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DIRICHLET-FINITE OUTER FUNCTIONS

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Abstract

Given a measurable function k non-negative a.e. on the circle $|z| = 1$, when is the outer function T_k (see (1.3)) continuous on the disk $|z| < 1$ and further, Dirichlet-finite: $\int\int_{|z|<1}|T_k'(z)|^2\,dx\,dy < \infty$? We shall show, among other results, that the answer is in the positive if $k \in \Lambda^2_\alpha$, $\frac{1}{2} < \alpha < 1$, with ess inf $k > 0$.

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1. Introduction

By the Dirichlet integral $D(f)$ of a function f analytic in $U = \{|z| < 1\}$ we mean

$$D(f) = \int\int_U |f'(z)|^2\,dx\,dy \qquad (z = x + iy).$$

Let $\mathfrak{D}$ be the family of Dirichlet-finite analytic functions f in U, namely, $D(f) < \infty$. Then $f \in \mathfrak{D}$ has the finite angular limit $f^*(e^{it})$ at a.e. (almost every(where)) point e^{it} of the unit circle $C = \{|z| = 1\}$, and that f^* is a member of complex $L^p(C)$ for all p, $0 < p < \infty$. J. A. Cima [3] raised the following question:

Given a $k \in \bigcap_{p>0} L^p(C)$, non-negative a.e., when can we find an $f \in \mathfrak{D}$ such that

(1.1) $$|f^*(e^{it})| = k(e^{it}) \quad a.e.\ on\ C?$$

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The above problem is a part of the following:

(Q1) *Find a necessary and sufficient condition for* $k \geqslant 0$, *a.e. on* C, *so that there exists* $f \in \mathfrak{D}$ *satisfying* (1.1).

This is essentially answered by L. Carleson [1].

THEOREM 1 (Carleson [1]). *A necessary and sufficient condition on* k *for* (Q1) *is that,* k *is measurable, that* $\log k \in L^1(C)$, *and that*

$$(1.2) \qquad J(k) \equiv \int_0^\pi \frac{dt}{\sin^2 \frac{t}{2}} \int_{-\pi}^{\pi} \left[\log \frac{k(e^{i(s+t)})}{k(e^{is})} \right] \left[k(e^{i(s+t)})^2 - k(e^{is})^2 \right] ds < \infty.$$

In effect, the necessity of (1.2) follows from Carleson's result [1, Theorem]. To prove the sufficiency we let Exp L^1 be the family of measurable functions $k > 0$ a.e. on C such that $\log k \in L^1(C)$. By an outer function we mean an analytic function T_k in U defined by

$$(1.3) \qquad T_k(z) = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log k(e^{it})\, dt \right], \qquad z \in U,$$

where $k \in \operatorname{Exp} L^1$. Then the angular limit $T_k^*(e^{it})$ exists and

$$(1.4) \qquad |T_k^*(e^{it})| = k(e^{it}) \quad \text{a.e. on } C.$$

Returning to Theorem 1 we assume (1.2). It follows from the inequality [1, (7)] (this is used, in effect, for the proof of his main theorem by Carleson, yet [1, Section 4] is self-contained) that $D(T_k) \leqslant (8\pi)^{-1} J(k) < \infty$.

Since the outer function is considered in the proof of the sufficiency of (1.2), it is natural to restrict the family $\mathfrak{D}$ within outer functions. More restrictively, let $\mathfrak{D}^*$ be the family of $T_k \in \mathfrak{D}$ such that T_k can be extended continuously to $\overline{U} = U \cup C$, or, continuous on $\overline{U}$, for short.

(Q2) *For what function* $k \geqslant 0$ *a.e. on* C, *is* $T_k \in \mathfrak{D}^*$ *true*?

We begin with continuous k. Let Λ_α $(0 < \alpha \leqslant 1)$ be the family of complex-valued functions k on C such that

$$|k(e^{it}) - k(e^{is})| \leqslant A_k |e^{it} - e^{is}|^\alpha \quad \text{on } C,$$

where $A_k \geqslant 0$ is a constant [4, p. 72]; apparently, $|e^{it} - e^{is}|^\alpha$ in the definition may be replaced by $|t - s|^\alpha$, $t, s \in (-\infty, +\infty)$.

THEOREM 2. *Let $k \in \Lambda_\alpha$ $(0 < \alpha < 1)$ be positive on C. Then T_k is continuous on $\bar{U}$ such that*

$$(1.5)\qquad |T_k^*(e^{it})| = k(e^{it}) \quad \textit{for all } e^{it} \in C.$$

Furthermore,

$$(1.6)\qquad T_k'(z) = O\big((1-|z|)^{-1+\alpha}\big) \quad \textit{as } |z| \to 1-0.$$

In the special case $\frac{1}{2} < \alpha < 1$, it follows from (1.6) that $T_k \in \mathfrak{D}^*$. K.-N. Chow and D. Protas [2, Theorem 2.2] proved that if $k \in \Lambda_\beta$ $(\frac{1}{2} < \beta \leqslant 1)$ is positive, then $D(T_k) < \infty$ and (1.4) holds. We can drop the word "almost" in (1.4) in the preceding sentence. Actually, choose $\frac{1}{2} < \alpha < \beta$. Then $k \in \Lambda_\alpha$ because $\Lambda_\beta \subset \Lambda_\alpha$. Therefore, (1.5) and $D(T_k) < \infty$ are true.

Let Λ_α^p $(1 \leqslant p < \infty,\ 0 < \alpha \leqslant 1)$ be the family of complex $k \in L^p(C)$ such that

$$\sup_{0<h\leqslant t} \int_{-\pi}^{\pi} |k(e^{i(s+h)}) - k(e^{is})|^p\, ds = O(t^{p\alpha}) \quad \text{as } t \to 0;$$

see [4, p. 72]. Then $\Lambda_\alpha \subset \Lambda_\alpha^p$ for $1 \leqslant p < \infty$, $0 < \alpha \leqslant 1$.

THEOREM 3. *Let $1 < p < \infty$ and $1/p < \alpha < 1$. Suppose that $k \in \Lambda_\alpha^p$ is real and there exists a constant $m > 0$ such that $k \geqslant m$ a.e. on C. Then T_k is continuous on $\bar{U}$ with* (1.4). *Furthermore,*

$$(1.7)\qquad \int_{-\pi}^{\pi} |T_k'(re^{it})|^p\, dt = O\big((1-r)^{-p+p\alpha}\big) \quad \textit{as } r \to 1-0.$$

Especially if $p = 2$ and $\frac{1}{2} < \alpha < 1$, then $T_k \in \mathfrak{D}^*$ by (1.7). Therefore, the best answer to (Q2) in the present paper is that $k \in \Lambda_\alpha^2$ $(\frac{1}{2} < \alpha < 1)$ with ess inf $k > 0$.

2. Proofs

For the proof of Theorem 2, let

$$0 < m \leqslant k(e^{it}) \leqslant M < \infty \quad \text{for all } e^{it} \in C.$$

Then, $T_k = M \exp g$, where

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{k(e^{it})}{M}\, dt.$$

Since $1/k \leqslant 1/m$, it follows that $\log(k/M) \in \Lambda_\alpha$. It further follows from [4, Theorem 5.8, p. 83] that g is continuous on $\bar{U}$ with $g^* \in \Lambda_\alpha$. Therefore T_k is continuous on $\bar{U}$ with $|T_k^*| = M \exp(\operatorname{Re} g^*) = k$ everywhere on C. To observe

(1.6) we note that

$$g'(z) = O\big((1 - |z|)^{-1+\alpha}\big)$$

by [4, Theorem 5.1, p. 74]. Since Re $g \leqslant 0$, it follows that $|T_k'| \leqslant M|g'|$, whence follows (1.6).

For the proof of Theorem 3 we note that, by the result [5, Theorem 5(ii), p. 627] due to G. H. Hardy and J. E. Littlewood, there exists $K \in \Lambda_{\alpha - 1/p}$ such that $k(e^{it}) = K(e^{it})$ a.e. on C. Consequently, K is bounded from above and from below,

$$0 < m \leqslant K(e^{it}) \leqslant M < \infty \quad \text{for all } e^{it} \in C,$$

where m is the number in Theorem 3. Now, $T_k = T_K = M \exp G$, where

$$G(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log\frac{K(e^{it})}{M}\, dt.$$

Since $K \in \Lambda_{\alpha - 1/p}$, it follows from Theorem 2 that $T_k = T_K$ is continuous on $\overline{U}$ and that $|T_k^*| = |T_K^*| = K = k$ a.e. on C. For the proof of (1.7), we note that Re G is bounded because $|\log(K/M)| \leqslant \log(M/m)$. Therefore, G is of Hardy class H^q for all $q > 0$ by [4, Theorem 4.1, p. 54]. Since $\log(k/M) \in \Lambda_\alpha^p$, it follows that $\log(K/M) \in \Lambda_\alpha^p$. Thus, by the Λ_α^p analogue of [4, Theorem 5.8, p. 83], being [4, Exercise 12, p. 91], one knows that $G^* \in \Lambda_\alpha^p$. Then by [4, Theorem 5.4, p. 78],

$$\int_{-\pi}^{\pi} |G'(re^{it})|^p\, dt = O\big((1 - r)^{-p+p\alpha}\big). \tag{2.1}$$

Again, by $|T_k'| = |T_K'| \leqslant M|G'|$, together with (2.1), one observes that (1.7) holds.

3. A concluding remark

As far as the condition of the type (1.7) is concerned, the following might be noteworthy.

Suppose that $k \in \Lambda_\alpha^p$ $(1 < p < \infty,\ 0 < \alpha < 1)$ *is real and that there exist constants* $m > 0$ *and* $M > 0$ *such that* $m \leqslant k \leqslant M$ *a.e. on* C. *Then*

$$\int_{-\pi}^{\pi} |T_k'(re^{it})|^p\, dt = O\big((1 - r)^{-p+p\alpha}\big) \quad \text{as } r \to 1 - 0. \tag{3.1}$$

For a sketch of the proof we set

$$\Gamma(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log\frac{k(e^{it})}{M}\, dt.$$

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Then, $T_k = M \exp \Gamma$ and $\log(m/M) \leqslant \operatorname{Re} \Gamma \leqslant 0$. We can show that

$$\int_{-\pi}^{\pi} |\Gamma'(re^{it})|^p \, dt = O((1-r)^{-p+p\alpha}),$$

which, together with $|T_k'| \leqslant M|\Gamma'|$, proves (3.1).

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