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ON KNOCK-OUT TOURNAMENTS

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1. **Introduction**. We define, as in [2], a random knock-out tournament with n players as a vector $(m_1, m_2, \ldots, m_k)$ of positive integers satisfying

$$\begin{aligned} &m_1+m_2+\cdots+m_k = n-1, \quad m_k = 1; \\ (1) \qquad &2m_1 \le n, \\ &2m_i \le n-m_1-m_2-\cdots-m_{i-1}, \quad i = 2, 3, \ldots, k. \end{aligned}$$

On the first round of the tournament $2m_1$ players, chosen at random, are paired off randomly; the remaining $n-2m_1$ players have a "bye". The m_1 losers are knocked out, leaving a tournament of $n-m_1$ players with vector $(m_2, m_3, \ldots, m_k)$.

We may argue heuristically that, since $n-1$ matches (losers) are necessary to locate the winner, the probability that a particular pair of players are matched during the tournament is $P_n^1=(n-1)\Big/\binom{n}{2}=2/n$. A formal proof by induction on n, using definition (1), is easy and hence left as an exercise for the reader.

In §2 we obtain recurrence relations for the probability P_n^i that a particular player A meets i specified other players $B_1, \ldots, B_i$ in the case of a tournament with minimum byes. These are applied in §3 to the classical case $n=2^t$.

2. **A recurrence relation for the tournament with minimum byes.** A minimum-byes tournament with N players has vector

$$(m_1, m_2, \ldots, m_t), m_i = \left[\frac{N+2^{i-1}-1}{2^i}\right], i = 1, 2, \ldots, t$$

where t is the smallest integer with $2^t \ge N$ ($[x]$ denotes the greatest integer $\le x$). Clearly $P_N^i=0$ for $i>t$. Noting $P_N^1=2/N$, consider first the even case $N=2n$. P_{2n}^i represents the sum of two exclusive and exhaustive cases:

(I) A plays one of $B_1, \ldots, B_i$, say B_j in round 1 and no two of the remaining B's are paired off in round 1;

(II) A does not meet any of $B_1, \ldots, B_i$ in round 1, nor do any two of the B's meet in round 1.

We evaluate the probabilities of cases I, II in randomly matching $2n$ players in Remarks 2, 3.

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REMARK 1. Let $T(2n)$ denote the number of ways of choosing n pairs from $2n$ players in round 1. Then

$$T(2n) = (2n-1)(2n-3)\ldots 3.1 = \frac{(2n)!}{n!\,2^n}.$$

Proof. Although this result is well known, we give a proof which is applicable to all remarks which follow. Let the $2n$ players be called for convenience $A, B_1, B_2, \ldots, B_i, C_{i+1}, \ldots, C_{2n-1}$.

An opponent for A can be chosen in $(2n-1)$ ways. Next, an opponent for the player with the smallest subscript among those remaining can be chosen in (2n−3) ways, and so on.

REMARK 2. The probability that A is paired off with B_j, $(1 \le j \le i)$, while no two of the remaining B's are paired off in round 1 is

$$\frac{[2n]_1^i}{T(2n)} = \frac{i(2n-i-1)(2n-i-2)\ldots(2n-2i+1)T(2n-2i)}{T(2n)}$$

(2)

$$= 2^i \frac{i}{n-i}\binom{n}{i+1}\Big/\binom{2n}{i+1}.$$

REMARK 3. The probability that none of $A, B_1, \ldots, B_i$ are paired off in round 1 is

$$\frac{[2n]_0^i}{T(2n)} = \frac{(2n-i-1)\ldots(2n-2i-1)T(2n-2i-2)}{T(2n)}$$

(3)

$$= 2^{i+1}\binom{n}{i+1}\Big/\binom{2n}{i+1}.$$

Application of the theorem of total probabilities to cases I, II, yields routinely from (2), (3) the even case of the following theorem.

THEOREM. *For* $2 \le i \le t$,

$$P_{2n}^i = \frac{\binom{n}{i+1}}{\binom{2n}{i+1}}\left[\frac{i}{(n-i)} P_n^{i-1} + P_n^i\right],$$

(4)

$$P_{2n-1}^i = \frac{\binom{n}{i+1}}{\binom{2n-1}{i+1}}\left[\frac{i}{(n-i)}\cdot\frac{n-1}{n} P_n^{i-1} + P_n^i\right].$$

Proof of case $N = 2n-1$. This is quite analogous to the even case by the following steps:

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REMARK 1′. Let $T(2n-1)$ denote the number of ways of playing round 1, i.e. giving a bye to 1 player and pairing off the remaining $(2n-2)$ players. Clearly $T(2n-1)=T(2n)$. $(n\geq 2)$

REMARK 2′. Case I of the even case is partitioned into two cases (the number of ways in which round 1 can be played in each case is indicated):

I_1 one of $B_1,\ldots,B_i$ has a bye... $i[2n-2]_1^{i-1}$;
I_2 none of $A, B_1,\ldots,B_i$ has a bye... $(2n-2-i)[2n-2]_1^i$.

REMARK 3′. Similarly case II is partitioned into the cases

II_1 A has a bye... $\dfrac{(2n-2-i)!}{(2n-2i-2)!}T(2n-2i-2)$;
II_2 one of $B_1,\ldots,B_i$ has a bye... $i[2n-2]_0^{i-1}$;
II_3 none of $A, B_1,\ldots,B_i$ has a bye... $(2n-2-i)[2n-2]_0^i$.

3. **The Classical Case $n=2^t$ and the Enumeration of Tournaments.** As a special case of our theorem, we consider the classical case with $n=2^t$ players, and vector $(2^{t-1}, 2^{t-2},\ldots,1)$. Only the even case of our theorem is applicable and we can easily verify by induction that

$$P_{2^t}^i = \frac{2}{2^t\binom{2^t-1}{i}}\left[2^t-\binom{t}{0}-\binom{t}{1}\cdots\binom{t}{i-1}\right], \tag{5}$$

a result first announced by Narayana [2].

We next remark that in the case where A wins any match with probability p, while the remaining players are equally matched amongst themselves, we have:

$$P_{2n}^i(p) = \frac{2p\binom{n}{i+1}}{\binom{2n}{i+1}}\left[\frac{i}{(n-i)}P_n^{i-1}(p)+P_n^i(p)\right],$$

(6)

$$P_{2n-1}^i(p) = \frac{\binom{n}{i+1}}{\binom{2n-1}{i+1}}\left[\frac{2pi}{(n-i)}\cdot\frac{(n-1)}{n}P_n^{i-1}(p)+\frac{1+2p(n-1)}{n}P_n^i(p)\right].$$

However, it should be noted that $P_n^1(p)$ is no longer given by $2/n\cdot P_n^1(p)$, at least in the classical case $n=2^t$, can be calculated (cf. [3]).

We conclude by enumerating the number of random tournaments as given by our definitions. Let T_n denote the number of random tournaments with n players, and $T_n(k)$ the number of such tournaments with exactly k pairs playing in round 1. Clearly

$$T_n(k) = T_{n-k},\ T_{2n} = \sum_{k=1}^{n} T_{2n}(k),\ T_{2n-1} = \sum_{k=1}^{n-1} T_{2n-1}(k), \tag{7}$$

so that

(8) $$T_n \leq T_k 2^{n-k} \quad (n \geq k).$$

Using this upper bound for T_n, the relations (7) and the table of values for T_n below, we can show, for example,

$$\frac{160}{256}\cdot 2^{n-3} < T_n \leq \frac{165}{256}\cdot 2^{n-3} \text{ for } n \geq 11.$$

More accurate bounds could be derived by the same method. Tables, 1, 2 below conclude our paper.

TABLE 1

Short table of values of T_n.

n	2	3	4	5	6	7	8	9	10	11
T_n	1	1	2	3	6	11	22	42	84	165

TABLE 2

Table of values for $P_n^i(p)$.

$n \backslash i$	1	2	3
3	0.58333	0.16667	
	0.6667	0.33333	
	0.75	0.5	
4	0.41667	0.08333	
	0.5	0.16667	
	0.58333	0.25	
5	0.31667	0.05	0.00833
	0.4	0.12222	0.03333
	0.5	0.21667	0.075
6	0.25833	0.03333	0.00417
	0.33333	0.08333	0.01667
	0.425	0.15	0.0375
7	0.21726	0.02381	0.00268
	0.28571	0.0619	0.01071
	0.37202	0.11429	0.02411
8	0.1875	0.01786	0.00179
	0.25	0.04762	0.00714
	0.33036	0.08929	0.01607

For each i, n in table 2, the three values given, are from left to right, for $p=0.25$, 0.5, 0.75.

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