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The hyper-archimedean kernel sequence of a lattice-ordered group

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The *hyper-archimedean kernel* $Ar(G)$ of a lattice-ordered group (hence forth l-group) is the largest hyper-archimedean convex l-subgroup of the l-group G . One defines $Ar^{\sigma}(G)$, for an ordinal σ as $\bigcup_{\alpha<\sigma} Ar^{\alpha}(G)$ if σ is a limit ordinal, and as the unique l-ideal with the property that

$$Ar^{\sigma}(G)/Ar^{\sigma-1}(G) = Ar\left(G/Ar^{\sigma-1}(G)\right) ,$$

otherwise. The resulting "Loewy"-like sequence of characteristic l-ideals, $Ar(G) \subseteq Ar^{2}(G) \subseteq \ldots \subseteq Ar^{\sigma}(G) \subseteq \ldots$, is called the *hyper-archimedean kernel sequence*. The first result of this note says that each $Ar^{\sigma}(G) \subseteq Ar(G)''$.

Most of the paper concentrates on archimedean l-groups; in particular, the hyper-archimedean kernels are identified for: $D(X)$, where X is a Stone space, a large class of free products of abelian l-groups, and certain l-subrings of a product of real groups.

It is shown that even for archimedean l-groups the hyper-archimedean kernel sequence may proceed past $Ar(G)$.

1. Introduction

The purpose of this note is to derive structure of an archimedean

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l-group using the notion of the hyper-archimedean kernel sequence defined in [8]. Our general terminology and notation is standard, as in [3]; the special notions to be discussed here are in the notation of [8].

An l-group H is *hyper-archimedean* if it is archimedean and every l-homomorphic image of H is archimedean. The following theorem encapsules the basic facts about the structure of hyper-archimedean l-groups. Many individuals have contributed to this well known theorem; for a fairly complete history see Theorem 1.1 in [5].

THEOREM 1.1. *For an l-group G the following are equivalent:*

(1) *G is hyper-archimedean;*

(2) *every proper prime subgroup of G is maximal, and hence minimal;*

(3) *the regular subgroups of G form a trivially ordered set;*

(4) *$G = G(g) \boxplus g'$, for each $g \in G$;*

(5) *if $0 < a$, $b \in G$ then $[a-(mb \wedge a)] \wedge b = 0$, for some positive integer m;*

(6) *if $0 < a$, $b \in G$ then $a \wedge nb = a \wedge (n+1)b$, for some positive integer n;*

(7) *G is l-isomorphic to an l-subgroup G' of $\prod\{R_i \mid i \in I\}$ so that for all $0 < x$, $y \in G'$, there exists an $n > 0$ such that $nx_i > y_i$ whenever $x_i > 0$. ($R_i = R$, the additive group of reals with the usual ordering, for each $i \in I$.)*

NOTES. (a) With reference to the notation in (4), if $x \in G$, $G(x)$ denotes the convex l-subgroup generated by x. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups then $G = \boxplus\{G_\lambda \mid \lambda \in \Lambda\}$ is the direct sum of the G_λ with coordinatewise ordering.

If x is a subset of an l-group G,

$$X' = \{g \in G \mid |g| \wedge |x| = 0 \text{, for all } x \in X\}$$

is the *polar* of X; $g' \equiv \{g\}' = G(g)'$.

(b) It should be noted that Conrad calls hyper-archimedean l-groups *epi-archimedean*; see [5].

If G is an l-group there is a convex l-subgroup $Ar(G)$ which is hyper-archimedean and contains every hyper-archimedean convex l-subgroup of G. $Ar(G)$ is characteristic; that is, invariant under all l-automorphisms of G, and $0 < g \in Ar(G)$ if and only if all its values are minimal prime subgroups. Further $Ar(G)$ is the intersection of all non-minimal primes of G. We call $Ar(G)$ the *hyper-archimedean kernel* of G, henceforth to be abbreviated *h.a. kernel*. It was first introduced and characterized as indicated in the lines of this paragraph in [8] by the author for representable l-groups; then in [5] Conrad removed the author's assumption of representability.

If σ is an ordinal, define $Ar^{\sigma}(G)$ as follows:

(a) $Ar^{\sigma}(G)/Ar^{\sigma-1}(G) = Ar\left(G/Ar^{\sigma-1}(G)\right)$, if σ is not a limit ordinal;

(b) $Ar^{\sigma}(G) = \bigcup_{\alpha<\sigma} Ar^{\alpha}(G)$, otherwise.

Then $Ar(G) \subseteq Ar^{2}(G) \subseteq \ldots \subseteq Ar^{\sigma}(G) \subseteq \ldots$, and all entries in this sequence are characteristic l-ideals. This is the *hyper-archimedean kernel sequence* (henceforth *h.a. kernel sequence*).

The following was not defined in [8]: by a standard cardinality argument $Ar^{\tau}(G) = Ar^{\tau+1}(G)$ for a suitable large ordinal τ. We define $Ar^{*}(G) = \bigcup_{\sigma} Ar^{\sigma}(G)$; thus $Ar^{*}(G) = Ar^{\tau}(G)$ for some ordinal τ.

THEOREM 1.2. *For any l-group G, $Ar^{*}(G) \subseteq Ar(G)''$.*

Proof. If suffices to show that if $Ar^{\sigma}(G) \subseteq Ar(G)''$ then $Ar^{\sigma+1}(G) \subseteq Ar(G)''$. If $Ar^{\sigma}(G) \subseteq Ar(G)''$ then $Ar^{\sigma}(G)' = Ar(G)'$.

So suppose $0 < x \in Ar^{\sigma+1}(G) \cap Ar(G)'$; then the values of $x + Ar^{\sigma}(G)$ are minimal prime subgroups of $G/Ar^{\sigma}(G)$. Any such value is of the form $N/Ar^{\sigma}(G)$ where N is a prime subgroup of G. Either N is itself a minimal prime of G, or else it contains a minimal prime subgroup

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M of G, and then $M \not\supseteq Ar^{\sigma}(G)$. We may then select $y \in Ar^{\sigma}(G)\backslash M$; by our assumption about σ, $x \wedge y = 0$, and this is absurd.

Therefore each prime subgroup N of G so that $N/Ar^{\sigma}(G)$ is a value of $x + Ar^{\sigma}(G)$, is a minimal prime of G, proving that $x \in Ar(G)$. This is once again a contradiction. Hence $Ar^{\sigma+1}(G) \cap Ar(G)' = 0$; that is, $Ar^{\sigma+1}(G) \subseteq Ar(G)''$ as promised.

2. The h.a. kernel sequence applied to archimedean l-groups

The central question here is naturally: how long can the h.a. kernel sequence be? Obviously, if one makes no restrictions on the types of l-groups one wishes to consider the answer is: as long as one pleases. Simply specify an ordinal σ and then construct a long enough lexicographic product of copies of the reals.

So let us ask the question again for archimedean l-groups. Let us in fact ask: if G is an archimedean l-group, is $Ar^*(G) = Ar(G)$? The answer is not, but most archimedean l-groups one considers have, in this sense, a trivial h.a. kernel sequence.

It is useful to start with the following characterization of $Ar(G)$.

LEMMA 2.1. *Suppose G is a representable l-group; $0 < x$ is in $Ar(G)$ if and only if for each $0 < a \in G$ there is a positive integer n so that $x \wedge na = x \wedge (n+1)a$.*

Proof. Suppose $0 < x \in Ar(G)$ and $0 < a \in G$; then $x \wedge a$ is in $Ar(G)$, so by Theorem 1.1 (6), $x \wedge n(x\wedge a) = x \wedge (n+1)(x\wedge a)$, for a suitable positive integer n. Since $k(x\wedge a) = kx \wedge ka$ in a representable l-group for all $k \geq 1$, we get $x \wedge n(x\wedge a) = x \wedge nx \wedge na = x \wedge na$, so that $x \wedge na = x \wedge (n+1)a$.

Conversely, if $x \wedge na = x \wedge (n+1)a$, for all $0 < a \in G$, and an appropriate $n = n(a)$, then $G(x)$ is hyper-archimedean by Theorem 1.1. Consequently, $x \in Ar(G)$

COROLLARY 2.1.1. *If G is representable,* $Ar(G) = \bigcap_{0<a} [G(a) \boxplus a']$.

Proof. By our lemma, $0 < x \in Ar(G)$ if and only if whenever

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$0 < a \in G$, $x \wedge na = x \wedge (n+1)a$, for a suitable n . This equation is valid if and only if $[x-(na \wedge x)] \wedge a = 0$; that is, if and only if $x - (na \wedge x) \in a'$. Since $na \wedge x \in G(a)$, it is clear that $0 < x \in AR(G)$ if and only if $x \in G(a) \boxplus a'$ for all $0 < a \in G$.

Now let us have a look at a few examples.

(1) $G = \prod\{R_\lambda \mid \lambda \in \Lambda\}$, where $R_\lambda = R$ for each $\lambda \in \Lambda$. From Lemma 2.1 it is clear that $Ar(G) = \boxplus_\lambda R_\lambda$. Now we wish to identify $Ar^2(G)$, so we look at $Ar(G/Ar(G))$: if $0 < x + Ar(G) \in Ar(G/Ar(G))$ then each value of x is either a minimal prime of G or else properly contains a minimal prime $M \not\supseteq Ar(G)$. However, each such minimal prime M is of the form $G_\lambda = \{g \in G \mid g_\lambda = 0\}$, since M will be the value of an element of $Ar(G)$. Thus M is maximal, giving us a contradiction. It follows that every value of x is a minimal prime, putting $x \in Ar(G)$, again a contradiction. The conclusion is then $Ar(G/Ar(G)) = 0$; that is, $Ar^2(G) = Ar(G) = Ar^*(G)$.

(2) $G = \prod\{Z_\lambda \mid \lambda \in \Lambda\}$, where $Z_\lambda = Z$, the additive group of integers with the usual ordering. Again using Lemma 2.1 we can see that $Ar(G)$ is the l-ideal of bounded integral functions. That $Ar(G/Ar(G)) = 0$ can be seen as follows. If $0 < x + Ar(G) \in Ar(G/Ar(G))$ then x is unbounded and - taking $x > 0$ without loss of generality - we can find a sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \in \Lambda$ such that the x_{λ_n} diverge. Define $u \in G$ as follows: u_{λ_i} is the largest integer $\le \sqrt{x_{\lambda_i}}$, for all $i = 1, 2, \ldots$, and $u_\lambda = 0$ otherwise; then $u \notin Ar(G)$.

For each positive integer m , $x \wedge (m+1)u - x \wedge mu$ is unbounded: note that

$$[x \wedge (m+1)u - x \wedge mu]_{\lambda_i} = \begin{cases} 0 & \text{, if } mu_{\lambda_i} \ge x_{\lambda_i} ; \\ x_{\lambda_i} - mu_{\lambda_i} & \text{, if } (m+1)u_{\lambda_i} \ge x_{\lambda_i} > mu_{\lambda_i} ; \\ u_{\lambda_i} & \text{, if } x_{\lambda_i} > (m+1)u_{\lambda_i} . \end{cases}$$

For each m, there is an $i = 1, 2, \ldots$ such that $(m+1)^2 < x_{\lambda_j}$, for all $j \geq i$. It is easy to see that this implies that $(m+1)u_{\lambda_j} < x_{\lambda_j}$, when $j \geq i$. It should now be clear that $x \wedge (m+1)u - x \wedge mu$ is indeed unbounded.

This is a contradiction, for according to Lemma 2.1 there is an $m > 0$ so that $(x \wedge mu) + Ar(G) = (x \wedge (m+1)u) + Ar(G)$. We conclude therefore that $Ar(G/Ar(G)) = 0$.

THEOREM 2.2. *Let G be an l-subring of $\prod R_\lambda$ (with $R_\lambda = R$ for each $\lambda \in \Lambda$) consisting of bounded functions. Then $Ar(G)$ is the subgroup generated by*

$$T = \{0 < g \in G \mid \text{g.l.b.}[g_\lambda \mid g_\lambda > 0] > 0 ,$$

$$\text{and each positive element } h < g \text{ also has this property}\} .$$

Moreover, $Ar(G/Ar(G)) = 0$.

Proof. From Lemma A in [3] it is clear that if $0 < g \in T$ then $g \in Ar(G)$. Conversely, suppose $0 < g \in Ar(G)$ but $\text{g.l.b.}[g_\lambda \mid g_\lambda > 0] = 0$; then we can find a sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ in Λ such that $\lim_{n\to\infty} g_{\lambda_n} = 0$. Let $s = g^2$; without any loss of generality we assume each $g_{\lambda_i} < 1$. By Lemma 2.1 there is an $m > 0$ so that $g \wedge ms = g \wedge (m+1)s$. For all but finitely many λ_i, $g_{\lambda_i} < 1/(m+1)$; thus $(m+1)s_{\lambda_i} = (m+1)g^2_{\lambda_i} < g_{\lambda_i}$. So $(g \wedge ms)_{\lambda_i} = ms_{\lambda_i}$ and $(g \wedge (m+1)s)_{\lambda_i} = (m+1)s_{\lambda_i}$, and then $g \wedge ms < g \wedge (m+1)s$, a contradiction. Therefore, $\text{g.l.b.}[g_\lambda \mid g_\lambda > 0] > 0$, and clearly $g \in T$.

Suppose now by way of contradiction that $0 < g + Ar(G)$ in $Ar(G/Ar(G))$. Then either $\text{g.l.b.}[g_\lambda \mid g_\lambda > 0] = 0$ or some element below g has this property. Without loss of generality we take $g > 0$ and $\text{g.l.b.}[g_\lambda \mid g_\lambda > 0] = 0$. We use the notation of the previous paragraph: $\lim_{n\to\infty} g_{\lambda_n} = 0$. By setting $s = g^2$ once more, notice that for each $m > 0$,

$(g\wedge(m+1)s-g\wedge ms)_{\lambda_i} = s_{\lambda_i}$ for all but finitely many λ_i . Since $\lim_{n\to\infty} s_{\lambda_n} = 0$ we have that $g \wedge (m+1)s - g \wedge ms \notin Ar(G)$; moreover $s \notin Ar(G)$, hence $g \wedge (m+1)s + Ar(G) > g \wedge ms + Ar(G)$, for all $m > 0$, contradicting the hypothesis that $g + Ar(G) \in Ar(G/Ar(G))$. Plainly then $Ar(G/Ar(G)) = 0$.

Let us continue with our examples.

(3) Suppose G is a free abelian l-group on two or more generators; Bleier [1] has shown that G has no non-trivial characteristic l-ideals. Since G is obviously not hyper-archimedean $Ar(G) = 0 = Ar^*(G)$.

(4) Let $G = C(X)$, the group of all real valued continuous functions on a compact, connected Hausdorff space X . It is a consequence of Theorem 2.2 that $Ar(G) = 0$; for if $0 < g \in G$ and g.l.b.$[g(x) \mid g(x) > 0] > 0$ then $g(x) > 0$, for all $x \in X$. To see this let $m =$ g.l.b.$[g(x) \mid g(x) > 0]$ and $U = \{x \in X \mid g(x) < m\}$; then $U = \{x \in X \mid g(x) = 0\}$, which implies that U is both open and closed. This is a contradiction unless U is void.

Now, if $0 < g \in Ar(G)$ we may assume without loss of generality that $g(x) \geq 1$ for all $x \in X$. Select two distinct points $a, b \in X$. By Urysohn's Lemma there is a continuous function $f \in G$ so that $f(X) \subseteq [0, 1]$, and $f(a) = 0$ while $f(b) = 1$. $0 < f \leq g$, and by our arguments of the previous paragraph g.l.b.$[f(x) \mid f(x) > 0] = 0$. This is a contradiction, and so $Ar(G) = 0$ as we had claimed.

(5) Let $G = Z \coprod Z$, the free product as abelian l-groups of two copies of Z . By Theorem 2.8 of [8], G is isomorphic to the l-group of continuous functions on $[0, 1]$ generated by $f(x) = x$ and $g(x) = 1 - x$. Applying Lemma 2.1 directly, $Ar(G) = 0$.

We shall return to this example shortly.

Next, we shall take a look at $D(X)$, the l-group of almost finite continuous functions from a Stone space X into the extended reals. (Recall: A *Stone space* is a compact, Hausdorff, extremally disconnected space.) We need to define a crucial concept first: a point p in a topological space X is a *p-point*, if whenever f is a real valued continuous function on X and $f(p) = 0$, then $f = 0$ on a neighbourhood of p . If f is a real valued continuous function on X , let $\operatorname{supp}(f)$ stand for the set $\{x \in X \mid f(x) \neq 0\}$.

THEOREM 2.3. *Let X be a Stone space and $G = D(X)$. Then* $Ar(G) = \{f \in G \mid \operatorname{supp}(f)$ *is closed and consists of* p*-points*$\}$. $Ar(G/Ar(G)) = 0$.

Proof. Suppose first that $0 < f \in G$ and $\operatorname{supp}(f)$ is a closed set consisting of p-points. Let

$$P_y = \{g \in G \mid g = 0 \text{ on a neighbourhood of } y\} ,$$

with $y \in X$; by Proposition 3.1 in [2] these are precisely the minimal primes of G . So if $f \notin P_y$ then $f(y) > 0$, and y is a p-point, or else $f(y) = 0$ but every neighbourhood of y contains a point of $\operatorname{supp}(f)$; that is, $y \in \overline{\operatorname{supp}(f)}$. This contradicts our hypothesis, and hence $f(y) > 0$. Using Theorem 3.11 in [2], P_y is a maximal l-ideal and hence a value of f ; clearly $f \in Ar(G)$.

Conversely, suppose $0 < f \in Ar(G)$ yet $f(z) > 0$ at the non p-point $z \in X$. Without loss of generality we may suppose $f(z) \geq 1$ since $Ar(G)$ is a real subspace of G . Let $V = \{x \in X \mid f(x) > 1/2\}$; then V is a neighbourhood of z . Since z is not a p-point there is a function $0 < g \in G$ such that $g(z) = 0$ yet each neighbourhood U of z contains a point s with $g(s) > 0$.

Let $V_n = \{x \in X \mid g(x) < 1/n\} \cap V$; V_n is a neighbourhood of z , so we may select an $s_n \in V_n$ such that $g(s_n) > 0$. Then $\lim_{n\to\infty} g(s_n) = 0$ while $f(s_n) > 1/2$, for all $n = 1, 2, \ldots$. Since $f \in Ar(G)$ there should be a positive integer k so that $f \wedge kg = f \wedge (k+1)g$; yet for each k , $(k+1)/n < 1/2$ if n is large enough. Thus $kg(s_n) < (k+1)g(s_n) < (k+1)/n < 1/2 < f(s_n)$, so that $(f\wedge kg)(s_n) < (f\wedge(k+1)g)(s_n)$; this is a contradiction. We conclude that f vanishes at all non p-points.

If $x \in \overline{\operatorname{supp}(f)}$ while $f(x) = 0$, there is a sequence of p-points $\{t_n\}$ so that $\lim_{n\to\infty} f(t_n) = 0$, while each $f(t_n) > 0$ and finite. Using f^2 as in the proof of Theorem 2.2 one can obtain a contradiction to the

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supposition that $f \in Ar(G)$. It follows that $f(x) > 0$, and $\mathrm{supp}(f)$ is closed.

Next, suppose $0 < h + Ar(G) \in Ar(G/Ar(G))$ with $h > 0$; then either

(1) $h(x) > 0$ at some non p-point $x \in X$, or else

(2) $\overline{\mathrm{supp}(h)}$ contains a non p-point.

We leave the second case to the reader.

In the first case we may suppose as earlier in the proof that $h(x) \geq 1$ and let $V = \{t \in X \mid h(t) > 1/2\}$. Choose a positive function d so that $d(x) = 0$, yet each neighbourhood of x contains a point s for which $d(s) > 0$. Again let $V_n = \{t \in X \mid d(t) < 1/n\} \cap V$, and select $s_n \in V_n$ so that $d(s_n) > 0$; then $d \notin Ar(G)$ since $\overline{\mathrm{supp}(d)}$ is not closed. As earlier $h \wedge kd < h \wedge (k+1)d$, for each $k \geq 1$; further $[h\wedge(k+1)-h\wedge kd](x) = 0$.

Finally, if $h \wedge (k+1)d - h \wedge kd$ were in $Ar(G)$ it would be real valued. Also $[h\wedge(k+1)d-h\wedge kd](t_n) = d(t_n)$ for large enough n ; the latter sequence converges to 0 , so that one can once again use the squaring method of the proof of Theorem 2.2 to get a contradiction. Hence $h \wedge (k+1)d + Ar(G) > h \wedge kd + Ar(G)$ for all $k = 1, 2, \ldots$; this contradicts our initial assumption, so it follows that $Ar(G/Ar(G)) = 0$.

To conclude this section let us observe that if G is any l-group which is a subdirect product of l-groups whose h.a. kernel is zero, then $Ar(G) = 0$; (see Proposition 1.8 in [8]). This enables us to show:

PROPOSITION 2.4. *If A and B are abelian l-groups and $G = A \coprod B$, the free product as abelian l-groups, then if G is a subdirect product of integers,* $Ar(G) = 0$.

Proof. By the proof of Proposition 3.4 in [7], G is then a subdirect product of copies of $Z \coprod Z$, whose h.a. kernel is zero (Example 5).

NOTE. G satisfies the hypotheses of Proposition 2.4 if A and B are both hyper-Z l-groups; recall from [8] that an l-group is *hyper-Z* if it is a subdirect product of integers and each l-homomorphic image has the same property.

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3. Two examples

Let us record the following result, Proposition 1.10 in [8].

THEOREM 3.1. *If* G *is a subdirect product of integers, say* $G \subseteq \prod\{Z_\lambda \mid \lambda \in \Lambda\}$ *, and* G *contains a bounded weak order unit, then* $\mathcal{A}r(G)$ *consists of all the bounded functions in* G *.*

(Recall that $0 < e \in G$ is a *weak order unit* if $e \wedge g > 0$ for all $0 < g \in G$.)

In [4] Conrad showed that a free abelian l-group on two or more generators had the property that in every representation as a subdirect product of integers there were no non-zero bounded functions. The question was then raised by him of how close this came to characterizing free abelian l-groups.

Consider a free product $G = A \coprod B$ of two abelian l-groups so that G is a subdirect product of integers. According to Proposition 2.4, $\mathcal{A}r(G) = 0$; moreover, in any subdirect product of integers a bounded functions is in the h.a. kernel. It follows that G has no non-zero bounded function in any representation by integers. A and B can be selected so that G is not free; for example let[1] $A = B = Z$.

Theorem 3.1 leaves open the question of what $\mathcal{A}r(G/\mathcal{A}r(G))$ is; we give an example of a subdirect product of integers so that $\mathcal{A}r(G) \subset \mathcal{A}r^2(G) = G$, and $\mathcal{A}r(G)$ is a prime subgroup.

Let $H = \prod_{n=1}^{\infty} Z_n$; $Z_n = Z$, for each $n = 1, 2, \ldots$. Let G be the l-subgroup generated by $H(u)$ and v , where $u = (1, 1, \ldots)$ and $v = (1, 2, 3, 4, 5, \ldots)$. By Theorem 3.1, $\mathcal{A}r(G) = H(u)$. It is not too hard to show that if $x \in H$, then $x \in G$ if and only if $x - nv$ is bounded for a suitable integer n . It is evident then that $G/\mathcal{A}r(G) \simeq Z$, so that $G = \mathcal{A}r^2(G)$.

This example also indicates how to construct an example of a subdirect

[1] The argument can also be presented by quoting Theorem 3.3 in [7], to the effect that these free products have no singular elements, and then using a result of Conrad in [4]: if a subdirect product of integers has no singular elements, then it has no non-zero bounded functions.

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product of integers G so that $Ar^m(G) = G$ and $Ar^{m-1}(G) \subset G$, for any predetermined integer m. Once again let $H = \prod_{n=1}^{\infty} Z_n$, $u = (1, 1, \ldots)$ and $v_k = (1, 2^k, 3^k, 4^k, \ldots)$, $1 \le k \le m-1$. Then define G to be the l-subgroup of G generated by $H(u)$ and $\{v_1, v_2, \ldots, v_m\}$.

4. Parting comments

It would be nice if the h.a. kernel were well behaved with respect to large subgroups; (recall that the l-subgroup H of G is *large* in G if for each non-zero convex l-subgroup K of G, $K \cap H \neq 0$). What we would like is to have $Ar(H) = H \cap Ar(G)$ if H is a large subgroup of G. Then we could use our theorem about the h.a. kernel of $D(X)$ to some advantage, in view of the so-called Bernau embedding theorem for archimedean l-groups. However, if $G = \prod_{n=1}^{\infty} R_n$, the l-group of all real sequences, and H is the l-subgroup of all eventually constant sequences, then $Ar(G) = \boxplus_{n=1}^{\infty} R_n$, while according to Theorem 2.2, $Ar(H) = H$; that is, H is hyper-archimedean. H is large in G, yet $Ar(H) \supset Ar(G) = Ar(G) \cap H$.

Another important question is the following. When is the h.a. kernel of an l-group dense in G? (Recall that the l-subgroup H of G is *dense* in G if for each $0 < g \in G$ there is an element $0 < h \le g$, with $h \in H$.) A convex l-subgroup A of G is dense in G if and only if $A'' = G$. So it is immediate from Theorem 1.2 that if $Ar^*(G) = G$ then $Ar(G)$ is dense in G.

If G is an archimedean l-group with basis then it is well known that G may be expressed as a subdirect product of reals in such a way that G contains the cardinal sum. Since the h.a. kernel of G contains this cardinal sum it follows that $Ar(G)$ is dense in G. However, our very first example shows that $Ar^*(G)$ may be a proper subgroup.

We should point out that if $Ar(G)$ is a cardinal summand of an l-group G, then $Ar^*(G) = Ar(G)$, but the converse is false.

This is a good place to mention a conjecture. If G is an archimedean l-group and $Ar(G)$ is dense (or large) in G, then G is a subdirect product of reals. In particular, if $Ar^*(G) = G$ the same conclusion is valid.

Finally, we mention two unpublished results of Conrad:

(a) if G is a finite valued l-group, then $Ar^*(G) = G$ if and only if the set of regular subgroups of G satisfies the descending chain condition;

(b) let Λ be a *root system*; that is, Λ is a p.o. set, and if $\lambda \parallel \mu$ in Λ they have no common lower bounds. Consider

$$V = V(\Lambda, R_\lambda) = \left\{v \in \prod\{R_\lambda \mid \lambda \in \Lambda\} \text{ the support of } v \text{ satisfies the ascending chain condition}\right\} ;$$

as is well known, V is an l-group if one declares $0 < v \in V$ if and only if each maximal non-zero component of v is positive. (For details the reader may consult [3] or [6].)

$$Ar^*(V) = \left\{v \in V \mid v \text{ is finitely non-zero, and if } v_\lambda \neq 0 \text{ then } \{\mu \in \Lambda \mid \mu \leq \lambda\} \text{ has finitely many maximal chains and satisfies the descending chain condition}\right\} .$$

References

[1] Roger D. Bleier, "Free l-groups and vector lattices", *J. Austral. Math. Soc.* (to appear).

[2] Donald Chambless, "Representations and extensions of lattice-ordered groups and rings", (Dissertation, Tulane University, Louisiana, 1971).

[3] Paul Conrad, *Lattice ordered groups* (Lecture Notes, Tulane University, Louisiana, 1970).

[4] Paul F. Conrad, "Free abelian l-groups and vector lattices", *Math. Ann.* **190** (1971), 306-312.

[5] Paul Conrad, "Epi-archimedean groups", preprint.

[6] Paul Conrad, John Harvey and Charles Holland, "The Hahn embedding theorem for abelian lattice ordered groups", *Trans. Amer. Math. Soc.* **108** (1963), 143-169.

[7] Jorge Martinez, "Free products of abelian l-groups", *Czechoslovak Math. J.* **23** (98) (1973), 349-361.

[8] Jorge Martinez, "Archimedean-like classes of lattice-ordered groups", *Trans. Amer. Math. Soc.* **186** (1973), 33-49.

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