},g_\varepsilon^{TX_{I,J}}\big)\to c\big(TZ_J,g^{TZ_J}\big)\pi^*c\big(TY_I,g^{TY_I}\big). \tag{3.57}$$

Recall that η_I, γ_J and $\gamma_{I,J}$ are linked by (3.55). As $|\cdot|_{K_{I,J},\varepsilon}$ is induced by ω_ε, as $\varepsilon\to 0$,

$$\log|\gamma_{I,J}|^2_{K_{I,J},\varepsilon}-\Big(\dim(Y)d+\sum_{j\in I}m_j\Big)\log\varepsilon\to\log|\gamma_J|^2_{K_J}+\log|\eta_I|^2_{K_I}. \tag{3.58}$$

Let $a_J(\gamma_Z,\omega_Z)$ be as in (3.6) with (X,γ,ω) replaced by (Z,γ_Z,ω_Z). More precisely,

$$a_J(\gamma_Z,\omega_Z)=\frac{1}{12}\int_{Z_J}c\big(TZ_J,g^{TZ_J}\big)\log|\gamma_Z|^{2/d}_{K_J}. \tag{3.59}$$

By (3.56)–(3.59), as $\varepsilon\to 0$,

$$\begin{aligned}&a_{I,J}(\mathcal{U},\gamma_X,\omega_\varepsilon)-\frac{\chi(Z_J)}{12}\Big(\dim(Y)+\frac{1}{d}\sum_{j\in I}m_j\Big)\log\varepsilon\int_{U\cap Y_I}c(TY_I,g^{TY_I})\\ &\quad\to\frac{\chi(Z_J)}{12}\int_{U\cap Y_I}c(TY_I,g^{TY_I})\log|\eta_I|^{2/d}_{K_I}+a_J(\gamma_Z,\omega_Z)\int_{U\cap Y_I}c(TY_I,g^{TY_I}).\end{aligned} \tag{3.60}$$

By (3.49) and (3.60), as $\varepsilon\to 0$,

$$\sum_{I\subseteq A}\sum_{J\subseteq B}w_d^Iw_d^Ja_{I,J}(\mathcal{U},\gamma_X,\omega_\varepsilon)\to\sum_{J\subseteq B}w_d^Ja_J(\gamma_Z,\omega_Z)\sum_{I\subseteq A}w_d^I\int_{U\cap Y_I}c(TY_I,g^{TY_I}). \tag{3.61}$$

The left-hand side of (3.61) yields a measure on X,

$$\mu_\varepsilon:\mathcal{U}\mapsto\sum_{I\subseteq A}\sum_{J\subseteq B}w_d^Iw_d^Ja_{I,J}(\mathcal{U},\gamma_X,\omega_\varepsilon), \tag{3.62}$$

The right-hand side of (3.61) yields a measure on Y,

$$\nu:U\mapsto\sum_{J\subseteq B}w_d^Ja_J(\gamma_Z,\omega_Z)\sum_{I\subseteq A}w_d^I\int_{U\cap Y_I}c(TY_I,g^{TY_I}). \tag{3.63}$$

The convergence in (3.61) is equivalent to the following: as $\varepsilon\to 0$,

$$\pi_*\mu_\varepsilon\to\nu. \tag{3.64}$$

By (3.49) and (3.62)–(3.64), as $\varepsilon\to 0$,

$$\sum_{I\subseteq A}\sum_{J\subseteq B}w_d^Iw_d^Ja_{I,J}(\gamma_X,\omega_\varepsilon)=\mu_\varepsilon(X)\to\nu(Y)=\chi_d(Y,\gamma_Y)\sum_{J\subseteq B}w_d^Ja_J(\gamma_Z,\omega_Z). \tag{3.65}$$

Step 3. We estimate $b_{I,J,j}(\omega_\varepsilon)$.

First we consider the case $j\in I$. We denote $I'=I\backslash\{j\}$. By (3.9), we have

$$b_{I,J,j}(\omega_\varepsilon)=\frac{1}{12}\int_{X_{I,J}}\tilde{c}\Big(TX_{I,J},TX_{I',J}|_{X_{I,J}},g_\varepsilon^{TX_{I',J}}|_{X_{I,J}}\Big). \tag{3.66}$$

By Proposition 1.9, as $\varepsilon\to 0$,

$$\tilde{c}\Big(TX_{I,J},TX_{I',J}|_{X_{I,J}},g_\varepsilon^{TX_{I',J}}|_{X_{I,J}}\Big)\to c\big(TZ_J,g^{TZ_J}\big)\pi^*\tilde{c}\Big(TY_I,TY_{I'}|_{Y_I},g^{TY_{I'}}|_{Y_I}\Big). \tag{3.67}$$

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By (3.66) and (3.67), as $\varepsilon \to 0$,

$$b_{I,J,j}(\omega_\varepsilon) \to \frac{\chi(Z_J)}{12} \int_{Y_I} \tilde{c}\Big(TY_I, TY_{I'}|_{Y_I}, g^{TY_{I'}}|_{Y_I}\Big). \tag{3.68}$$

By (3.49) and (3.68), as $\varepsilon \to 0$,

$$\sum_{I\subseteq A} \sum_{J\subseteq B} \sum_{j\in I} w_d^I w_d^J \frac{m_j + d}{d} b_{I,J,j}(\omega_\varepsilon) \to 0. \tag{3.69}$$

Now we consider the case $j \in J$. We denote $J' = J\backslash\{j\}$. By (3.9), we have

$$b_{I,J,j}(\omega_\varepsilon) = \frac{1}{12} \int_{X_{I,J}} \tilde{c}\Big(TX_{I,J}, TX_{I,J'}|_{X_{I,J}}, g_\varepsilon^{TX_{I,J'}}|_{X_{I,J}}\Big). \tag{3.70}$$

By Proposition 1.9, as $\varepsilon \to 0$,

$$\tilde{c}\Big(TX_{I,J}, TX_{I,J'}|_{X_{I,J}}, g_\varepsilon^{TX_{I,J'}}|_{X_{I,J}}\Big) \to \tilde{c}\Big(TZ_J, TZ_{J'}|_{Z_J}, g^{TZ_{J'}}|_{Z_J}\Big)\pi^* c(TY_I, g^{TY_I}). \tag{3.71}$$

Let $b_{J,j}(\omega_Z)$ be as in (3.9) with (X,γ,ω) replaced by (Z,γ_Z,ω_Z). More precisely,

$$b_{J,j}(\omega_Z) = \frac{1}{12} \int_{Z_J} \tilde{c}\Big(TZ_J, TZ_{J'}|_{Z_J}, g^{TZ_{J'}}|_{Z_J}\Big). \tag{3.72}$$

By (3.70)–(3.72), as $\varepsilon \to 0$,

$$b_{I,J,j}(\omega_\varepsilon) \to \chi(Y_I) b_{J,j}(\omega_Z). \tag{3.73}$$

By (3.49) and (3.73), as $\varepsilon \to 0$,

$$\sum_{I\subseteq A} \sum_{J\subseteq B} \sum_{j\in J} w_d^I w_d^J \frac{m_j + d}{d} b_{I,J,j}(\omega_\varepsilon) \to \chi_d(Y,\gamma_Y) \sum_{J\subseteq B} \sum_{j\in J} w_d^J \frac{m_j + d}{d} b_{J,j}(\omega_Z). \tag{3.74}$$

Step 4. We conclude.

Taking $\varepsilon \to 0$ on the right-hand side of (3.46) and applying (3.50), (3.65), (3.69) and (3.74), we obtain

$$\tau_d(X,\gamma_X) = \chi_d(Y,\gamma_Y) \sum_{J\subseteq B} w_d^J \bigg(\tau_{\mathrm{BCOV}}(Z_J,\omega_Z) - a_J(\gamma_Z,\omega_Z) - \sum_{j\in J} \frac{m_j + d}{d} b_{J,j}(\omega_Z)\bigg). \tag{3.75}$$

On the other hand, by Definition 3.2 and (3.10), we have

$$\tau(Z,\gamma_Z) = \sum_{J\subseteq B} w_d^J \bigg(\tau_{\mathrm{BCOV}}(Z_J,\omega_Z) - a_J(\gamma_Z,\omega_Z) - \sum_{j\in J} \frac{m_j + d}{d} b_{J,j}(\omega_Z)\bigg). \tag{3.76}$$

From (3.75) and (3.76), we obtain (3.42). This completes the proof. □

3.4 Proof of Theorem 0.5

Now we are ready to prove Theorem 0.5.

Proof of Theorem 0.5. The proof consists of several steps.

Step 1. Following [BFM75, § 1.5], we introduce a deformation to the normal cone.

Let $\mathscr{X} \to X \times \mathbb{C}$ be the blow-up along $Y \times \{0\}$. Let $\Pi : \mathscr{X} \to \mathbb{C}$ be the composition of the canonical projections $\mathscr{X} \to X \times \mathbb{C}$ and $X \times \mathbb{C} \to \mathbb{C}$. For $z \in \mathbb{C}^*$, we denote

$$X_z = \Pi^{-1}(z). \tag{3.77}$$

Let $\mathbb{1}$ be the trivial line bundle over Y. Recall that N_Y is the normal bundle of $Y \hookrightarrow X$. Recall that X' is the blow-up of X along Y. The variety $\Pi^{-1}(0)$ consists of two irreducible

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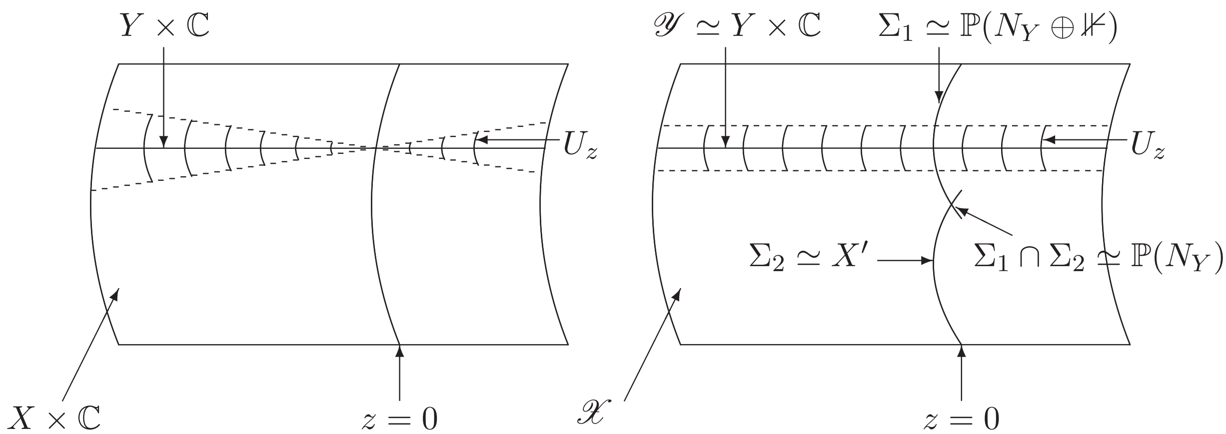

FIGURE 1. Deformation to the normal cone.

components: $\Pi^{-1}(0) = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 \simeq \mathbb{P}(N_Y \oplus \mathbb{1})$ and $\Sigma_2 \simeq X'$. We denote

$$X_0 = \Sigma_1. \tag{3.78}$$

For $j = 1, \ldots, l$, let $\mathscr{D}_j \subseteq \mathscr{X}$ be the closure of $D_j \times \mathbb{C}^* \subseteq \mathscr{X}$. For $z \in \mathbb{C}$, we denote

$$D_{j,z} = \mathscr{D}_j \cap X_z. \tag{3.79}$$

Let $\mathscr{Y} \subseteq \mathscr{X}$ be the closure of $Y \times \mathbb{C}^* \subseteq \mathscr{X}$. For $z \in \mathbb{C}$, we denote

$$Y_z = \mathscr{Y} \cap X_z. \tag{3.80}$$

See Figure 1.

Let g^{TX} be a Hermitian metric on TX. Let $d(\cdot,\cdot) : X \times X \to \mathbb{R}$ be the geodesic distance associated with g^{TX}. For $x \in X$, we denote

$$d_Y(x) = \inf_{y \in Y} d(x, y). \tag{3.81}$$

For $z \in \mathbb{C}^*$, set

$$U_z = \{x \in X : d_Y(x) < |z|\} \times \{z\} \subseteq X_z. \tag{3.82}$$

We identify the fiber of $\mathbb{1}$ with $\mathbb{C}$. For $v \in N_Y$ and $s \in \mathbb{C}$ such that $(v, s) \neq (0, 0)$, we denote by $[v : s]$ the image of (v, s) in $\mathbb{P}(N_Y \oplus \mathbb{1})$. Let $|\cdot|$ be the norm on N_Y induced by g^{TX}. Set

$$U_0 = \{[v : s] \in \mathbb{P}(N_Y \oplus \mathbb{1}) : |v| < |s|\} \subseteq X_0. \tag{3.83}$$

For $\varepsilon > 0$ small enough, we have smooth families

$$(U_z)_{|z|<\varepsilon}, \quad (Y_z)_{|z|<\varepsilon}, \quad (U_z \cap D_{j,z})_{|z|<\varepsilon} \quad \text{with } j = 1, \ldots, l. \tag{3.84}$$

We remark that $Y_z \subseteq U_z$ for $z \in \mathbb{C}$.

Let $\mathscr{F} : \mathscr{X}' \to \mathscr{X}$ be the blow-up along $\mathscr{Y}$. For $z \in \mathbb{C}$, we denote

$$X'_z = \mathscr{F}^{-1}(X_z). \tag{3.85}$$

Set

$$f_z = \mathscr{F}|_{X'_z} : X'_z \to X_z, \tag{3.86}$$

which is the blow-up along Y_z. For $z \in \mathbb{C}$, set

$$D'_{0,z} = f_z^{-1}(Y_z) \subseteq X'_z. \tag{3.87}$$

For $z \in \mathbb{C}$ and $j = 1, \ldots, l$, let $D'_{j,z} \subseteq X'_z$ be the strict transformation of $D_{j,z} \subseteq X_z$.

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For $z \in \mathbb{C}$, set

$$U'_z = f_z^{-1}(U_z). \tag{3.88}$$

For $\varepsilon > 0$ small enough, we have smooth families

$$(U'_z)_{|z|<\varepsilon}, \quad (U'_z \cap D'_{j,z})_{|z|<\varepsilon} \quad \text{with } j = 0, \dots, l. \tag{3.89}$$

We remark that $D'_{0,z} \subseteq U'_z$ for $z \in \mathbb{C}$.

Step 2. We introduce a family of meromorphic pluricanonical sections.

Denote

$$m = m_1 + \cdots + m_q, \tag{3.90}$$

which is the vanishing order of γ on Y. Recall that r is the codimension of $Y \hookrightarrow X$. Recall that $\gamma \in \mathscr{M}(X, K_X^d)$. For $z \neq 0$, we identify X_z with X in the obvious way. For $z \neq 0$, set

$$\gamma_z = z^{-m-rd}\gamma \in \mathscr{M}(X_z, K_{X_z}^d). \tag{3.91}$$

There is a unique $\gamma_0 \in \mathscr{M}(X_0, K_{X_0}^d)$ such that for $\varepsilon > 0$ small enough,

$$(\gamma_z|_{U_z})_{|z|<\varepsilon} \tag{3.92}$$

is a smooth family. Now we briefly explain the existence of γ_0. We take a holomorphic local chart

$$\varphi : \mathbb{C}^n \supseteq V \to X \tag{3.93}$$

such that:

- $0 \in V$ and $\varphi(0) \in Y$;
- $\varphi^{-1}(Y) = \{(z_1, \dots, z_n) \in V : z_1 = \cdots = z_r = 0\}$;
- $\varphi^*\gamma = \theta(z_1, \dots, z_n) z_1^{m_1} \cdots z_q^{m_q} (dz_1 \wedge \cdots \wedge dz_n)^d$, where θ is a holomorphic function on V such that $\theta(0, \dots, 0, z_{r+1}, \dots, z_n) \neq 0$ for generic $z_{r+1}, \dots, z_n$.

For $z \neq 0$, let $\varphi_z : V \to X_z$ be the composition of $\varphi : V \to X$ and the identification $X = X_z$. We take a holomorphic local chart

$$\phi : \mathbb{C}^n \times \{z \in \mathbb{C} : |z| < \varepsilon\} \supseteq W \to \mathscr{X} \tag{3.94}$$

such that for $0 < |z| < \varepsilon$:

- $\phi(z_1, \dots, z_n, z) \in \varphi_z(V) \subseteq X_z$;
- $\varphi_z^{-1}(\phi(z_1, \dots, z_n, z)) = (zz_1, \dots, zz_r, z_{r+1}, \dots, z_n)$.

Then a direct calculation yields

$$\begin{aligned} z^{-m-rd}\phi^*\gamma &= \theta(zz_1, \dots, zz_r, z_{r+1}, \dots, z_n) z_1^{m_1} \cdots z_q^{m_q} (dz_1 \wedge \cdots \wedge dz_n)^d \\ &\to \theta(0, \dots, 0, z_{r+1}, \dots, z_n) z_1^{m_1} \cdots z_q^{m_q} (dz_1 \wedge \cdots \wedge dz_n)^d \end{aligned} \tag{3.95}$$

as $z \to 0$. Moreover, the calculation above shows that the hypothesis in § 3.3 holds with (X, γ_X) replaced by (X_0, γ_0). In particular, (X_0, γ_0) is a d-Calabi–Yau pair.

Step 3. We introduce a family of Kähler forms.

Let $\mathscr{U} \subseteq \mathscr{X}$ be such that $\mathscr{U} \cap X_z = U_z$ for any $z \in \mathbb{C}$. Then $\mathscr{U}$ is an open subset of $\mathscr{X}$. Set $\mathscr{U}' = \mathscr{F}^{-1}(\mathscr{U}) \subseteq \mathscr{X}'$. We have $\mathscr{U}' \cap X'_z = U'_z$ for any $z \in \mathbb{C}$.

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Let ω be a Kähler form on $\mathscr{X}$. Let ω' be a Kähler form on $\mathscr{X}'$ such that

$$\omega'|_{\mathscr{X}'\backslash\mathscr{U}'} = \mathscr{F}^*(\omega|_{\mathscr{X}\backslash\mathscr{U}}). \tag{3.96}$$

For $z \in \mathbb{C}$, set

$$\omega_z = \omega|_{X_z}, \quad \omega'_z = \omega'|_{X'_z}. \tag{3.97}$$

By (3.86), (3.96) and (3.97), we have

$$\omega'_z|_{X'_z\backslash U'_z} = f_z^*(\omega_z|_{X_z\backslash U_z}) \quad \text{for } z \in \mathbb{C}. \tag{3.98}$$

For $\varepsilon > 0$ small enough, we have smooth families

$$(\omega_z|_{U_z})_{|z|<\varepsilon}, \quad (\omega'_z|_{U'_z})_{|z|<\varepsilon}. \tag{3.99}$$

Step 4. We show that the function $z \mapsto \tau_d(X'_z, f_z^*\gamma_z) - \tau_d(X_z, \gamma_z)$ is continuous at $z = 0$.

Denote

$$m_0 = m_1 + \cdots + m_q + (r-1)d. \tag{3.100}$$

For $z \in \mathbb{C}$, by (3.79), (3.86), (3.87) and (3.92), we have

$$\mathrm{div}(\gamma_z) = \sum_{j=1}^{l} m_j D_{j,z}, \quad \mathrm{div}(f_z^*\gamma_z) = \sum_{j=0}^{l} m_j D'_{j,z}. \tag{3.101}$$

Here $D_{j,0}$ and $D'_{j,0}$ may be empty for certain j. Let $(D_{J,z})_{J\subseteq\{1,\ldots,l\}}$ be as in (0.9) with X replaced by X_z and D_j replaced by $D_{j,z}$. Let $(D'_{J,z})_{J\subseteq\{0,\ldots,l\}}$ be as in (0.9) with X replaced by X'_z and D_j replaced by $D'_{j,z}$. By Definition 3.2 and (3.10), we have

$$\begin{aligned}
&\tau_d(X'_z, f_z^*\gamma_z) - \tau_d(X_z, \gamma_z) \\
&\quad = \sum_{0\in J\subseteq\{0,\ldots,l\}} w_d^J \Big(\tau_{\mathrm{BCOV}}(D'_{J,z}, \omega'_z) - a_J(f_z^*\gamma_z, \omega'_z) - \sum_{j\in J} \frac{m_j + d}{d} b_{J,j}(\omega'_z) \Big) \\
&\qquad - \sum_{J\subseteq\{1,\ldots,l\}} w_d^J \big(a_J(f_z^*\gamma_z, \omega'_z) - a_J(\gamma_z, \omega_z) \big) \\
&\qquad - \sum_{J\subseteq\{1,\ldots,l\}} \sum_{j\in J} w_d^J \frac{m_j + d}{d} \big(b_{J,j}(\omega'_z) - b_{J,j}(\omega_z) \big) \\
&\qquad + \sum_{J\subseteq\{1,\ldots,l\}} w_d^J \big(\tau_{\mathrm{BCOV}}(D'_{J,z}, \omega'_z) - \tau_{\mathrm{BCOV}}(D_{J,z}, \omega_z) \big).
\end{aligned} \tag{3.102}$$

For $0 \in J \subseteq \{0,\ldots,l\}$, we have $D'_{J,z} \subseteq U'_z$. Thus,

$$(D'_{J,z})_{z\in\mathbb{C}} \tag{3.103}$$

is a smooth family. Hence, the first summation in (3.102) is continuous at $z = 0$.

For $J \subseteq \{1,\ldots,l\}$, we denote

$$D_{J,z} = D^{\mathrm{in}}_{J,z} \sqcup D^{\mathrm{ex}}_{J,z} \tag{3.104}$$

such that each irreducible component of $D^{\mathrm{in}}_{J,z}$ (respectively, $D^{\mathrm{ex}}_{J,z}$) lies in (respectively, does not lie in) Y_z. As $D^{\mathrm{in}}_{J,z} \subseteq Y_z \subseteq U_z$, the family

$$(D^{\mathrm{in}}_{J,z})_{z\in\mathbb{C}} \tag{3.105}$$

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is smooth. On the other hand, we have

$$D^{\mathrm{ex}}_{J,z} = f_z(D'_{J,z}). \tag{3.106}$$

Moreover, the map $f_z|_{D'_{J,z}} : D'_{J,z} \to D^{\mathrm{ex}}_{J,z}$ is the blow-up along $D^{\mathrm{ex}}_{J,z} \cap Y_z$.

Recall that

$$K_J, \quad \gamma_J, \quad g^{TD_J}_{\omega}, \quad |\cdot|_{K_J,\omega} \tag{3.107}$$

were constructed in §§ 3.1 and 3.2 for a d-Calabi–Yau pair (X,γ) together with a Kähler form ω on X. Let

$$K_{J,z}, \quad \gamma_{J,z}, \quad g^{TD_{J,z}}_{\omega_z}, \quad |\cdot|_{K_{J,z},\omega_z} \tag{3.108}$$

be as in (3.107) with (X,γ) replaced by (X_z,γ_z) and ω replaced by ω_z. Let

$$K'_{J,z}, \quad \gamma'_{J,z}, \quad g^{TD'_{J,z}}_{\omega'_z}, \quad |\cdot|_{K'_{J,z},\omega'_z} \tag{3.109}$$

be as in (3.107) with (X,γ) replaced by $(X'_z, f^*_z\gamma_z)$ and ω replaced by ω'_z. By (3.6), (3.98), (3.104) and (3.106), for $J \subseteq \{1,\dots,l\}$, we have

$$\begin{aligned} a_J(f^*_z\gamma_z,\omega'_z) - a_J(\gamma_z,\omega_z) = {} & \frac{1}{12}\int_{D'_{J,z}\cap U'_z} c_{n-|J|}\Big(TD'_{J,z}, g^{TD'_{J,z}}_{\omega'_z}\Big) \log|\gamma'_{J,z}|^{2/d}_{K'_{J,z},\omega'_z} \\ & - \frac{1}{12}\int_{D^{\mathrm{ex}}_{J,z}\cap U_z} c_{n-|J|}\Big(TD_{J,z}, g^{TD_{J,z}}_{\omega_z}\Big) \log|\gamma_{J,z}|^{2/d}_{K_{J,z},\omega_z} \\ & - \frac{1}{12}\int_{D^{\mathrm{in}}_{J,z}} c_{n-|J|}\Big(TD_{J,z}, g^{TD_{J,z}}_{\omega_z}\Big) \log|\gamma_{J,z}|^{2/d}_{K_{J,z},\omega_z}. \end{aligned} \tag{3.110}$$

By (3.89), each integration in (3.110) depends continuously on z. Thus, the second summation in (3.102) is continuous at $z=0$. The same argument shows that the third summation in (3.102) is continuous at $z=0$.

By (3.104), we have the obvious identity

$$\begin{aligned} & \tau_{\mathrm{BCOV}}(D'_{J,z},\omega'_z) - \tau_{\mathrm{BCOV}}(D_{J,z},\omega_z) \\ & \quad = \tau_{\mathrm{BCOV}}(D'_{J,z},\omega'_z) - \tau_{\mathrm{BCOV}}(D^{\mathrm{ex}}_{J,z},\omega_z) - \tau_{\mathrm{BCOV}}(D^{\mathrm{in}}_{J,z},\omega_z). \end{aligned} \tag{3.111}$$

As the families in (3.99) are smooth, by Theorem 2.7 and (3.98), the function $z \mapsto \tau_{\mathrm{BCOV}}(D'_{J,z},\omega'_z) - \tau_{\mathrm{BCOV}}(D^{\mathrm{ex}}_{J,z},\omega_z)$ is continuous at $z=0$. As the families in (3.99) and (3.105) are smooth, the function $z \mapsto \tau_{\mathrm{BCOV}}(D^{\mathrm{in}}_{J,z},\omega_z)$ is continuous at $z=0$. Hence, the fourth summation in (3.102) is continuous at $z=0$.

Step 5. We conclude.

By Step 4, we have

$$\lim_{z\to 0}\big(\tau(X'_z, f^*_z\gamma_z) - \tau(X_z,\gamma_z)\big) = \tau(X'_0, f^*_0\gamma_0) - \tau(X_0,\gamma_0). \tag{3.112}$$

On the other hand, by Proposition 3.4 and (3.91), for $z \neq 0$, we have

$$\begin{aligned} \tau_d(X_z,\gamma_z) &= \tau_d(X,\gamma) - \frac{\chi_d(X,\gamma)}{12}\log|z|^{-2(m+rd)/d}, \\ \tau_d(X'_z, f^*_z\gamma_z) &= \tau(X', f^*\gamma) - \frac{\chi_d(X',f^*\gamma)}{12}\log|z|^{-2(m+rd)/d}. \end{aligned} \tag{3.113}$$

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Note that $(m + rd)/d > 0$, by (3.112) and (3.113), we have

$$\begin{aligned} \chi_d(X', f^*\gamma) - \chi_d(X, \gamma) &= 0, \\ \tau_d(X', f^*\gamma) - \tau_d(X, \gamma) &= \tau_d(X'_0, f_0^*\gamma_0) - \tau_d(X_0, \gamma_0). \end{aligned} \tag{3.114}$$

Note that X_0 is a $\mathbb{C}\mathrm{P}^r$-bundle over $Y_0 \simeq Y$, by Theorem 3.6, we have

$$\tau_d(X_0, \gamma_0) = \chi_d(Y, D_Y)\tau_d\big(\mathbb{C}\mathrm{P}^r, \gamma_{r,m_1,\ldots,m_q}\big). \tag{3.115}$$

Recall that $E = f^{-1}(Y)$. Note that X'_0 is a $\mathbb{C}\mathrm{P}^1$-bundle over $D'_{0,0} \simeq E$, by Theorem 3.6, we have

$$\tau_d(X'_0, f_0^*\gamma_0) = \chi_d(E, D_E)\tau_d\big(\mathbb{C}\mathrm{P}^1, \gamma_{1,m_0}\big). \tag{3.116}$$

From (3.114)–(3.116), we obtain (0.22). This completes the proof. □

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