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ON CYCLES AND CONNECTIVITY IN PLANAR GRAPHS

BY

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1. **Introduction**. Let G be a graph and $\zeta(G)$ be the greatest integer n such that every set of n points in G lies on a cycle [8]. It is clear that $\zeta(G)\geq 2$ for 2-connected planar graphs. Moreover, it is easy to construct arbitrarily large 2-connected planar graphs for which $\zeta=2$. On the other hand, by a well-known theorem of Tutte [5], [6], if G is planar and 4-connected, it has a Hamiltonian cycle, i.e., $\zeta(G)=|V(G)|$ for all 4-connected (and hence for all 5-connected) planar graphs.

In this paper we settle the one remaining case by showing that $\zeta(G)\geq 5$ for 3-connected planar graphs and this is best possible in the sense that there are arbitrarily large 3-connected planar graphs with $\zeta=5$.

2. **Additional terminology.** For any graphical concepts not defined here the reader is referred to Harary [4]. All graphs in this paper are finite, undirected, and loopless.

Let P be a path in G and H a subgraph of G. Following Watkins [7] we say P and H are *openly disjoint* (abbreviated o.d.) if they have at most endpoints of P in common. A family of paths $P_1, P_2, \ldots, P_n$ is openly disjoint if they have at most endpoints in common.

We shall have occasion to denote paths by their endpoints. In this case $P[a, b]$, or simply $[a, b]$ when unambiguous, will denote that section of path P with endpoints a and b. We denote $P[a, b]-a-b$ by $P(a, b)$ or by (a, b), with similar definitions for $P(a, b]$ and $P[a, b)$. $P[a, x, b]$ and $[a, x, b]$ will denote a path with endpoints a and b and intermediate point x. If H is a subgraph of G and w a point of G not on H, a (w, H) *path* is any path joining w and H in G but having no intermediate points in H.

3. **Main results.** The following generalization of Menger's theorem is in turn a special case of a result of Dirac [2, Theorem 1]. We shall appeal to it repeatedly and shall call it GMT for brevity.

THEOREM 1. *If G is n-connected and if $u, v_1, \ldots, v_n$ are $n+1$ distinct points in G, then there exist n openly disjoint paths $P_1, \ldots, P_n$ in G, where P_i joins u and v_i, for all i.*

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THEOREM 2. *If G is planar and 3-connected, then any given three points a, b, c and line $x=\alpha\beta$ of G lie on a cycle.*

Proof. As an immediate corollary to another theorem of Dirac [3, Theorem 9] there is a cycle C in G containing a, b, and x. If $c \in C$ we are done. Otherwise by GMT there are three openly disjoint (c, C) paths P_1, P_2, and P_3 ending on C at three distinct points γ_1, γ_2, and γ_3 respectively. Without loss of generality we may assume that C is oriented as $[\alpha, \beta, a, b, \alpha]$. Let $C[\beta, \alpha)=C_1$, $C(a, b)=C_2$, and $C(b, \alpha]=C_3$. Unless one γ_i lies on each of these sections of C, say $\gamma_i \in C_i$, we are done (cf. Figure 1).

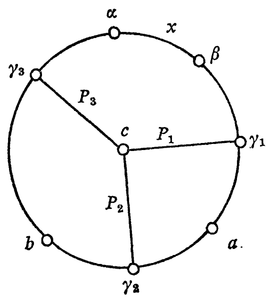

Figure 1.

Now let $C'=(C-C(\gamma_1, \gamma_2)) \cup P_1 \cup P_2$. Since G is planar, C' separates the plane into two regions, one of which contains a, the other P_3. Again by GMT, there are three o.d. (a, C') paths Q_1, Q_2, Q_3 meeting C' at δ_1, δ_2, δ_3 respectively.

Let the sections of C' be $C_1'=C'[b, \alpha]$, $C_2'=C'[\beta, c]$, $C_3'=C'[c, b]$. As before, one δ_i must occur in each C_i' or we are done. Moreover, no $\delta_i=b$ or c, or again the desired cycle is obtained.

There are three possibilities, $\delta_1 \in C'(b, \gamma_3)$, $\delta_1 \in C'(\gamma_3, \alpha]$, and $\delta_1=\gamma_3$. In the first case we have a cycle $[a, \delta_2, \beta, \alpha, \gamma_3, c, \delta_3, b, \delta_1, a]$. In the second case we have a cycle $[a, \delta_1, \alpha, \beta, \delta_2, c, \gamma_3, b, \delta_3, a]$. The third case is shown in Figure 2.

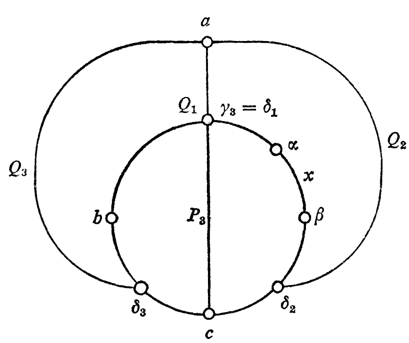

Figure 2.

Now define cycle $C''=P_3 \cup Q_1 \cup Q_3 \cup C'[\delta_3, c]$. Then C'' separates b from $\Lambda=C'[\gamma_3, x, c] \cup Q_2$. Once again by GMT there are three o.d. (b, C'') paths meeting C'' at points $\lambda_1, \lambda_2, \lambda_3$. Let the three parts of C'' be $C_1''=C_1''[a, \gamma_3]$, $C_2''=C''[\gamma_3, c]$, and $C_3''=C''[c, a]$. No two of the λ_i can lie in one of the C_i'' or else the desired cycle is obtained. Hence in particular, no $\lambda_i=\gamma_3$, a or c. But then $[a, \lambda_1, \gamma_3, \alpha, \beta, \delta_2, c, \lambda_2, b, \lambda_3, a]$ is the desired cycle.

As an immediate consequence of this theorem we have

COROLLARY 2.1. *If G is planar and 3-connected, then any four points of G lie on a cycle.*

Thus $\zeta(G)\geq 4$ for 3-connected planar graphs.

At this point we are ready to prove the main theorem of this paper.

THEOREM 3. *If G is planar, 3-connected, and $G\neq K_4$, then $\zeta(G)\geq 5$.*

Proof. Let a, b, c, d, e be any five points of G. We know from Corollary 2.1 that $\zeta(G)\geq 4$. Thus there is a cycle C in G containing a, b, c, d. If $e \in C$ we are done, so suppose $e \notin C$.

Since G is 3-connected there are, by GMT, three o.d. paths from e to three distinct points $\gamma_1, \gamma_2, \gamma_3$ of C. Clearly if any two γ_i's lie on $C[a, b]$, or $C[b, c]$, or $C[c, d]$ or on $C[d, a]$ we are done. Hence at most one γ_i lies in each of these four sections of C. There are then, up to homeomorphism two cases to consider:

(I) $\gamma_1 \in C(d, a)$, $\gamma_2 \in C(a, b)$, $\gamma_3 \in C(b, c)$.

(II) $\gamma_1=$d, $\gamma_2 \in C(a, b)$, $\gamma_3 \in C(b, c)$.

Now delete (γ_1, a, γ_2) from both I and II. In each case we obtain the graph shown in Figure 3.

Let D denote the cycle $[e, \gamma_2, b, \gamma_3, c, d, e]$ and H denote $D \cup (e, \gamma_3)$. Now by GMT, there are three o.d. (a, H) paths Q_1, Q_2, Q_3. Since D separates a from (e, γ_3), these paths end at points $\delta_1, \delta_2, \delta_3$ on D. If we follow $C[a, \gamma_2]$ from γ_2 to the first

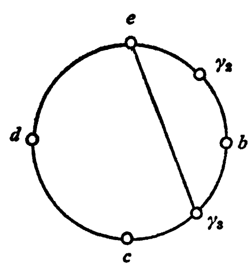

Figure 3.

point λ on one of the Q's, say Q_1, then the paths $C[\gamma_2, \lambda] \cup Q_1[\lambda, a]$, Q_2, Q_3 are three o.d. (a, D) paths. Thus we may assume without loss of generality that $\delta_1 = \gamma_2$.

It is now a simple matter to show that there are five ways in which Q_2 and Q_3 may be drawn without producing a cycle containing a, b, c, d, and e. (Note that $c \neq \delta_2$ or δ_3, for if $c = \delta_2$, say, the cycle $[a, \gamma_2, b, \gamma_3, e, d, c, a]$ suffices.)

1. $\delta_1 = \gamma_2$ $\quad \delta_2 \in [d, e)$ $\quad \delta_3 = \gamma_3$
2. $\delta_1 = \gamma_2$ $\quad \delta_2 \in (d, e)$ $\quad \delta_3 \in (c, d)$
3. $\delta_1 = \gamma_2$ $\quad \delta_2 \in (b, \gamma_3]$ $\quad \delta_3 \in (c, d)$

We now treat each of these cases.

Case 1. Delete (γ_2, b, γ_3) and call the resulting graph H_1. By GMT there are three o.d. paths from b to H_1. Furthermore, the cycle $M = [e, \gamma_2, a, \gamma_3, e]$ separates b from the rest of H_1. Thus the endpoints of these three paths must lie on M. It is easily verified that if two paths end on $M[e, a]$, on $M[a, \gamma_3]$, or on $M[\gamma_3, e]$ we are done. On the other hand, if one path ends on each of these, a cycle containing a, b, c, d, and e is easily discovered.

Case 2. We have the configuration of Figure 4.

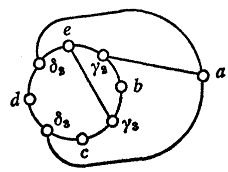

Figure 4.

Delete (δ_3, c, γ_3) from this graph and call the resulting graph H_3.

There are three o.d. paths from c to H_3, ending on the cycle $[e, \gamma_3, b, \gamma_2, a, d, \delta_2, e]$. As before we may assume, without loss of generality, that one of these paths ends at γ_3.

There are, up to homeomorphism, three ways in which the other two paths can be drawn without producing a cycle containing a, b, c, d, and e. That is when the three paths end on

(i) $\gamma_3, \delta_2, \gamma_2$

(ii) $\gamma_3, \delta_2, (a, d)$

or

(iii) $\gamma_3, \gamma_2, (\delta_2, e)$.

(i) If we delete (γ_3, b, γ_2) from the graph and then consider the three o.d. paths from b to the cycle $[e, \gamma_3, c, \gamma_2, e]$, we find that these three paths must produce a cycle containing a, b, c, d, and e.

(ii) Let the endpoint on (a, d) be denoted μ. If we delete (δ_2, d, μ) from the graph and consider the three o.d. paths from b to the cycle $[a, \delta_2, c, \mu, a]$ we find that these paths produce the desired cycle containing a, b, c, d, and e.

(iii) If we delete (γ_3, b, γ_2) from the graph and consider the three o.d. paths from b to the cycle $[e, \gamma_3, c, \gamma_2, e]$, we find that these paths produce the desired cycle containing a, b, c, d, and e.

Case 3. By deleting (γ_2, b, δ_2), we obtain the graph shown in Figure 5.

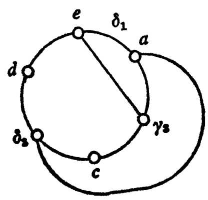

Figure 5.

Call this graph H_4. The cycle $[e, \delta_1, a, \gamma_3, e]$ separates b from the rest of H_4. For this reason the three o.d. paths from b to H_4 end on this cycle. It is easily shown that, unless all three paths end on $[e, \gamma_3]$, a cycle containing a, b, c, d, and e is present. So we need consider only situations shown in Figure 6.

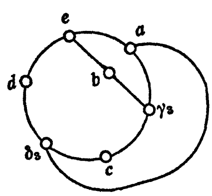

Figure 6.

Delete (γ_3, c, δ_3) and call the resulting graph H_5.

The cycle $E=[e, b, \gamma_3, a, \delta_3, d, e]$ separates c from (e, a). Thus the three o.d. paths R_1, R_2, R_3 from c to H_5 end at points σ_1, σ_2, σ_3 of E. It is easily shown that we have a cycle containing a, b, c, d, and e unless, up to homeomorphism, $\sigma_1=e$, $\sigma_2 \in (b, \gamma_3, a)$, and $\sigma_3 \in (a, \delta_3, d)$.

Delete the path (e, b, σ_2) from $H_5 \cup R_1 \cup R_2 \cup R_3$ and call the resulting graph H_6. The cycle $F=[c, \sigma_2, a, \delta_1, e, c]$ separates b from the rest of H_6. Thus the three o.d. paths from c to H_6 end on F. It is easily verified that there is a cycle containing a, b, c, d, and e regardless of the location of these three endpoints. This completes the proof of the theorem.

Since there exist planar 3-connected graphs with five points, Theorem 3 is, in a trivial sense, best possible. However, Theorem 3 cannot be improved even by

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excluding those graphs with a sufficiently small number of points. To see this, the reader may verify that the 3-connected planar graphs G_n, shown in Figure 7, have $n>10$ points, but each has $\zeta(G_n)=5$. In each G_n it is easily seen that there is no cycle containing the six points a, b, c, d, e, and f.

G_n:

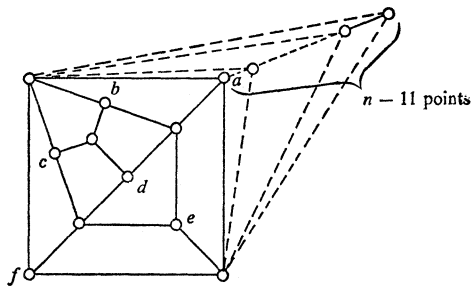

Figure 7.

The graph G_{11} has been previously described by Barnette and Jucovič [1] who show that it is also the smallest 3-connected planar graph containing no Hamiltonian cycle.

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