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VARIATIONAL FORMULATION OF HIGHER ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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Let A be an elliptic (partial) differential operator of order $2m$ on a compact manifold $\overline{M}$ with boundary Γ. Let B be a normal system of m differential boundary operators on Γ. Assume all manifolds and coefficients are arbitrarily smooth. We construct sesquilinear forms J in terms of which there are equivalent variational formulations of the natural boundary value problems determined by A and B with solutions in Sobolev spaces $H^s(M)$, $0 < s < 2m$. Such forms are also constructed for problems with mixed boundary conditions. The variational formulation permits localization of *a priori* estimates and the interchange of existence and uniqueness questions between the boundary value problem and an associated adjoint problem.

1. Introduction

Let A be a linear elliptic differential operator of order $2m$ with smooth (complex) coefficients on a smooth n-dimensional compact Riemannian manifold $\overline{M}$ with boundary Γ. Set $M = \overline{M} \sim \Gamma$ and denote by $H^s(M)$ and $H^s(\Gamma)$ the corresponding Sobolev spaces of order s.

Associated with A are bounded sesquilinear forms

$$J : H^m(M) \times H^m(M) \to \mathbb{C}$$

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whose associated operators A_J agree with A on $C_0^\infty(M)$ the space of smooth functions with compact support in M. The operator

$$A_J : H^m(M) \to \overset{\circ}{H}{}^{-m}(M)$$

is defined by

$$\langle A_J u, v\rangle = J[u, v]$$

where $\langle\ ,\ \rangle$ denotes the conjugate linear pairing between $H^m(M)$ and its (conjugate) dual $\overset{\circ}{H}{}^{-m}(M)$ extending the L^2 inner product of smooth functions.

For each $m \le j \le 2m-1$ let C_j be a normal linear differential boundary operator of order $2m - j - 1$ with smooth coefficients. Associated with J and the C_j are unique normal boundary operators B_j of order j with smooth coefficients such that for u, v smooth

$$J[u, v] = (Au, v) - \sum_{j=m}^{2m-1} \langle B_j u, C_j v\rangle \tag{1.1}$$

where $(\ ,\ )$ and $\langle\ ,\ \rangle$ denote the inner products in $L^2(M)$ and $L^2(\Gamma)$ respectively.

From the B_j and C_j one can construct various natural boundary value problems as follows. Firstly, if X is an index set, let $E_X = \{E_j : j \in X\}$ when this has meaning. Then for each partition $P' \cup Q'$ of $\{m, m+1, \ldots, 2m-1\}$ consider the problem

(1.2) given $f \in L^2(M)$ and $t_{P'} \in \prod_{j\in P'} H^{m-j-\frac{1}{2}}(\Gamma)$ find $u \in H^m(M)$ such that $Au = f$ on M and $B_{P'}u = t_{P'}$, $C_{Q'}u = 0$ on Γ.

By (1.1) an equivalent variational formulation is

(1.3) given $f \in L^2(M)$ and $t_{P'} \in \prod_{j\in P'} H^{m-j-\frac{1}{2}}(\Gamma)$ find $u \in V$ such that for all $v \in V$, $J[u, v] = (f, v) - \langle t_{P'}, C_{P'}v\rangle$

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where $\langle\ ,\ \rangle$ denotes the pairing on $\prod_{j\in P'} H^{m-j-\frac{1}{2}}(\Gamma) \times H^{-m+j+\frac{1}{2}}(\Gamma)$ and

$V = \left\{ v \in H^m(M) : C_{Q'}v = 0 \right\}$.

The boundary conditions $C_{Q'}u = 0$ are often called forced or stable, whereas the conditions $B_{P'}u = t_{P'}$ are called natural. Since the operators C_j, $j \in Q'$, have orders $2m-j-1 \leq m-1$, inhomogeneous boundary conditions $C_{Q'}u = t_{Q'}$ can be readily replaced by homogeneous ones.

This variational approach was largely developed by Lions. See for example [4]. In [1] Agmon obtained a condition on principal symbols which characterized coercivity for the problem (1.2) when A is strongly elliptic. Grubb [3] obtained a global condition on the operators, when A is properly elliptic and $B_{P'}$ and $C_{Q'}$ are normal boundary operators, which characterized those problems (1.2) which are associated with a sesquilinear form J as in (1.1). In his book [2], Aubin presents an account of approximation methods for solutions of problems with variational formulation (1.3).

In this paper we consider the problem of determining which normal boundary value problems for an elliptic operator A have a variational formulation via an associated sesquilinear form J. We widen slightly the concept of variational formulations (1.3). Following McIntosh [6] and [7] we consider forms J defined on spaces $V \times W$ where W is not necessarily the same as V. This permits the study of a wider class of boundary conditions than the forced and natural conditions of (1.2). In fact we show that all normal boundary value problems for an elliptic operator A have equivalent variational formulations.

Let s be an integer, $1 \leq s \leq 2m-1$, and $P \cup Q$ a partition of $\{0, 1, \ldots, 2m-1\}$, each of P and Q having m elements. If $X = P$ or Q let $X' = X \cap \{s, s+1, \ldots, 2m-1\}$ and $X'' = X \sim X'$. Let B_P be a normal system of differential boundary operators B_j of order j with smooth coefficients. Then we consider the following problem

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(1.4) given $f \in L^2(M)$ and $t_{P'} \in \prod_{j \in P'} H^{s-j-\frac{1}{2}}(\Gamma)$ find $u \in H^s(M)$ such that $Au = f$ on M and $B_{P'}u = t_{P'}$, $B_{P''}u = 0$ on Γ.

Extend B_P to a normal system $\{B_j : 0 \le j \le 2m-1\}$, B_j of order j with smooth coefficients. Without additional conditions, we construct a boundary system $\{C_j : 0 \le j \le 2m-1\}$ of normal operators C_j of order $2m - j - 1$ with smooth coefficients, and a bounded form

$$J : H^s(M) \times H^{2m-s}(M) \to \mathbb{C}$$

such that, for smooth u, v,

$$J[u, v] = (Au, v) - \sum_{j=m}^{2m-1} \langle B_j u, C_j v \rangle . \tag{1.5}$$

Let $V = \left\{u \in H^s(M) : B_{P''}u = 0\right\}$ and $W = \left\{v \in H^{2m-s}(M) : C_{Q'}v = 0\right\}$. Problem (1.4) has the equivalent variational formulation

(1.6) given $f \in L^2(M)$ and $t_{P'} \in \prod_{j \in P'} H^{s-j-\frac{1}{2}}(\Gamma)$ find $u \in V$ such that for all $v \in W$, $J[u, v] = (f, v) - \langle t_{P'}, C_{P'}v \rangle$.

An important reason for having variational formulations of boundary value problems, especially in the general case where coercivity fails, is that extimates can be localized. Consider for example the following estimate for problem (1.4):

(1.7) $\|u\|_s \le c\left(\|Au\|_0 + \|B_P u\| + \|u\|_{s-1}\right)$ for all $u \in H^s(M)$ such that $Au \in L^2(M)$.

The norms are the appropriate Sobolev space norms. If A is properly elliptic, B_P normal and covering A, then (1.7) is satisfied. To prove this (see Lions and Magenes [5]) a similar estimate is proved for $s \ge 2m$, and then for $s \le 0$ by duality, and finally for $0 < s < 2m$ by interpolation.

The reason for the indirect approach when $0 < s < 2m$ is the difficulty of localizing the estimate. Freezing the coefficients of A

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for example is awkward because the space of functions on which (1.7) is to be proved depends on these coefficients. However, with the variational formulation this difficulty is removed and the estimate can be proved directly. Moreover, less smoothness is required of the coefficients of A - smoothness of order $m + |s-m|$ is sufficient, instead of order $2m$ as required by the interpolation argument. This will be the content of a forthcoming paper in which we solve the problem [5, 2.11.2] posed by Lions and Magenes of finding a proof of (1.7) that avoids interpolation.

Further, using the variational formulation, the question of existence for the boundary value problem can be transferred to the usually simpler question of uniqueness for an adjoint problem. This technique is well-known. See for example Schechter [11] or Lions and Magenes [5].

The results of this paper will also be used in our work on higher order elliptic equations with mixed boundary conditions. For, suppose $\{B_j : 0 \le j \le 2m-1\}$ is a system of normal differential boundary operators, B_j of order j with smooth coefficients. Then we can construct the adjoint system $\{C_j : 0 \le j \le 2m-1\}$ and the form J satisfying (1.5). Suppose Γ_1 and Γ_2 are disjoint open subsets of Γ with common smooth boundary ω satisfying $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \omega$. Take P and Q as before. For $1 \le j \le 2$ let r_j denote restriction to Γ_j. For $1 \le s \le 2m-1$, consider the mixed boundary value problem (1.8) given $f \in L^2(M)$,
$t_{P'} \in \prod_{j \in P'} H^{s-j-\frac{1}{2}}(\Gamma_1)$ and

$t_{Q'} \in \prod_{j \in Q'} H^{s-j-\frac{1}{2}}(\Gamma_2)$, find $u \in H^2(M)$ such that $Au = f$ on M, $r_1 B_{P'} u = t_{P'}$ and $r_1 B_{P''} u = 0$ on Γ_1, $r_2 B_{Q'} u = t_{Q'}$ and $r_2 B_{Q''} u = 0$ on Γ_2.

Let $V = \left\{u \in H^s(M) : r_1 B_{P'} u = 0,\ r_2 B_{Q'} u = 0\right\}$ and
$W = \left\{v \in H^{2m-s}(M) : r_2 C_{P'} v = 0,\ r_1 C_{Q'} v = 0\right\}$. Then (1.8) has the equivalent variational formulation

(1.9) given $f \in L^2(M)$, $t_{P'} \in \prod_{j \in P'} H^{s-j-\frac{1}{2}}(\Gamma_1)$ and

$t_{Q'} \in \prod_{j \in Q'} H^{s-j-\frac{1}{2}}(\Gamma_2)$, find $u \in V$ such that for all $v \in W$,

$J[u, v] = (f, v) - \langle t_{P'}, C_{P'}v \rangle - \langle t_{Q'}, C_{Q'}v \rangle$.

The last two brackets denote respectively the pairings on

$\prod_{j \in P'} H^{s-j-\frac{1}{2}}(\Gamma_1) \times \overset{\circ}{H}{}^{-s+j+\frac{1}{2}}(\Gamma_1)$ and $\prod_{j \in Q'} H^{s-j-\frac{1}{2}}(\Gamma_2) \times \overset{\circ}{H}{}^{-s+j+\frac{1}{2}}(\Gamma_2)$.

2. Existence of associated forms

We make the following assumption on the manifold M :

(2.1) M is an open subset of an n-dimensional compact smooth Riemannian manifold M' without boundary in which M has $(n-1)$-dimensional smooth boundary Γ and closure $\overline{M}$.

Let $\rho(p)$ denote the signed geodesic distance from Γ of point p in M' near Γ . We take $\rho > 0$ in M and $\rho < 0$ in $M' \sim \overline{M}$. For some $\varepsilon > 0$ there is a neighbourhood $U_\varepsilon = \{p \in M' : -\varepsilon < \rho(p) < \varepsilon\}$ of Γ with a finite covering by coordinate patches U_ε^j with coordinates of the form $x = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1})$ is a coordinate system in Γ and $x_n = \rho$.

Define a vector field on U_ε by

$$n(p) = \left.\frac{\partial}{\partial\rho}\right|_p \in T_pM' .$$

For $p \in \Gamma$, $n(p)$ is the unit inward normal to Γ . Define the unit inward conormal at $p \in \Gamma$ to be the unique covector $\eta(p) \in T_p^*M'$ which annihilates $T_p\Gamma$ and is unity on $n(p)$.

The trace operators γ_j for $j \geq 0$ are defined as usual by $\gamma_0 u = u|\Gamma$ and $\gamma_j u = \gamma_0 D_n^j u$ where $D_n = (1/i)(\partial/\partial\rho)$.

We assume that

(2.2) $A = A(p, D)$ is a linear elliptic partial differential operator of order $2m$ with smooth coefficients on M'

and

(2.3) $B_j = B_j(p, D)$ for each $0 \le j \le 2m-1$ is a normal linear partial differential boundary operator of order j with smooth coefficients on Γ .

If $\sigma(B_j, p, \xi)$ denotes the principal symbol of B_j at $(p, \xi) \in T^*(M')$, then B_j normal means that $\sigma(B_j, p, \eta(p))$ is non-zero for each $p \in \Gamma$.

Let $A' = A'(p, D)$ be the formal adjoint of A . That is, A' is the unique elliptic operator satisfying

(2.4) $$(Au, v) = (u, A'v) \quad \text{for all } u, v \in C_0^\infty(M) .$$

According to Green's theorem (see for example Lions and Magenes [5, Theorem 2.2.1]) there is a uniquely determined adjoint system of differential boundary operators $C_j = C_j(p, D)$ of orders $2m - j - 1$, $0 \le j \le 2m-1$, such that, for smooth u, v ,

(2.5) $$(Au, v) - (u, A'v) = \sum_{j=0}^{2m-1} \langle B_j u, C_j v \rangle .$$

Furthermore, we prove the following result.

THEOREM 2.6. *Let A, B_j and M satisfy conditions* (2.1), (2.2) *and* (2.3), *and let s be an integer, $1 \le s \le 2m-1$. There exists a bounded sesquilinear form J on $H^s(M) \times H^{2m-s}(M)$ such that, for smooth u, v ,*

$$J[u, v] = (Au, v) - \sum_{j=s}^{2m-1} \langle B_j u, C_j v \rangle .$$

Before proving the theorem we require the following

LEMMA 2.7. *Under the conditions of Theorem* 2.6 *there exists a bounded sesquilinear form J on $H^s(M) \times H^{2m-s}(M)$ and normal differential boundary operators F_j of order j and G_j of order $2m - j - 1$ for $0 \le j \le 2m-1$, with smooth coefficients, such that, for smooth u, v ,*

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$$(2.8) \qquad J[u, v] = (Au, v) - \sum_{j=s}^{2m-1} \langle F_j u, G_j v\rangle$$
$$= (u, A'v) + \sum_{j=0}^{s-1} \langle F_j u, G_j v\rangle .$$

Proof. Using a partition of unity, the problem is reduced to constructing J, F_j, G_j on a coordinate patch $\mathcal{U} = \mathcal{U}_\varepsilon^i$ near the boundary and showing that (2.8) holds for $u, v \in C_0^\infty(\mathcal{U} \cap \overline{M})$.

On $\mathcal{U}$ we can write A in the form $A = \sum D^p a_{pq} D^q$, where $D_j = (1/i)(\partial/\partial x_j)$ and the summation is over multi-indices p, q such that $|p| \le 2m-s$ and $|q| \le s$. Hence $A' = D^q \bar{a}_{pq} D^p$.

Define J on $C_0^\infty(\mathcal{U} \cap \overline{M})$ by

$$J[u, v] = \int_{\mathcal{U}\cap M} \sum a_{pq}(D^q u)\overline{(D^p v)} .$$

Integrating by parts we obtain

$$J[u, v] = \int_{\mathcal{U}\cap M} (Au)\bar{v} + i\int_\Gamma \sum_{p,q} \sum_{t=1}^{p_n} \left(D_n^{t-1} D^{(p',0)} a_{pq} D^q u\right)\overline{D_n^{p_n-t} v}$$

where $p = (p_1, \ldots, p_n) = (p', p_n)$. Collecting terms we obtain

$$J[u, v] = \int_{\mathcal{U}\cap M} (Au)\bar{v} - \int_\Gamma \sum_{j=s}^{2m-1} (F_j u)\overline{\gamma_{2m-j-1} v}$$

where F_j is a differential operator of order j with smooth coefficients.

The coefficient of D_n^j in F_j is $i a_{\tilde{p}\tilde{q}}$ where $\tilde{p} = (0, 2m-s)$ and $\tilde{q} = (0, s)$. By the ellipticity of A, $a_{\tilde{p}\tilde{q}} \ne 0$ on M' and so F_j is normal as required for $s \le j \le 2m-1$, as is $G_j = \gamma_{2m-j-1}$. Since $a_{\tilde{p}\tilde{q}}$ is independent of the particular choice of boundary coordinate system, the partition of unity argument preserves normality.

Integrating by parts in the other direction, the second part of (2.8) is obtained with $F_j = \gamma_j$ for $0 \le j \le s-1$.

Proof of Theorem 2.6. For $s \le j \le 2m-1$, $C_j = \sum_{t=s}^{2m-1} c_{jt}\gamma_{2m-t-1}$ where c_{jt} is a differential operator in the boundary of order $t - j$. We use the convention that the only differential operator of negative order is the zero operator. By (2.5) and (2.8) we have, for smooth u, v ,

$$\sum_{j=0}^{2m-1} \langle B_j u, C_j v\rangle = \sum_{j=0}^{2m-1} \langle F_j u, G_j v\rangle .$$

If $\gamma_j u = 0$ for $0 \le j \le s-1$ then

$$\sum_{j=s}^{2m-1} \langle B_j u, C_j v\rangle = \sum_{t=s}^{2m-1} \langle F_t u, G_t v\rangle .$$

Hence $F_t - \sum_{j=s}^{2m-1} C'_{jt} B_j$ has normal order at most $s - 1$ and can be expressed in the form $\sum_{r=0}^{s-1} S_{tr}\gamma_r$ where S_{tr} is a differential operator in the boundary of order $t - r$. It follows that there exists a bounded sesquilinear form $J^1 : H^s(M) \times H^{2m-s}(M) \to \mathbf{C}$ such that, for smooth u, v ,

$$J^1[u, v] = \sum_{t=s}^{2m-1} \sum_{r=0}^{s-1} \langle S_{tr}\gamma_r u, \gamma_{2m-t-1} v\rangle .$$

This result is proved for the case $s = m$ by Grubb [3, Proposition 5.1]. Her proof readily adapts to our case. If J^0 denotes a form satisfying (2.8) then $J = J^0 + J^1$ satisfies the claims of the theorem.

3. Variational formulation

We now turn to the problem (1.4) posed in the introduction. To use Theorem 2.6, extend B_p to a system $(B_j)_0^{2m-1}$ of normal boundary operators B_j of order j and let $(C_j)_0^{2m-1}$ be the adjoint system relative to A . Let J^1 be the corresponding sesquilinear form on

$H^s(M) \times H^{2m-s}(M)$ given by Theorem 2.6. If J denotes the restriction of J^1 to $V \times W$ then (1.5) holds as required and it follows that (1.4) has the equivalent variational formulation (1.6). We investigate this equivalence more fully.

Let $H_A^s(M) = \{u \in H^s(M) : Au \in L^2(M)\}$ with the usual graph norm. Then B_j maps $H_A^s(M)$ to $H^{s-j-\frac{1}{2}}(\Gamma)$ which we abbreviate to Π_j. Thus $B_P : H_A^s(M) \to \Pi_P$ is a bounded operator.

As the orders of $B_{P''}$ are less than s, $B_{P''} : H^s(M) \to \Pi_{P''}$ is also bounded. Since C_j has order $2m - j - 1$, if $j \geq s$ then C_j maps $H^{2m-s}(M)$ to $H^{-s+j+\frac{1}{2}}(\Gamma) = \Pi_j^*$.

If $\overset{\circ}{H}{}^s(M)$ denotes the subspace of $H^s(M')$ consisting of distributions with support in $\overline{M}$ then $\overset{\circ}{H}{}^s(M)^* = H^{-s}(M)$ with the pairing on $\overset{\circ}{H}{}^s(M) \times H^{-s}(M)$ extending by continuity the L^2 inner product on $C_0^\infty(M) \times C^\infty(\overline{M})$.

Further, by the normality of C_j, the following is an exact sequence of bounded operators

(3.1) $$0 \to \overset{\circ}{H}{}^{2m-s}(M) \xrightarrow{i} W \xrightarrow{C_{P'}} \Pi_{P'}^* \to 0 .$$

We use these facts to relate existence/uniqueness or, more generally, Fredholm properties of (A, B_P) with similar properties of the form J.

A bounded linear operator between Hilbert spaces is called left semi-Fredholm, respectively right semi-Fredholm, if it has finite dimensional kernel, respectively finite codimension, and closed range. The operator is Fredholm if both left semi-Fredholm and right semi-Fredholm.

A bounded sesquilinear form $J : V \times W \to \mathbb{C}$ we will call left semi-Fredholm, right semi-Fredholm, Fredholm, left invertible, right invertible or invertible if its associated operator $A_J : V \to W^*$ defined by $\langle A_J v, w \rangle = J[v, w]$ has the same property.

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Consider the following diagrams, where the subscript $\ker A$ denotes the kernel of A in the mentioned space, and M has been dropped from the names of various spaces:

(3.2)
$$\begin{array}{ccccccc}
0 \to V_{\ker A} & \to & H^s_A & \xrightarrow{(A, B_{P''})} & L^2 \times \Pi_{P''} & & \\
\downarrow B_{P'} & & \downarrow (A, B_P) & & \downarrow I & & \\
0 \to \Pi_{P'} & \to & L^2 \times \Pi_P & \longrightarrow & L^2 \times \Pi_{P''} & \to & 0 ,
\end{array}$$

(3.3)
$$\begin{array}{ccccccc}
0 \to V_{\ker A} & \longrightarrow & V & \xrightarrow{A} & H^{s-2m} & & \\
\downarrow B_{P'} & & \downarrow A_J & & \downarrow I & & \\
0 \to \Pi_{P'} & \xrightarrow[-C^*_{P'}]{} & W^* & \xrightarrow[i^*]{} & H^{s-2m} & \to & 0 .
\end{array}$$

These diagrams commute and the horizontal sequences are exact. The commutativity of (3.3) follows from Theorem 2.6 using the fact that smooth functions are dense in $H^s(M)$, $\overset{\circ}{H}{}^s(M)$ and $H^s_A(M)$. (Lions and Magenes [5].)

If in addition the maps $(A, B_{P''})$ and A are surjective, which is the case if, for example, the Dirichlet problem for A is uniquely solvable, or more generally if these maps are right semi-Fredholm, then we can apply the five lemma (Pryde [9]) to relate the Fredholm properties of (A, B_P) and J .

THEOREM 3.4. *Let* A, B_j *and* M *satisfy conditions* (2.1), (2.2) *and* (2.3), *and let* s *be an integer,* $1 \le s \le 2m-1$. *If* $(A, B_{P''}) : H^s(M) \to H^{s-2m}(M) \times \Pi_{P''}$ *is right semi-Fredholm, then* $(A, B_P) : H^s_A(M) \to L^2(M) \times \Pi_P$ *is left semi-Fredholm (right semi-Fredholm) if and only if* $J : V \times W \to \mathbb{C}$ *is left semi-Fredholm (right semi-Fredholm). If* $(A, B_{P''})$ *is onto then* (A, B_P) *is left (right) invertible if and only*

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If J is left (right) invertible.

Proof. If $(A, B_{P''}) : H^s \to H^{s-2m} \times \Pi_{P''}$ is right semi-Fredholm (onto) then so too are $(A, B_{P''}) : H_A^s \to L^2 \times \Pi_{P''}$ (since $L^2 \subset H^{s-2m}$) and $A : V \to H^{s-2m}$ (since $V = \ker B_{P''}$). The theorem follows from the preceding discussion.

We remark that since A is elliptic, $A : H^s \to H^{s-2m}$ is right semi-Fredholm and since $B_{P''}$ is normal, $B_{P''} : H^s \to \Pi_{P''}$ is onto. It follows from the five lemma for example that $(A, B_{P''}) : H_A^s \to L^2 \times \Pi_{P''}$ is right semi-Fredholm if and only if $A : V \to H^{s-2m}$ is right semi-Fredholm. Hence

COROLLARY 3.5. *Let A, B_j and M satisfy conditions* (2.1), (2.2) *and* (2.3), *and let s be an integer, $1 \le s \le 2m-1$. Then (A, B_P) is Fredholm if and only if J is Fredholm.*

This corollary expresses the equivalence, in terms of Sobolev spaces, of the boundary value problem (1.4) and the variational formulation (1.6). Analogous results can be proved to show the equivalence of problems (1.8) and (1.9) for the case of mixed boundary conditions.

4. Localization

Because the domain of J depends on $B_{P''}$ and $C_{Q'}$, it is not suited to localization. We overcome this problem by replacing J with a related form J^2 whose domain is independent of the operators and which has the same Fredholm properties.

THEOREM 4.1. *Suppose H, K, X, Y are Hilbert spaces, $J^1 : H \times K \to \mathbf{C}$ is a bounded sesquilinear form and $B : H \to X$, $C : K \to Y$ are bounded surjections. Let J be the restriction of J^1 to $\ker B \times \ker C$ and define J^2 on $(H \times Y^*) \times (K \times X^*)$ by*

$$J^2[(u, \phi), (v, \psi)] = J^1[u, v] + \langle Bu, \psi \rangle + \langle \phi, Cv \rangle .$$

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Then J is left semi-Fredholm, right semi-Fredholm, left invertible or right invertible if and only if J^2 has the same property.

Proof. Let $j_H : \ker B \to H$ and $j_K : \ker C \to K$ be the injections. Then the following diagrams commute and the horizontal sequences are exact, so the five lemma gives the result immediately:

(4.2)
$$\begin{array}{ccccccccc}
0 & \to & \ker B & \xrightarrow{j_H} & H & \xrightarrow{B} & X & \to & 0 \\
 & & \downarrow{\scriptstyle A_J} & & \downarrow{\scriptstyle (j_K^* A_{J^1}, B)} & & \downarrow{\scriptstyle I} & & \\
0 & \to & (\ker C)^* & \longrightarrow & (\ker C)^* \times X & \longrightarrow & X & \to & 0 ,
\end{array}$$

(4.3)
$$\begin{array}{ccccccccc}
0 & \to & Y^* & \longrightarrow & H \times Y^* & \longrightarrow & H & \to & 0 \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle A_{J^2}} & & \downarrow{\scriptstyle (j_K^* A_{J^1}, B)} & & \\
0 & \to & Y^* & \xrightarrow[(C^*,0)]{} & K^* \times X & \xrightarrow[j_K^* \times I]{} & (\ker C)^* \times X & \to & 0 .
\end{array}$$

In our case, J^2 on $\left(H^s(M) \times \Pi_{Q'}\right) \times \left(H^{2m-s}(M) \times \Pi^*_{P''}\right)$ is defined by

$$J^2\left[(u, \phi_{Q'}), (v, \psi_{P''})\right] = J^1[u, v] + \langle B_{P''}u, \psi_{P''}\rangle + \langle \phi_{Q'}, C_{Q'}v\rangle .$$

Write u for a typical element of $H^s(M) \times \Pi_{Q'}$ with norm $\|\mathrm{u}\|_s$ and v for an element of $H^{2m-s}(M) \times \Pi^*_{P''}$ with norm $\|\mathrm{v}\|_{2m-s}$. Let $\|\mathrm{u}\|_{s-1}$ and $\|\mathrm{v}\|_{2m-s-1}$ denote the norms in the larger spaces obtained by replacing each Sobolev space H^t by the larger space H^{t-1} in which it is compactly embedded.

By Peetre's lemma [8, p. 728], J^2 is left semi-Fredholm if and only if

(4.4)
$$\|\mathrm{u}\|_s \le c\left(\sup \frac{|J^2[\mathrm{u},\mathrm{v}]|}{\|\mathrm{v}\|_{2m-s}} + \|\mathrm{u}\|_{s-1}\right) \quad \text{for all } \mathrm{u} ;$$

and $\mathcal{J}^2$ is right semi-Fredholm if and only if

$$(4.5)\qquad \|v\|_{2m-s} \le c \sup\left(\frac{|\mathcal{J}^2[u,v]|}{\|u\|_s} + \|v\|_{2m-s-1}\right) \text{ for all } v .$$

Using standard localization techniques, these estimates are replaced by similar ones for homogeneous operators with constant coefficients in $\mathbb{R}^n_+$. In Pryde [*10*] for example, the localized estimates are shown to be satisfied if and only if A is properly elliptic and covered by B_P.

5. Adjoint problems

Correspondences between the boundary value problem defined by an elliptic A and normal B_P and the adjoint problem defined by A' and C_Q are well known. See for example Schechter [*11*, Lemma 2.2]. Using the form $\mathcal{J}$ and the five lemma, we can readily relate Fredholm properties of the two problems when solutions lie in $H^s(M)$ or $H^{2m-s}(M)$ and $0 \le s \le 2m$.

THEOREM 5.1. *Let* A, B_j *and* M *satisfy conditions* (2.1), (2.2) *and* (2.3), *and let* s *be an integer,* $1 \le s \le 2m-1$. *If* $(A, B_{P''}) : H^s(M) \to H^{s-2m}(M) \times \Pi_{P''}$ *and* $(A', C_{Q'}) : H^{2m-s}(M) \to H^{-s}(M) \times \Pi^*_{Q'}$, *are right semi-Fredholm then* $(A, B_P) : H^s_A(M) \to L^2(M) \times \Pi_P$ *is left semi-Fredholm (right semi-Fredholm) if and only if*

$$(A', C_Q) : H^{2m-s}(M) \to L^2(M) \times \Pi^*_Q$$

is right semi-Fredholm (left semi-Fredholm). In particular (A, B_P) *is Fredholm if and only if* (A', C_Q) *is Fredholm.*

Proof. Analogous to diagrams (3.1), (3.2) and (3.3) we have the following diagrams:

$$(5.2)\qquad 0 \to \mathring{H}^s(M) \xrightarrow{i} V \xrightarrow{B_{Q''}} \Pi_{Q''} \to 0 ,$$

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(5.3)
$$\begin{array}{ccccccc}
0 \to W_{\ker A'} & \to & H^{2m-s}_{A'} & \xrightarrow{(A',C_{Q'})} & L^2 \times \Pi^*_{Q'} & & \\
\downarrow C_{Q''} & & \downarrow (A',C_Q) & & \downarrow I & & \\
0 \to \Pi^*_{Q''} & \to & L^2 \times \Pi^*_Q & \longrightarrow & L^2 \times \Pi^*_{Q'} & \to & 0 \ ,
\end{array}$$

(5.4)
$$\begin{array}{ccccccc}
0 \to W_{\ker A'} & \longrightarrow & W & \xrightarrow{A'} & H^{-s} & & \\
\downarrow C_{Q''} & & \downarrow A^*_J & & \downarrow I & & \\
0 \to \Pi^*_{Q''} & \xrightarrow[B^*_{Q''}]{} & V^* & \xrightarrow[i^*]{} & H^{-s} & \to & 0 \ .
\end{array}$$

By the same arguments as in Theorem 3.4, (A', C_Q) is right semi-Fredholm (left semi-Fredholm) if and only if A^*_J is right semi-Fredholm (left semi-Fredholm) if and only if (A, B_P) is left semi-Fredholm (right semi-Fredholm). For the last statement of the theorem, compare the proof of Corollary 3.5.

With regard to the mixed boundary value problem (1.8) one can show similarly that $(A, r_1B_P, r_2B_Q) : H^s_A(M) \to L^2(M) \times r_1\Pi_P \times r_2\Pi_Q$ is Fredholm if and only if $(A', r_2C_P, r_1C_Q) : H^{2m-s}_A(M) \to L^2(M) \times r_2\Pi^*_P \times r_1\Pi^*_Q$ is Fredholm.

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