Remarks on the Algebra of the 4-nodal Cubic Surface.

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The derivation of the 4-nodal Cubic Surface and the Quartic Surface,[1] of which it is a particular case, are well known: certain new results of interest from the point of view of symmetry, and extension to n-fold space, are provided by the symbolic algebra. In particular a simple proof is given, in § 2, of the theorem that a symmetry exists between the four vertices of the tetrahedron and the fifth point whose locus is the cubic surface, and this property can be extended to the case of $n + 1$ points in n-fold space with one additional point.

§ 1 Denote a quadric symbolically by the point equation $a_x{}^2 = a'_x{}^2 = 0$, or in plane coordinates by $u_\alpha{}^2 = u_{\alpha'}{}^2 = \ldots . 0$. The line equation to the quadric is $(Ap)^2 = (aa'p)^2 = 0$ and two lines q_{ij}, q'_{ij} $(ij = 1, 2, 3, 4, \; i \neq j)$ are conjugate lines to the quadric if

$$(Aq)\,(Aq') = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with analogous expressions for the tangential equation to the quadric.

The four-nodal cubic surface is the locus of a point F such that the feet of the perpendiculars from F to the faces of a tetrahedron $BCDE$ are coplanar. Or again—take the poles of the faces of the tetrahedron with respect to the quadric and join these poles to F: such lines will meet the corresponding faces in four points which are coplanar if F describes a quartic surface, which surface reduces to the cubic surface when the quadric is a conic in the plane at infinity. In n-fold space, $(n + 1)$ points $BCD \ldots . E$ are taken; and the poles of the faces with respect to a quadric: such points are joined to a further point F, and the lines meet the corresponding faces in points lying on a *prime*: F will then describe a *primal* of order $(n + 1)$, which reduces to a primal of order n—analogous to the 4-nodal cubic surface—when the quadric degenerates to a quadric of a prime.

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[1] H. F. Baker, *Principles of Geometry*, **4**, 26.

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Algebraically this means that the quadric surface, in plane coordinates, $u_\alpha{}^2 = 0$, is such that its discriminant vanishes, viz.,

$$(\alpha\, \alpha'\, \alpha''\, \alpha''')^2 = 0; \qquad \dots\dots\dots\dots\dots\dots \quad (2)$$

and, taking $l_x = 0$ to be the plane which contains the conic, then

$$l_\alpha\, \alpha_i = 0 \qquad (i = 1, 2, 3, 4), \qquad \dots\dots\dots\dots\dots\dots (3)$$

with corresponding expressions for n-space. A general quadric in n-space is a *primal* of order 2, while a *prime* is a primal of order unity.

Take the points $BCDE$ with coordinates β, γ, δ, ϵ, and equations,

$$u_\beta = 0, \;\; u_\gamma = 0, \;\; u_\delta = 0, \;\; u_\epsilon = 0,$$

where $$u_\beta = u_1\,\beta_1 + u_2\,\beta_2 + u_3\,\beta_3 + u_4\,\beta_4,$$

and take any given plane $p_x = 0$; then the equation of the quadric S' which passes through the curve of intersection of $S = 0 = a_x{}^2$ and $p_x = 0$ and also through the given four points is given by

$$a_x{}^2\,(\beta\gamma\delta\epsilon) - p_x \left\{ (x\gamma\delta\epsilon)\frac{a_\beta{}^2}{p_\beta} - (x\delta\epsilon\beta)\frac{a_\gamma{}^2}{p_\gamma} + (x\epsilon\beta\gamma)\frac{a_\delta{}^2}{p_\delta} - (x\beta\gamma\delta)\frac{a_\epsilon{}^2}{p_\epsilon} \right\} = 0;$$

as is easily verified. Calling this $a_x{}^2\,(\beta\gamma\delta\epsilon) - p_x\, p'_x = 0$, the second factor p'_x thus gives the second plane of intersection of the two quadrics $S = 0$ $S' = 0$.

Reciprocally if $u_\alpha{}^2 = 0$ be the quadric and four planes be given by $b_x = 0$, $c_x = 0$, $d_x = 0$, $e_x = 0$, and a point $F, u_\zeta = 0$, there is a second quadric envelope which touches the four planes and the cone, of vertex F, enveloping $u_\alpha{}^2 = 0$. The vertex of the second enveloping cone, viz., F', to the two quadrics is given by $u_{\zeta'}$, or,

$$(ucde)\,\frac{b_\alpha{}^2}{b_\zeta} - (udeb)\,\frac{c_\alpha{}^2}{c_\zeta} + (uebc)\,\frac{d_\alpha{}^2}{d_\zeta} - (ubcd)\,\frac{e_\alpha{}^2}{c_\zeta} = 0 \dots\dots (4)$$

with analagous expressions for n-fold space. Then there will be $n + 1$ such terms, and $n + 1$ letters in each bracket factor.

Now take the poles of the four faces of $BCDE$ with respect to $u_\alpha{}^2 = 0$: call these points $B'C'D'E'$, B' being the pole of CDE, and so on for the others. Such a point as B' has coordinates $(\gamma\delta\epsilon\alpha)\,\alpha_i$, or $b_\alpha\, \alpha_i$, where $b_x = 0$ is the face CDE. The point, where the line joining B' to F, $u_\zeta = 0$, meets the plane $b_x = 0$, is given by,

$$\zeta_i\, b_\alpha{}^2 - b_\zeta\, b_\alpha\, \alpha_i\,.$$

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Hence the four such points will be coplanar if the determinant,

$$(\zeta_i\, b_a{}^2 - b_\zeta\, b_a\, a_i\,,\ \zeta_i\, c_a{}^2 - c_\zeta\, c_a\, a_i\,,\ \zeta_i\, d_a{}^2 - d_\zeta\, d_a\, a_i\,,\ \zeta_i\, e_a{}^2 - e_\zeta\, e_a\, a_i\,) = 0.$$

Or, on expanding,

$$\frac{1}{4!}(a\,a'\,a''\,a''')^2 (bcde)\, b_\zeta\, c_\zeta\, d_\zeta\, e_\zeta$$

$$= \frac{1}{3!} \sum_{bcde} (\zeta a\, a'\, a'')\,(cde \mid a\, a'\, a'')\, c_\zeta\, d_\zeta\, e_\zeta\, b_a{}^2 \ldots\ldots\ldots\ldots (5)$$

The right hand side involves 4 terms got by cyclical interchange of *bcde*, but with positive and negative signs alternately.

Taking the point-form of the quadric $u_a{}^2 = 0$ as $a_x{}^2 = 0$, (5) can be written as

$$a_a{}^2\,(bcde)\, b_\zeta\, c_\zeta\, d_\zeta\, c_\zeta = \sum_{bcde} a_\zeta\,(acde)\, c_\zeta\, d_\zeta\, e_\zeta\, b_a{}^2 \ldots\ldots\ldots\ldots (6)$$

or

$$1 = \sum_{bcde} \frac{a_\zeta\,(acde)\, b_a{}^2}{b_\zeta\,(bcde)\, a_a{}^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

all the signs on the right being positive, the four terms being got by cyclical interchange of *bcde*.

It is evident that ζ describes a quartic surface: in the case of n-fold space the relationship is exactly of the same form, viz.,

$$1 = \sum_{bcde\ldots} \frac{a_\zeta\,(acde\ldots.)\, b_a{}^2}{b_\zeta\,(bcde\ldots.)\, a_a{}^2} \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Expressing (7) in terms of the points *BC DE*, since by definition, $(bcdu) = u_\epsilon$, the equation can be written as

$$1 = \frac{\Sigma\, a_\zeta\, a_\beta\,(\gamma\delta\epsilon a)^2}{a_a{}^2\,(\beta\gamma\delta\epsilon(\,(\gamma\delta\epsilon\zeta)} \,.$$

So the locus of ζ is

$$1 = \Sigma\, \frac{a_x\, a_\beta\,(\gamma\delta\epsilon a)^2}{a_a{}^2(\beta\gamma\delta\epsilon)\,(\gamma\delta\epsilon x)} \,. \quad \ldots\ldots\ldots\ldots\ldots (9)$$

It is a straightforward application to show that, if condition (5) holds, then FB', $F'B'$ are conjugate lines to the quadric, where F' is the point whose equation is (4).

Let F' have coordinates ζ'_i.

The lines FB' and $F'B'$ having coordinates

$$(\zeta a)_{ij}\, b_a\,,\ (\zeta' a)_{ij}\, b_a\,,$$

are conjugate with regard to $u_a{}^2 = 0$ if

$$(a\,a'\,a''\,\zeta)\,(a\,a'\,a'''\,\zeta')\,b_{a''}\,b_{a'''} = 0,$$

or $$(a\,a'\,a''\,\zeta)\,(a\,a'\,a''\,\zeta')\,.\,b_a{}^2 - \tfrac{1}{4}\,(a\,a'\,a''\,a''')^2\,b_\zeta\,b'_{\zeta'} = 0.$$

Substituting for ζ' its coordinates from (4), and dividing by $b_a{}^2 \neq 0$, the condition (5) is obtained.

Similarly FC', $F'C'$ are conjugate lines, etc. It is clear that the proof holds for higher dimensions.

Let now $u_a{}^2 = 0$ reduce to a conic in the plane $l_x = 0$, and hence

$$(a\,a'\,a''\,a''')^2 = 0, \quad l_a\,a_i = 0.$$

Equation (5) becomes

$$\sum_{bcde} (a\,a'\,a''\zeta)\,(a\,a'\,a''\,|\,cde)\,\frac{b_a{}^2}{b_\zeta} = 0.$$

Multiply both sides of the non-vanishing quantity l_ζ and notice that

$(a\,a'\,a''\,|\,cde)\,l_\zeta = (a\,a'\,a''\,\zeta)\,(lcde) +$ terms containing l_a, which vanish. Thus a factor in each term is $(a\,a'\,a''\,\zeta)^2$, and the remaining factor is

$$\frac{b_a{}^2(lcde)}{b_\zeta} - \frac{c_a{}^2\,(ldeb)}{c_\zeta} + \frac{d_a{}^2\,(lebc)}{d_\zeta} - \frac{e_a{}^2\,(lbcd)}{e_\zeta} = 0,$$

or

$$\frac{(\beta\gamma\delta a)^2\,l_\epsilon}{(\beta\gamma\delta\zeta)} - \frac{(\gamma\delta\epsilon a)^2\,l_\beta}{(\gamma\delta\epsilon\zeta)} + \frac{(\delta\epsilon\beta a)^2\,l_\gamma}{(\delta\epsilon\beta\zeta)} - \frac{(\epsilon\beta\gamma a)\,l_\delta}{(\epsilon\beta\gamma\zeta)} = 0. \quad \ldots.(10)$$

thus the locus of F, or ζ, is the known four-nodal cubic surface.

An analogous equation holds for the corresponding surface in n-space.

§2. It was first pointed out by W. Mantel[1] that each of the five points in (10), $BCDEF$, is in the same relation to the other four. This property holds similarly for n-space, as will now be shown from equation (10).

Now by the usual symbolic identity

$$(\gamma\delta\epsilon a)^2\,l_\beta = (\gamma\delta\epsilon a)\,\{(\gamma\delta\epsilon\beta)\,l_a + (\gamma\delta\beta a)\,l_\epsilon + (\gamma\beta\epsilon a)\,l_\delta + (\beta\delta\epsilon a)\,l_\gamma\}$$

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[1] H. F. Baker, *loco. cit.*

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Now the first term on the right vanishes by § 1 (3); take each of the other terms with the appropriate denominator of (10) from a corresponding one in (10): thus for example,

$$\frac{(\gamma\delta\epsilon\alpha)(\beta\gamma\delta\alpha)\,l_\epsilon}{(\gamma\delta\epsilon\zeta)} - \frac{(\beta\gamma\delta\alpha)^2\,l_\epsilon}{(\beta\gamma\delta\zeta)}$$

$$= \frac{(\beta\gamma\delta\alpha)\,l_\epsilon\{(\gamma\delta\epsilon\alpha)(\beta\gamma\delta\zeta) - (\beta\gamma\delta\alpha)(\gamma\delta\epsilon\zeta)\}}{(\gamma\delta\epsilon\zeta)(\beta\gamma\delta\zeta)}$$

$$= \frac{(\beta\gamma\delta\alpha)\,l_\epsilon\,.\,(\beta\gamma\delta\epsilon)(\gamma\delta\zeta\alpha)}{(\gamma\delta\epsilon\zeta)(\beta\gamma\delta\zeta)}.$$

Each factor thus contains the non-vanishing $(\beta\gamma\delta\epsilon)$ and the total expression is

$$\frac{(\beta\gamma\delta\alpha)(\gamma\delta\zeta\alpha)\,l_\epsilon}{(\beta\gamma\delta\zeta)} + \frac{(\epsilon\beta\gamma\alpha)(\epsilon\gamma\zeta\alpha)\,l_\delta}{(\epsilon\beta\gamma\zeta)} + \frac{(\delta\epsilon\beta\alpha)(\delta\epsilon\zeta\alpha)\,l_\gamma}{(\delta\epsilon\beta\zeta)} = 0.$$

This expression is symmetrical in ζ and β and thus the points F and B are interchangeable, which proves the result. For n-space there would be n terms, each bracket factor containing $n+1$ symbols.