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ZERO DIVISORS AND IDEMPOTENTS IN GROUP RINGS

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1. Introduction. We consider the following problem: If KG is the group ring of a torsion free group over a field K, show that KG has no divisors of zero. At characteristic zero, major progress was made by Brown [**2**], who solved the problem for G abelian-by-finite, and then by Farkas and Snider [**4**], who considered G polycyclic-by-finite. Here we present a solution at nonzero characteristic for polycyclic-by-finite groups. We also show that if K has characteristic $p > 0$ and G is polycyclic-by-finite with only p-torsion, then KG has no idempotents other than 0 or 1. Finally we show that if R is a commutative ring of nonzero characteristic without nontrivial idempotents and G is polycyclic-by-finite such that no element different from 1 in G has order invertible in R, then RG has no nontrivial idempotents. This is proved at characteristic zero in [**3**].

We denote by $\Delta(G)$ the augmentation ideal of a group ring RG. We denote conjugate elements g and h of G by $g \sim h$, and for $a = \sum a_g g \in RG$ and $g \in G$ we write

$$t_g a = \sum_{h \sim g} a_h.$$

We denote by $M_d(RG)$ the ring of d by d matrices with entries in RG, and for $\alpha \in M_d(RG)$, we write $\operatorname{tr} \alpha$ for the sum of the diagonal entries of α.

We cite [**6**] as a general reference.

2. Idempotents. For a ring A, we set

$$[A, A] = \{\sum(a_i b_i - b_i a_i) \colon a_i, b_i \in A\}.$$

If A has prime characteristic p, it is well-known that if $a_1, a_2, \ldots, a_m \in A$ then

$$(\sum a_i)^p = \sum a_i^p + \beta$$

where $\beta \in [A, A]$. We need the following analogue at characteristic p^n.

LEMMA 1. *Let A be a ring of prime-power characteristic p^n. If k is an integer, $k \geqq n$, and $a_1, a_2, \ldots, a_m \in A$, then for $s = p^{n-1}$ we have*

$$(\sum a_i)^{p^k} = \beta + \sum (a_{i_1} a_{i_2} \ldots a_{i_s})^{p^{k-n+1}}$$

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where the sum on the right is over all s-tuples $(i_1, i_2, \ldots, i_s)$ *with* $1 \leqq i_j \leqq m$, *and where* $\beta \in [A, A]$.

Proof. Set $t = p^k$. Then

$$(\sum a_i)^{p^k} = \sum a_{i_1} a_{i_2} \ldots a_{i_t}$$

where the sum on the right is over all t-tuples $(i_1, i_2, \ldots, i_t)$ with $1 \leqq i_j \leqq m$. Let $\sigma(a_{i_1} a_{i_2} \ldots a_{i_t}) = a_{i_2} a_{i_3} \ldots a_{i_t} a_{i_1}$; then if $\sigma(x) = y$, we have $x - y \in [A, A]$. If x is not of the form $(a_{i_1} a_{i_2} \ldots a_{i_s})^{p^{k-n+1}}$, then

$$\sigma^{p^{n-1}}(x) \neq x,$$

and so the σ-orbit containing x has a multiple of p^n members, each congruent to x modulo $[A, A]$. This completes the proof.

The following result is proved, at characteristic p, by Formanek [**5**, Lemmas 6 and 7].

LEMMA 2. *Let R be a commutative ring of prime-power characteristic* p^n. *Let G be a group with the property that if* $x \in G$ *has infinite order and* $x \sim x^{p^i}$ *for some i, then* $i = 0$. *Let* $e = \sum_{i=1}^{m} a_i g_i$ *be an idempotent element of RG. If x has infinite or p-power order, then* $t_x e = 0$.

Proof. Let $s = p^{n-1}$. Then by Lemma 1, we have, for any integer $k > n$,

$$e = e^{p^k} = \beta + \sum (a_{i_1} a_{i_2} \ldots a_{i_s})^{p^{k-n+1}} (g_{i_1} g_{i_2} \ldots g_{i_s})^{p^{k-n+1}},$$

where $\beta \in [RG, RG]$, and the sum is over all s-tuples $(i_1, i_2, \ldots, i_s)$ with $1 \leqq i_j \leqq m$. Pick k sufficiently large so that no $(g_{i_1} g_{i_2} \ldots g_{i_s})^{p^{k-n+1}}$ is conjugate to x. Then since $t_x \beta = 0$, the proof is complete.

We define, for an ideal I of a ring,

$$I^{\omega} = \bigcap_{n=1}^{\infty} I^n \quad \text{and} \quad I^{\omega^{n+1}} = (I^{\omega^n})^{\omega}.$$

LEMMA 3 ([**7**, I.3.15]). *Let H be a poly-infinite-cyclic group with Hirsch number n. Then for a field K, we have* $\Delta(H)^{\omega^n} = 0$.

Proof. Pick $H_1 \lhd H$, with H/H_1 infinite cyclic, and H_1 poly-infinite-cyclic. Then $\Delta(H/H_1)^{\omega} = 0$. Now use induction on n.

We can now prove our result on idempotents. This is proved, at characteristic 0, by Formanek [**5**, Theorem 1].

THEOREM 1. *Let G be a polycyclic-by-finite group with only p-torsion and let K be a field of characteristic p. If* $e \in KG$ *and* $e^2 = e$, *then e is* 0 *or* 1.

Proof. We claim that it suffices to prove the theorem for finite K. For if $e = \sum_{i=1}^{m} a_i g_i$ with each $a_i \neq 0$, then [**6**, 2.2.6] there exists a valua-

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tion ring A in K containing all the a_i and a homomorphism ϕ from A into an algebraic closure of $GF(p)$ such that each $\phi(a_i) \neq 0$. Then $\sum\phi(a_i)g_i$ is an idempotent in the group ring of G over the finite field generated over $GF(p)$ by all the $\phi(a_i)$. We now assume that K is finite. There exists a discrete valuation ring R of characteristic zero, unramified over the p-local integers $\mathbf{Z}_{(p)}$, such that $R/pR = K$ [**8**, II, Theorem 3].

We have $e(1 - e) = 0$, so we may assume that the augmentation of e is 0, that is, $e \in \Delta(G)$. We may lift e to an idempotent e_n of $(R/p^nR)G$ by [**6**, 2.3.7], since $(pR/p^nR)G$ is a nilpotent ideal of $(R/p^nR)G$. We choose e_n so that e_{n+1} is a lifting of e_n, for $n > 1$. Let H be a normal poly-infinite-cyclic subgroup of G of finite index, and let $\bar{e}_n$ denote the image of e_n in $(R/p^nR)(G/H)$. Then if $e_n = \sum a_g g$, with $a_g \in R/p^nR$, $g \in G$, we have

$$t_1\bar{e}_n = \sum_{g\in H} a_g = \sum_h t_h e_n,$$

where the sum on the right is over certain $h \in H$. By Lemma 2, $t_h e_n = 0$ if $h \neq 1$, whence $t_1\bar{e}_n = t_1 e_n$. Now for $g \in G$, $g \neq 1$, we have $t_g e_n = 0$, and since the augmentation of e_n is 0, it follows that $t_1 e_n = t_1\bar{e}_n = 0$. Let

$$\epsilon = \lim \bar{e}_n \in \varprojlim (R/p^nR)(G/H).$$

Then $t_1\epsilon = 0$, and since $\varprojlim (R/p^nR)$ is an integral domain of characteristic 0, and G/H is finite, this implies that $\epsilon = 0$. Thus e is in the kernel of $KG \to K(G/H)$, namely $KG\Delta(H)$. However for large n,

$$e \in (KG\Delta(H))^{\omega^n} = KG(\Delta(H)^{\omega^n})$$

which is 0 by Lemma 3. This completes the proof.

3. Euler characteristics. We state some elementary facts about Euler characteristics of projective modules over group rings. Proofs can be found in Chapter 13, § 4 of [**6**], and in [**1**].

Let R be a commutative ring and let P be a finitely generated projective RG-module. Choose a projective module Q such that $P \oplus Q$ is free over RG, of finite rank, say d. Let $\alpha: P \oplus Q \to P \oplus Q$ be the projection onto P, and let $e \in M_d(RG)$ be a matrix which represents α with respect to some ordered basis of $P \oplus Q$. Thus $e^2 = e$. Define the Euler characteristic of P, denoted $\chi(P)$, to be $t_1(\operatorname{tr} e)$; this is independent of the choice of Q and of e. If H is a subgroup of G of finite index, then the restriction of P to RH, denoted by P_H, is finitely generated and projective over RH, and

$$\chi(P_H) = [G: H]\chi(P).$$

If G is finite and R is local, then P_1 is free, and $\chi(P_1)$ is the rank of P_1 over R.

4. Zero divisors. We need a generalization of Lemma 2, proved at characteristic p by Farkas-Snider [**4**] (see [**6**], Lemma 13.4.15).

LEMMA 4. *Let R and G be as in Lemma 2. Let $\alpha = (\alpha(i, j))$ be an idempotent element of $M_d(RG)$. If x has infinite or p-power order, then*

$$t_x(\operatorname{tr} \alpha) = 0.$$

Proof. Let $s = p^{n-1}$ and let

$$X = \bigcup \operatorname{supp}\{\alpha(i_1, i_2)\alpha(i_2, i_3) \ldots \alpha(i_s, i_1)\}$$

where the union is over all s-tuples $(i_1, i_2, \ldots, i_s)$ with $1 \leqq i_j \leqq d$. Let

$$Y = \{g_1 g_2 \ldots g_s : g_i \in X, 1 \leqq i \leqq s\},$$

so Y is a finite subset of G. It follows from the hypothesis on G that $x \sim y^{p^t}$ for some $y \in Y$ for only finitely many integers t. Let t_0 be the largest such t, and let k be an integer such that $k - 2n + 2 > t_0$.

Let $\{e_{ij}\}$ be the matrix units of $M_d(RG)$, so that $\alpha = \sum_{i,j} \alpha(i, j)e_{ij}$. From Lemma 1, we have

$$\alpha = \alpha^{p^k} = \beta + \sum (\alpha(i_1, j_1)\alpha(i_2, j_2) \ldots \alpha(i_s, j_s))^{p^{k-n+1}} (e_{i_1 j_1} \ldots e_{i_s j_s})^{p^{k-n+1}},$$

where the sum is over all s-tuples of pairs $((i_1, j_1), \ldots, (i_s, j_s))$, with $1 \leqq i_h, j_h \leqq d$, and $\beta \in [M_d(RG), M_d(RG)]$. Using the facts that $\operatorname{tr} \beta = 0$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta, we have

$$\operatorname{tr} \alpha = \sum (\alpha(i_1, i_2)\alpha(i_2, i_3) \ldots \alpha(i_s, i_1))^{p^{k-n+1}}$$

where the sum is over all s-tuples $(i_1, \ldots, i_s)$ with $1 \leqq i_j \leqq d$. Consider a typical term in this sum, and suppose that

$$\alpha(i_1, i_2)\alpha(i_2, i_3) \ldots \alpha(i_s, i_1) = \sum_{j=1}^{m} a_j g_j \in RG.$$

Then

$$(\sum a_j g_j)^{p^{k-n+1}} = \gamma + \sum (a_{j_1} a_{j_2} \ldots a_{j_s})^{p^{k-2n+2}} (g_{j_1} g_{j_2} \ldots g_{j_s})^{p^{k-2n+2}},$$

$\gamma \in [RG, RG]$, and the sum is over all s-tuples $(j_1, j_2, \ldots, j_s)$ with $1 \leqq j_l \leqq m$. By our choice of k, we know that $(g_{j_1} g_{j_2} \ldots g_{j_s})^{p^{k-2n+2}}$ is never conjugate to x, and since $t_x(\gamma) = 0$, we conclude that $t_x(\operatorname{tr} \alpha) = 0$.

THEOREM 2. *Let G be a torsion free polycyclic-by-finite group and let K be a field of characteristic $p > 0$. Then KG has no zero divisors.*

Proof. As in the proof of Theorem 1, we may assume that K is finite, and that $K = R/pR$ where R is an integral domain of characteristic 0. Fix an integer $n > 1$ and set $S = R/p^n R$.

Now Theorem 13.4.11 of [**6**] (which is a version of Theorem 1 of Farkas-Snider [**4**]) states the following: KG has no zero divisors provided that for every finitely generated projective KG-module P, and any poly-infinite-cyclic normal subgroup H of G of finite index, we have that $[G:H]$ divides $\dim_K(P_H/\Delta(H)P_H)$. Accordingly, let P be a finitely generated projective KG-module, and let $e \in M_d(KG)$ be an idempotent matrix such that $\chi(P) = t_1(\text{tr } e)$. Since $pM_d(SG)$ is a nilpotent ideal of $M_d(SG)$, and

$$M_d(SG)/pM_d(SG) \simeq M_d(SG/pSG) \simeq M_d(KG)$$

then [**6**, 2.3.7] e may be lifted to an idempotent matrix $e' \in M_d(SG)$. If

$$\pi: M_d(SG) \rightarrow M_d(KG)$$

is the extension of the natural map $S \rightarrow S/pS = K$, then $\pi(e') = e$. Let P' be the projective SG-module given by the kernel of

$$1 - e': (SG)^d \rightarrow (SG)^d,$$

so $\chi(P') = t_1(\text{tr } e')$. Let H be a normal poly-infinite-cyclic subgroup of G of finite index; then

(1) $\chi(P_H') = [G: H]\chi(P')$.

We claim that $\chi(P'/\Delta(G)P') = \chi(P')$. Let $\text{tr } e' = \sum a_g g \in SG$. Then

$$\chi(P'/\Delta(G)P') = \sum a_g = a_1 + \sum_{x_i} \sum_{g\sim x_i} a_g$$

for certain $x_i \in G$. Since G satisfies the hypothesis of Lemma 4, we deduce that $\sum_{g\sim x_i} a_g = 0$ for each x_i, and therefore

$$\sum a_g = a_1 = \chi(P'),$$

and our claim is valid. By the same argument, we have

$$\chi(P_H'/\Delta(H)P_H') = \chi(P_H').$$

Therefore (1) becomes

(2) $\chi(P_H'/\Delta(H)P_H') = [G: H]\chi(P'/\Delta(G)P')$.

Now $P_H'/\Delta(H)P_H'$ is a finitely generated projective S-module, and is therefore free since S is local. Moreover,

$$\chi(P_H'/\Delta(H)P_H') = \text{rank}_S(P_H'/\Delta(H)P_H') = \dim_K(P_H/\Delta(H)P_H)$$

and

$$\chi(P'/\Delta(G)P') = \text{rank}_S(P'/\Delta(G)P') = \dim_K(P/\Delta(G)P)$$

where the right sides of these equations are interpreted as elements of

S. We then have, from (2),

$$\dim_K(P_H/\Delta(H)P_H) \equiv [G: H] \dim_K P/\Delta(G)P \pmod{p^n}$$

and since n was arbitrary, this congruence may be replaced by an equality. The theorem now follows from Theorem 1 of [**4**].

5. Idempotents again. We remark that the proof of Theorem 2 may be used to give an alternate proof of Theorem 1. For if $e \in KG$ is an idempotent in $\Delta(G)$, we have, with $P = KGe$,

$$\dim_K((KGe)_H/\Delta(H)(KGe)_H) = [G: H]\dim_K(KGe/\Delta(G)KGe).$$

Since $e \in \Delta(G)$, the right side is 0; hence so is the left side, and $e \in KG\Delta(H)$, which implies that $e = 0$ from Lemma 3.

Our final result was proved at characteristic 0 in [**3**, Theorem 2].

THEOREM 3. *Let R be a commutative ring of characteristic $n > 0$, having no idempotent other than 0 or 1. Let G be a polycyclic-by-finite group, having no element $\neq 1$ whose order is a unit of R. Then RG has no nontrivial idempotent.*

Proof. Let $e \in RG$ be a nontrivial idempotent. Since R has no nontrivial idempotent, its characteristic must be a p-power for some prime p. We may factor out the nil radical of R, and thus assume that R has no nilpotent element; in particular, R has characteristic p. We may further assume that R is generated (as a ring) by the finitely many coefficients of $e \in RG$, so R is Noetherian, and

$$R \subset \prod_{i=1}^{m} F_i,$$

a direct product of fields of characteristic p. Then $RG \subset \prod F_iG$, and by Theorem 1, we have

$$e = (1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \prod F_iG.$$

Let I be the ideal of R generated by the coefficients of e. Then $I^2 = I$, so

$$\bigcap_{n=1}^{\infty} I^n = I;$$

by Krull's Theorem, [**9**, p. 216, Theorem 12], there exists $x \in I$ with $I(1 - x) = 0$. Then $x^2 = x$, so x is 0 or 1. Since $e \neq 0$, then $x \neq 0$. Therefore $x = 1$ so $I = R$, which is impossible, since

$$e = (1, 1, \ldots, 1, 0, \ldots, 0) \neq (1, 1, \ldots, 1, \ldots, 1).$$

This completes the proof.

REFERENCES

1. H. Bass, *Euler characteristics and characters of discrete groups*, Inv. Math. *35* (1976), 155–196.
2. K. A. Brown, *On zero divisors in group rings*, Bull. London Math. Soc. *8* (1976), 251–256.
3. G. Cliff and S. Sehgal, *On the trace of an idempotent in a group ring*, Proc. Amer. Math. Soc. *62* (1977), 11–14.
4. D. Farkas and R. Snider, K_0 *and Noetherian group rings*, J. Algebra *42* (1976), 192–198.
5. E. Formanek, *Idempotents in Noetherian group rings*, Can. J. Math. *15* (1973), 366–369.
6. D. Passman, *The algebraic structure of group rings* (Wiley-Interscience, New York, 1977).
7. S. Sehgal, *Topics in group rings* (Marcel Dekker, New York, 1978).
8. J.-P. Serre, *Corps locaux* (Hermann, Paris, 1962).
9. O. Zariski and P. Samuel, *Commutative algebra* (Vol. I) (Van Nostrand, Princeton, 1958).

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