HYPERBOLIC CONVOLUTION OPERATORS

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1. Introduction. Hyperbolic differential operators with constant coefficients introduced and studied systematically by Gårding **(4)**, were characterized by the existence of the fundamental solution with some cone condition, according to Hörmander **(6)**. Recently Ehrenpreis, extending the notion of hyperbolicity due to Gårding, has defined hyperbolic operators for distributions with compact support in the convolution sense. Under the hypothesis that the operator is invertible as a distribution, he has established a theorem analogous to the theorem of Hörmander mentioned above **(3)**. Motivated by these results, we shall define "hyperbolic convolution operators" which are similar to (but slightly different from) semi-hyperbolic operators in **(3)**. In Section 2 we shall show that hyperbolicity for convolution operators can be reduced to that for "truncations" of those operators. In Section 3 we shall discuss particularly hyperbolicity for finite difference–differential operators and characterize them in terms of their Fourier transforms. We shall give in Section 4 an algebraic condition for convolution operators (distributions with compact support) to be hyperbolic. In Section 5 we shall introduce some convolution operators with a leading linear differential operator $P(D)$ and prove that the convolution operator is hyperbolic if and only if its support is contained in a cone and $P(D)$ is hyperbolic in the sense of Gårding. Finally in Section 6 we shall show how smoothness of the fundamental solution for the operator in Section 5 depends on that of the fundamental solution for $P(D)$.

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2. Hyperbolic operators.

2.1. Definitions. We shall fix a real vector $N \in R^n$ throughout the paper. A differential operator (of order m) $P(D)$ is defined by

$$P(D) = \sum_{|\alpha| \leqslant m} a_\alpha D^\alpha$$

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where all a_α are constants, and particularly a_α for $|\alpha| = m$ are not all zero and

$$D^\alpha = \left(i^{-1}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(i^{-1}\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(i^{-1}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

with $\alpha_k \geqslant 0$ integers. Let us denote by p the principal part of the polynomial $P(\zeta)$, that is

$$p(\zeta) = \sum_{|\alpha|=m} a_\alpha \zeta^\alpha.$$

A polynomial P is said to be hyperbolic with respect to N if $p(N) \neq 0$ and there is a real number t_0 such that

$$P(\xi + itN) \neq 0 \quad \text{when } \xi \in R^n \text{ and } t < t_0.$$

By a Γ-cone we mean a closed cone having no points $\neq O$ in common with the half-space $x \cdot N \leqslant 0$, where the dot denotes the inner product operation in R^n. Then a theorem of Hörmander may be stated as follows.

THEOREM 2.1.1. *A polynomial P is hyperbolic with respect to N if and only if there exists a fundamental solution E, for the differential operator $P(D)$, whose support is contained in a Γ-cone.*

Now this theorem makes it natural to define "hyperbolic convolution operators" as in the following, where $\mathscr{E}'$ denotes the space of distributions with compact support (**8**, vol. 1, p. 88).

DEFINITION 2.1.1. *Let $S \in \mathscr{E}'$. Then S is said to be hyperbolic with respect to N if there is a fundamental solution E for S*

$$S * E = \delta$$

such that the support of E contains O and is contained in a certain Γ-cone.

Since all distributions with support limited to the left with respect to a Γ-cone are associative and commutative for convolution (**8**), we have a uniqueness theorem on fundamental solutions.

THEOREM 2.1.2. *Let $S \in \mathscr{E}'$ be hyperbolic with respect to N. Then there exists one and only one fundamental solution for S, with support in a Γ-cone.*

Proof. Assume that E_1 and E_2 are two fundamental solutions, supports of which are contained in a Γ-cone. Then the equalities

$$\begin{aligned} E_1 &= \delta * E_1 \\ &= (S * E_2) * E_1 \\ &= (E_2 * S) * E_1 \\ &= E_2 * (S * E_1) \\ &= E_2 * \delta = E_2, \end{aligned}$$

imply our assertion.

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THEOREM 2.1.3. *If $S \in \mathscr{E}'$ is hyperbolic with respect to N, then the support of S is contained in a Γ-cone.*

Proof. From Definition 2.1.1 we find a fundamental solution E for S with support in a Γ-cone. Let K be the smallest convex Γ-cone containing supp E. (When T is a distribution, we denote by supp T the support of T.) We denote by K_a the translation of K, with vertex at a, and by H_T the convex closure of the set $\cup_x K_x$ where x runs through supp T. Then a theorem of Lions on supports **(7)** gives

$$H_\delta = \text{convex closure of } (H_S + H_E). \tag{2.1.1}$$

Now we have $H_E \supset K$ since supp $E \ni 0$. Conversely, for any $x \in \text{supp}\, E$, $K_x \subset K$, so that $K \supset H_E$. Hence we have $H_E = K = H_\delta$. Combining this fact and (2.1.1) we obtain that $H_S \subset K$ and so supp $S \subset K$, which proves the theorem.

THEOREM 2.1.4. *If $S \in \mathscr{E}'$ is hyperbolic with respect to N, then so is S with respect to N' for all N' in a neighbourhood of N.*

Proof. By our assumption, there is a fundamental solution E for S with support in K defined above. Let $U(N)$ be the set

$$\{N' \in R^n | x \cdot N' > 0 \text{ for all } x \in K;\, x \neq 0\}.$$

Then K is also a Γ-cone with respect to $N' \in U(N)$ and hence, by definition, S is hyperbolic with respect to $N' \in U(N)$.

2.2. Singularity at the origin. We say that a distribution S has a *singular point* P or that S is singular at P if S is not equal to any C^∞-function in any neighbourhood of P. Then we have

THEOREM 2.2.1. *If $S \in \mathscr{E}'$ is hyperbolic with respect to N, then S must be singular at the origin.*

Proof. Suppose that S is equal to a C^∞-function in a neighbourhood of O. Take $\alpha \in C_0^\infty$ such that $\alpha(x) = 1$ in a smaller neighbourhood and that

$$S = \alpha S + S_\epsilon,$$

where $\alpha S \in C_0^\infty$ and supp $S_\epsilon \subset \{x \cdot N \geqslant \epsilon\}$ for some $\epsilon > 0$. Now let U be a neighbourhood of O contained in the half-space $x \cdot N < \epsilon$. Hence for any $\phi \in C_0^\infty(U)$,

$$\langle S, \phi\rangle = \langle \alpha S, \phi\rangle.$$

Since S is hyperbolic with respect to N, we can find a fundamental solution E for S so that

$$\text{supp}(S_\epsilon * E) \subset \text{supp}\, S_\epsilon + \text{supp}\, E \subset \{x \cdot N \geqslant \epsilon\}.$$

Hence we have for any $\phi \in C_0^\infty(U)$

$$\langle S * E, \phi \rangle = \langle \alpha S * E, \phi \rangle.$$

Now let $\psi \in C_0^\infty$ be 1 in a neighbourhood of O, with support in U. Then we get for any $\phi \in C_0^\infty$

$$\langle \delta, \phi \rangle = \langle \psi(\alpha S * E), \phi \rangle$$

so that

$$\delta = \psi(\alpha S * E) \in C_0^\infty,$$

which is a contradiction. Thus S should be singular at O.

2.3. Truncation. If S is hyperbolic with respect to N, it can be easily seen that $\operatorname{supp} S \ni 0$. Let $\alpha \in C_0^\infty$ be 1 in a neighbourhood of O. We call a distribution αS a truncation of S. Then in view of the following theorem we may reduce hyperbolicity of convolution operators to a property of a small neighbourhood of O.

THEOREM 2.3.1. *If S is hyperbolic with respect to N, then so is any truncation of S. Conversely, if a truncation of S is hyperbolic with respect to N, so is S.*

Proof. By our definition, we have a decomposition of S:

$$S = S_1 + S_2,$$

where S_1 is a truncation of S and $\operatorname{supp} S_2 \subset \{x \cdot N > 0\}$. First assume that S is hyperbolic with respect to N. Hence S has to have a fundamental solution E with support in a Γ-cone. Let us consider a geometrical series of convolutions $\{E_\nu\}$ defined by

$$E_{\nu+1} = E * \sum_{k=0}^{\nu} (E * S_2)^{*k},$$

where $E_1 = E$ and T^{*k} denotes the k-tuple convolution of T. Since there is a positive number ϵ such that

$$\operatorname{supp} S_2 \subset \{x \cdot N \geqslant \epsilon\},$$

we may see, using a theorem on supports, that

$$\operatorname{supp} \sum_{k=0}^{\nu} (E * S_2)^{*k} \subset \{x \cdot N \geqslant \nu\epsilon\},$$

from which there follows that

$$\lim_{\nu \to \infty} E_\nu \tag{2.3.1}$$

exists in $\mathscr{D}'$ (**8**, vol. 2, p. 71). Now define a distribution E^1 by (2.3.1). Convoluting $E_{\nu+1}$ by S_1 and using associativity and commutativity for S_1, S_2, and E, we obtain that

$$\begin{aligned} S_1 * E_{\nu+1} &= (S - S_2) * E_{\nu+1} \\ &= (S - S_2) * E * \sum_{k=0}^{\nu} (E * S_2)^{*k} \\ &= (\delta - E * S_2) * \sum_{k=0}^{\nu} (E * S_2)^{*k} \\ &= \delta - (E * S_2)^{*(\nu+1)}. \end{aligned}$$

Since the last member in the above equalities tends to δ when $\nu \to \infty$, we have

$$\lim_{\nu\to\infty} (S * E_{\nu+1}) = S_1 * E^1 = \delta.$$

Thus we have constructed the fundamental solution E^1 for S_1. To see that S_1 is hyperbolic with respect to N it suffices to show that supp E^1 is contained in a certain Γ-cone. However, it can be verified readily that

$$\text{supp}\{E * (E * S_2)^{*k}\} \subset (k + 1)(\text{supp}\, E) + k(\text{supp}\, S_2) \subset \Gamma_1 + \Gamma_2,$$

where Γ_1 and Γ_2 are both Γ-cones containing supp E and supp S respectively. Therefore we may conclude that

$$\text{(2.3.2)} \qquad \text{supp}\, E^1 \subset \Gamma_1 + \Gamma_2.$$

The second member of (2.3.2) being certainly a Γ-cone, we have proved the first part of our theorem.

The same argument as above can be applied to prove the remaining part. Actually a fundamental solution E for S may be obtained by defining

$$\text{(2.3.3)} \qquad E = E^1 * \sum_{k=0}^{\infty} (-1)^k (E^1 * S_2)^{*k}$$

provided that S_1 is hyperbolic with respect to N, where E^1 is a fundamental solution for S_1, with support in a Γ-cone. That E given by (2.3.3) satisfies $S * E = \delta$ is clear. Thus the proof has been completed.

2.4. Examples. By Theorem 2.1.1 a hyperbolic differential operator $P(D)$ is hyperbolic as a convolution operator $P(D)\delta$. For completeness we shall construct the fundamental solution with support in the "wave cone" by a method due to Hörmander **(6)**. Let $P(D)$ be a hyperbolic linear differential operator with respect to N, and let $\Gamma_P(N)$ be the set of all N' such that

$$p(N' + tN) = 0$$

has only negative zeros. We call $\Gamma_P(N)$ the "normal cone" of $P(D)$. Then it can be shown that there exist numbers t and C such that

$$\text{(2.4.1)} \qquad |P(\xi + itN + isN')| \geqslant C \qquad \text{for all } N' \in \Gamma_P(N)$$

when $\text{Re}\, t \leqslant t_1$ and $\text{Re}\, s \leqslant 0$. Let us define a linear form E on C_0^∞ by

$$(2.4.2)\qquad \langle E, \phi\rangle = (2\pi)^{-n}\int \frac{\tilde{\phi}(\zeta)}{P(\zeta)}\,d\xi$$

where $\zeta = \xi + itN$. (The Fourier transform of $\phi \in \mathscr{S}$ is defined by $\phi\hat{\ }(\xi) = \langle e^{-ix\cdot\xi}, \phi(x)\rangle$. The Fourier inverse transform of $\psi \in \mathscr{S}$ is given by $\tilde{\psi}(x) = \langle e^{-i\xi\cdot x}, \psi(\xi)\rangle$.) For the definition of $\mathscr{S}$ see (**8**, vol. 2, p. 89). Since for $\phi \in C_0^\infty$,

$$\zeta^\alpha\tilde{\phi}(\zeta) = \int (D^\alpha\phi)(x)e^{ix\cdot\zeta}dx,$$

we have for some a and C

$$|\tilde{\phi}(\zeta)| \leqslant C\frac{\sum_{|\alpha|\leqslant n+1}||D^\alpha\phi||_\infty}{(1+|\xi|)^{n+1}}e^{at},$$

where $\zeta = \xi + itN$. Thus the second member of (2.4.2) is convergent in view of (2.4.1) and the above inequality, and then E defines a distribution. More precisely, the linear form on C_0^∞

$$\langle e^{tx\cdot N}E, \phi\rangle = (2\pi)^{-n}\int\frac{\tilde{\phi}(\xi)}{P(\zeta)}\,d\xi$$

defines a temperate distribution $F = e^{tx\cdot N}E$. In other words, E is a product of an exponential function growing in the N-direction and a temperate distribution. That E is a fundamental solution for $P(D)$ is readily verified. We remark here that E is independent of the choice of t if $t \leqslant t_1$. Now if $\operatorname{supp}\phi \subset \{x\cdot N < -\epsilon\}$ we obtain

$$|\langle E, \phi\rangle| \leqslant Ce^{\epsilon t} \qquad (\phi \in C_0^\infty)$$

with a suitable constant C independent of t, from which follows $\langle E, \phi\rangle = 0$ making $t \to -\infty$. Since we may take $\epsilon > 0$ arbitrarily small provided that $\operatorname{supp}\phi \subset \{x\cdot N < 0\}$, we conclude that $\operatorname{supp} E \subset \{x\cdot N \geqslant 0\}$. It follows from the above remark and (2.4.1) that the contour in the integration (2.4.2) can be shifted to a contour $\zeta = \xi + itN + isN'$, where $s \leqslant 0$ and $N' \in \Gamma_P(N)$. An argument similar to the above gives

$$|\langle E, \phi\rangle| \leqslant Ce^{at}e^{\epsilon s},$$

where $\phi \in C_0^\infty$ with support in $\{x\cdot N' < -\epsilon\}$ and C and a are constants independent of s. Hence we have $\langle E, \phi\rangle = 0$ after making $s \to -\infty$. Consequently we obtain that $\operatorname{supp} E \subset W_P(N)$, where

$$W_P(N) = \{x\,|x\cdot N' \geqslant 0 \qquad \text{for all } N' \in \Gamma_P(N)\}$$

which we call the wave cone of $P(D)$.

Next we shall give a simple example of a hyperbolic convolution operator as a function in R^2, for simplicity.

Let $\chi(x, y)$ be the characteristic function of the square domain in R^2:

$[0, 1] \times [0, 1]$. We shall prove that $\chi(x, y)$ is hyperbolic with respect to $N = (a, b)$, where $a, b > 0$. Since

$$\chi(x, y) = c(x)c(y)$$

where $c(\cdot)$ is the characteristic function of the interval $[0, 1]$, the Fourier transform is given by

$$\begin{aligned}\chi^{\wedge}(\xi, \eta) &= c^{\wedge}(\xi)c^{\wedge}(\eta)\\ &= -\frac{(1 - e^{-i\xi})(1 - e^{-i\eta})}{\xi\eta}.\end{aligned}$$

Now let us expand formally $\chi^{\wedge}(\xi, \eta)^{-1}$. Since

$$(1 - e^{-i\xi})^{-1} = \sum_{l=0}^{\infty} e^{-il\xi},$$

we obtain

$$\chi^{\wedge}(\xi, \eta)^{-1} = (i\xi)(i\eta) \sum_{l,m=0}^{\infty} e^{-i(l\xi+m\eta)}.$$

Taking the Fourier inverse transform of the second member, we have

$$\sum_{l,m=0}^{\infty} \frac{\partial^2}{\partial x \partial y} \delta(x - l, y - m).$$

This expression suggests a fundamental solution E for χ as follows:

$$E = \lim_{k\to\infty} \sum_{l,m=0}^{k} \frac{\partial^2}{\partial x \partial y} \delta(x - l, y - m).$$

That the second member is convergent in $\mathscr{D}'$ is clear. In order to check that E has the required property, we shall compute $\chi * E_k$, putting

$$E_k = \sum_{l,m=0}^{k} \frac{\partial^2}{\partial x \partial y} \delta(x - l, y - m).$$

From the relation

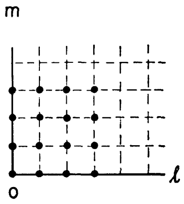
$$\chi(x, y) = \{H(x) - H(x - 1)\}\{H(y) - H(y - 1)\},$$

where $H(\cdot)$ is the Heaviside function, it follows that

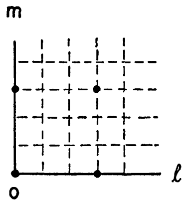
$$\begin{aligned}&\chi * \frac{\partial^2}{\partial x \partial y} \delta(x - l, y - m)\\ &\quad = \{\delta(x) - \delta(x - 1)\} \times \{\delta(y) - \delta(y - 1)\} * \{\delta(x - l, y - m)\}\\ &\quad = \{\delta(x, y) - \delta(x - 1, y) - \delta(x, y - 1) + \delta(x - 1, y - 1)\}\\ &\qquad\qquad * \{\delta(x - l, y - m)\}\\ &\quad = \delta(x - l, y - m) - \delta(x - l - 1, y - m) - \delta(x - l, y - m - 1)\\ &\qquad\qquad + \delta(x - l - 1, y - m - 1).\end{aligned}$$

Now it can be easily seen that

(2.4.5) $$\chi * E_k = \delta(x, y) - \delta(x - k, y) - \delta(x, y - k) + \delta(x - k, y - k).$$



supp E_3



supp $\chi * E_3$

Then the second member of (2.4.5) tends to $\delta(x, y)$ as $k \to \infty$. Thus we have proved that $E = \lim E_k$ is a fundamental solution for χ. Also we have proved that supp E consists of all lattice points (l, m), where $l, m \geqslant 0$ are integers, and that the singularity located at each lattice point is uniformly of order 4 **(2)**.

3. Hyperbolicity of finite difference–differential operators.

3.1. Finite difference–differential operators. Let us consider a finite difference–differential operator

(3.1.1) $$S = \sum_{k=0}^{l} P_k(D)\delta_{a_k}.$$

If S is hyperbolic with respect to N, then from Theorem 2.1.3 it follows that

(3.1.2) $$a_k = 0 \text{ for some } k, \qquad a_{k'} N > 0 \text{ for all } k' \neq k.$$

Further, since $P_k(D)\delta$ is a truncation of S, Theorem 2.3.1 implies that $P_k(D)$ is hyperbolic with respect to N as a differential operator. Conversely, if S, given by (3.1.1), satisfies (3.1.2) and if the differential operator $P_k(D)$ is hyperbolic with respect to N, then using again Theorem 2.3.1 we conclude that S is hyperbolic with respect to N. Hence we have

THEOREM 3.1.2. *A finite difference–differential operator S, given by* (3.1.1), *is hyperbolic with respect to N if and only if there exists a k $(0 \leqslant k \leqslant 1)$ such that $a_k = 0$ and $a_{k'} \cdot N > 0$ for all $k' \neq k$, and that $P_k(D)$ is hyperbolic with respect to N.*

Now we shall give a precise description of the fundamental solution E for S with support in a Γ-cone. We may assume $k = 0$ without loss of generality.
In view of Theorem 2.3, E is given by

(3.1.2) $$E = E_0 * \sum_{m=0}^{\infty} (-1)^m \left(E_0 * \sum_{k=1}^{l} P_k(D)\delta_{a_k}\right)^{*m}$$
$$= E_0 * \sum_{m=0}^{\infty} (-1)^m \left(\sum_{k=1}^{l} P_k(D)\tau_{a_k}E_0\right)^{*m},$$

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where E_0 is the fundamental solution for $P_0(D)$ just constructed in 2.4. However, since

$$\left(\sum_{k=1}^{l} P_k(D)\tau_{a_k}E_0\right)^{*m} = \sum_{|q|=m} \frac{m!}{q!} P_1(D)^{q_1} \ldots P_l(D)^{q_l}\tau_{q_1a_1+\ldots+q_la_l}(E_0^{*m}),$$

we have

$$E = \sum_{m=0}^{\infty} (-1)^m \sum_{|q|=m} \frac{m!}{q!} E_q,$$

where

$$E_q = P_1(D)^{q_1} \ldots P_l(D)^{q_l}\tau_{q_1a_1+\ldots+q_la_l}(E_0^{*m}).$$

Then it is obvious that

$$\operatorname{supp} E_q \subset \sum_{k=1}^{l} q_k a_k + W_{P_0}(N),$$

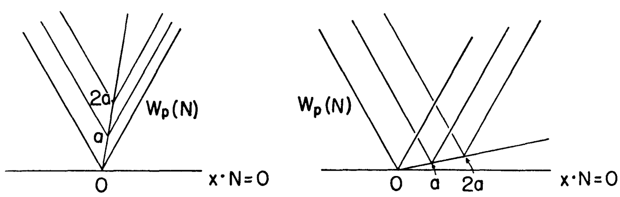
where $W_{P_0}(N)$ is the wave cone of $P_0(D)$. Therefore we have

THEOREM 3.1.3. *To the hyperbolic finite difference–differential operator S defined by* (3.1.1) *corresponds a fundamental solution E with support in the sum of all the cones, each of which is congruent to $W_{P_0}(N)$ and with its vertex at some lattice point $\sum_{k=1}^{l} q_k a_k$ consisting of vectors $a_1, \ldots, a_l$ and integers $q_k \geqslant 0$, $k = 1, \ldots, l$.*

Example. If $S = P(D)\delta + \delta_a$ is hyperbolic with respect to N, then $a \cdot N > 0$ and supp E is contained in the Γ-cone

$$\bigcup_{m=0}^{\infty} \{ma + W_P(N)\}.$$

$a \in W_P(N)$ $\qquad\qquad$ $a \notin W_P(N)$



3.2. Some algebraic conditions. Let S, defined by (3.1.1), be hyperbolic with respect to N. Then there is a k ($0 \leqslant k \leqslant l$), say 0, such that $a_0 = 0$, $P_0(D)$ is hyperbolic with respect to N, and $a_k \cdot N > 0$ for $k = 1, 2, \ldots, l$. Now the Fourier–Laplace transform of S is given by

$$S^\wedge(\zeta) = P_0(\zeta) + \sum_{k=1}^{l} P_k(\zeta)e^{-ia_k\cdot\zeta}. \tag{3.2.1}$$

Let Γ be the set $\{N' | a_k \cdot N' > 0$ for $k = 1, 2, \ldots, l\}$. Bearing in mind that

$$|P_0(\xi + itN + isN')| \geqslant C_1$$

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for some constant C_1 when $t \leqslant t_1$, $s \leqslant 0$, and $N' \in \Gamma_{P_0}(N)$, we have for another constant C_2

$$|S^\wedge(\xi + itN + isN')| \geqslant C_1\{1 - C_2(1 + |\xi + itN + isN'|)^m e^{\epsilon t + \delta s}\}$$

when $N' \in \Gamma_{P_0}(N) \cap \Gamma$, where

$$m = \max_{1 \leqslant k \leqslant l} \deg P_k, \qquad \epsilon = \min_{1 \leqslant k \leqslant l} a_k \cdot N, \qquad \text{and } \delta = \min_{1 \leqslant k \leqslant l} a_k \cdot N'.$$

Let us choose a constant C_0 such that

$$C_0 > C_2 \sup_{s \leqslant 0} (1 + |s|)^m e^{\delta s}.$$

Since

$$(1 + |\xi + itN + isN'|)^m \leqslant (1 + |\xi + itN|)^m (1 + |s|)^m,$$

we obtain

$$|S^\wedge(\xi + itN + isN')| \geqslant C_1\{1 - C_0(1 + |\xi + itN|)^m e^{\epsilon t}\}.$$

Now we can find a constant $K > 0$ such that $-K \leqslant t_1$ and that

$$t \leqslant -K[1 + \log(1 + |t| + |\xi|)] \tag{3.2.2}$$

implies

$$e^{-\epsilon t} \geqslant 2C_0(1 + |\xi + itN|)^m.$$

Hence below a contour γ:

$$t = -K(1 + \log[1 + |t| + |\xi|)], \tag{3.2.3}$$

we have

$$|S^\wedge(\xi + itN + isN')| \geqslant C_{N'} \tag{3.2.4}$$

for some constant $C_{N'}$ depending only on N' and S if $N' \in \Gamma_{P_0}(N) \cap \Gamma$.

Next we assume that there exists a convex neighbourhood $U(N)$ of N such that if $N' \in U(N)$, then for some K and $C_{N'}$ as above, (3.2.2) implies (3.2.4). In order to construct a fundamental solution for S we define a linear functional on C_0^∞:

$$\langle E, \phi \rangle = (2\pi)^{-n} \int_\gamma \frac{\tilde{\phi}(\zeta)}{S^\wedge(\zeta)} d\zeta, \tag{3.2.5}$$

where $\phi \in C_0^\infty$. We note that $|\zeta|/|\xi|$ and $|d\zeta|/d\xi$ are bounded on γ. Since for any $\nu > 0$,

$$|\tilde{\phi}(\xi + itN + isN')| \leqslant M_\nu(1 + |\xi + itN + isN'|)^{-\nu},$$

the integration on the above converges. Thus we may see that E defines a distribution. Now let $\phi \in C_0^\infty$, with support in $\{x \cdot N < 0\}$. Then

$$\operatorname{supp} \phi \subset \{x | \, x \cdot N \leqslant -\epsilon\}$$

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for some $\epsilon > 0$ and we have

$$|\tilde{\phi}(\xi + itN + isN)| \leqslant M_\nu(1 + |\xi + itN + isN|)^{-\nu} e^{\epsilon(t+s)}$$

for some constant M_ν. Hence we obtain for any $s \leqslant 0$ that

$$|\langle E, \phi\rangle| \leqslant (2\pi)^{-n} M_{n+1} C_N^{-1} e^{\epsilon s} \int_\gamma (1 + |\xi + itN + isN|)^{-n-1} e^{\epsilon t} |d\zeta|$$
$$\leqslant Me^{\epsilon s},$$

where M is a constant independent of s. Making $s \to -\infty$, we get $\langle E, \phi\rangle = 0$. Thus we conclude that $\operatorname{supp} E \subset \{x \cdot N \geqslant 0\}$.

Similarly, if t is fixed and $\phi \in C_0^\infty$ with

$$\operatorname{supp} \phi \subset \{x | x \cdot N' < -\delta\} \qquad (\delta > 0),$$

then there exist constants a and M both independent of s such that

$$|\langle E, \phi\rangle| \leqslant Me^{at} e^{\delta s}.$$

Thus it follows that $\operatorname{supp} E \subset \{x \cdot N' \geqslant 0\}$. Consequently we have proved that

$$\operatorname{supp} E \subset \bigcap_{N' \in U(N)} \{x \mid x \cdot N' \geqslant 0\}.$$

The second member being a Γ-cone, (3.2.5) defines a distribution with support in a Γ-cone. It is obvious that E is a fundamental solution for S. Thus the following theorem has been proved.

THEOREM 3.2.1. *Let S be a finite difference–differential operator. Then a necessary and sufficient condition that S be hyperbolic with respect to N is that a convex neighbourhood of N, $U(N)$, exist such that for some constants K and $C_{N'}$ depending only on N' and S,* (3.2.2) *implies* (3.2.4) *when $N' \in U(N)$.*

4. An algebraic condition for hyperbolic convolution operators.

In this section we shall suppose that $S \in \mathscr{E}'$ is hyperbolic with respect to N, and discuss the variety in which $S^\wedge(\xi + itN)$ is zero-free. A result of the previous section may suggest to us that $S^\wedge(\xi + itN) \neq 0$ below some contour like (3.2.3). Actually we shall prove the following theorem.

THEOREM 4.1. *Let $S \in \mathscr{E}'$ be hyperbolic with respect to N. Then for any positive number σ there exist positive constants m and C such that*

$$|S^\wedge(\xi + itN)| \geqslant e^{\sigma t} \tag{4.2.1}$$

when

$$e^{-t} \geqslant C(1 + |\xi + itN|)^m \qquad (t < 0). \tag{4.2.2}$$

Proof. We shall carry out the proof following the ideas of Hörmander **(6)**. Suppose that our theorem is false. Then we may find a triple of sequences

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$\{m_j\}$, $\{C_j\}$, and $\{\xi_j + it_j N\}$ $(t_j < 0)$ such that the following conditions are satisfied for some $\sigma > 0$:

(1) $$C_j, m_j \to \infty \quad \text{as} \quad j \to \infty;$$

(2) $$|S^\wedge(\xi_j + it_j N)| < e^{\sigma t_j};$$

(3) $$e^{-t_j} \geqslant C_j(1 + |\xi_j + it_j N|)^{m_j}.$$

The hypothesis on S implies that there exists a fundamental solution E for S with support in a Γ-cone, say Γ_0. Let ω be an open set with compact closure $\subset \Gamma_0$ and let

$$\sup_{x \in \omega} x \cdot N = \delta.$$

Now let us introduce a Banach space $C_0(\bar{\omega})$, the set of continuous functions vanishing outside $\bar{\omega}$. Also we introduce the set of C^∞-functions vanishing when $x \cdot N \geqslant \eta + \delta$, say $C^\infty_{\eta+\delta}$ where

$$\eta = \max_{x \in \operatorname{supp} S} x \cdot N.$$

Then the mapping

(4.2.3) $$f \in C_0(\bar{\omega}) \to \langle f, \phi \rangle$$

is continuous with the norm $||f||_\infty$ for each fixed $\phi \in C^\infty_{\eta+\delta}$. On the other hand, for each fixed $f \in C_0(\bar{\omega})$ there exists a distribution u such that

$$S * u = f,$$
$$\operatorname{supp} u \subset \Gamma_0.$$

In fact, $u = E * f$ has the required properties. Hence we have the following equalities, for $\phi \in C^\infty_{\eta+\delta}$:

(4.2.4) $$\langle f, \phi \rangle = \langle S * u, \phi \rangle = \langle u, S^\vee * \phi \rangle.$$

Since

$$\operatorname{supp}(S^\vee * \phi) \subset \operatorname{supp} S^\vee + \operatorname{supp} \phi \subset \{x \cdot N \leqslant \eta + \delta\},$$

u should be integrated over the compact set

(4.2.5) $$\Gamma_0 \cap \{x \cdot N \leqslant \eta + \delta\}$$

in the last member of (4.2.4). Now let Ω be a neighbourhood of the set (4.2.5), contained in the half-space $\{x \cdot N > -\epsilon\}$ and with compact closure, and let Ω_1 be another neighbourhood of the set (4.2.5) such that $\bar{\Omega}_1 \subset \Omega$. Let us take $\alpha \in C_0^\infty$ such that $\alpha(x) = 1$ on $\bar{\Omega}_1$ and 0 outside Ω. Then we define a metrizable topological linear space $C^\infty_{\eta+\delta,\alpha}$ by

$$C^\infty_{\eta+\delta,\alpha} = \{\alpha(S^\vee * \phi) \mid \phi \in C^\infty_{\eta+\delta}\}$$

with the topology introduced by semi-norms:

$$\rho_k(\psi) = \sum_{|\beta|=k} ||D_\beta \psi||_\infty, \qquad \psi \in C^\infty_{\eta+\delta,\alpha}.$$

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Since

$$\langle f, \phi\rangle = \langle u, \alpha(S^{\vee} * \phi)\rangle$$

for each fixed $f \in C_0(\bar{\omega})$, there exist an integer $k > 0$ and a constant M such that

$$|\langle f, \phi\rangle| \leqslant M \sum_{j=1}^{k} \rho_j(\psi), \qquad \psi = \alpha(S^{\vee} * \phi). \tag{4.2.6}$$

Hence the linear mapping $\psi \to \langle f, \phi\rangle$ is continuous on $C^{\infty}_{\eta+\delta,\alpha}$. Combining (4.2.3) and (4.2.6) we observe that the bilinear mapping

$$(f, \psi) \to \langle f, \phi\rangle \tag{4.2.7}$$

from the product space $C_0(\bar{\omega}) \times C^{\infty}_{\eta+\delta,\alpha}$ to C^1 is separately continuous. However, since a separate continuous bilinear form on the product of a Fréchet space and a metrizable space is continuous **(1)**, the mapping (4.2.7) is continuous. Therefore there exist an integer $k > 0$ and a constant K such that

$$|\langle f, \phi\rangle| \leqslant K \sum_{j=1}^{k} \rho_j(\psi)\, ||f||_{\infty}. \tag{4.2.8}$$

We shall construct a couple of sequences $\phi_j \in C^{\infty}_{\eta+\delta}, f_j \in C_0(\bar{\omega}), j = 1, 2, \dots,$ such that

$$\sum_{\nu=1}^{k} \rho_\nu(\psi_j)\, ||f_j||_{\infty} \to 0, \qquad \psi_j = \alpha(S^{\vee} * \phi_j)$$

and

$$\langle f_j, \phi_j\rangle \to 1$$

as $j \to \infty$, which contradicts the inequality (4.2.8). We fix a $y \in \omega$. Then for sufficiently small $\epsilon > 0$,

$$y\cdot N + \epsilon < \delta.$$

Now let us define $\phi_j(x)$ by

$$\phi_j(x) = \lambda(x\cdot N - \eta)\exp[i(y - x)\cdot(\xi_j + it_j N)], \tag{4.2.9}$$

where $\lambda(\theta) \in C^{\infty}(R^1)$ has the value 1 when $\theta < y\cdot N + \epsilon$ and 0 when $\theta \geqslant \delta$. It is easy to see that $\phi_j \in C^{\infty}_{\eta+\delta}$. Also define $f_j(x)$ by

$$f_j(x) = \exp(\tfrac{1}{3}\epsilon|t_j|)F[(y - x)\exp(\epsilon|t_j|/3n)], \tag{4.2.10}$$

where $F \in C_0^{\infty}$ satisfies the condition

$$\int F(x)dx = 1.$$

That $f_j \in C_0(\bar{\omega})$ for sufficiently large j is clear. Now we shall estimate $|S^{\vee} * \phi_j(x)|$.

By the definition of ϕ_j we have

$$S^{\vee} * \phi_j(x) = \langle S_z, \phi_j(z + x)\rangle = \exp[i(y - x)\cdot(\xi_j + it_j N)] \times \langle S_z, \lambda((x + z)\cdot N - \eta)\exp[-iz\cdot(\xi_j + it_j N)]\rangle. \tag{4.2.11}$$

First, consider the case $x \cdot N \geqslant y \cdot N + \epsilon$. Since S is represented as a linear combination of Dh, where D is some differential monomial and $h(x) \in L^\infty$ with support in a neighbourhood of supp S and in $\{x \cdot N \geqslant -\frac{1}{2}\epsilon\}$, it follows from (4.2.11) that for a differential operator D there exist constants C_1 and A such that

$$|D\psi_j(x)| \leqslant C_1 \exp[t_j(x-y)\cdot N](1 + |\xi_j + it_j N|)^A \exp(-\tfrac{1}{2}\epsilon t_j)$$
$$\leqslant C_1 \exp\left(\frac{\epsilon}{2} - \frac{A}{m_j}\right) t_j.$$

Next, let us discuss the case $x \cdot N < y \cdot N + \epsilon$. When $z \in \operatorname{supp} S$, we have

$$\lambda(x \cdot N + z \cdot N - \eta) = 1. \tag{4.2.12}$$

For, if $z \in \operatorname{supp} S$, $z \cdot N \leqslant \eta$, and hence

$$x \cdot N + z \cdot N - \eta \leqslant y \cdot N + \epsilon + z \cdot N - \eta \leqslant y \cdot N + \epsilon.$$

From the definition of λ, (4.2.12) follows. Hence combining (4.2.11) and (4.2.12) we obtain

$$S^\vee * \phi_j(x) = \exp[i(y-x)\cdot(\xi_j + it_j N)]S^\wedge(\xi_j + it_j N). \tag{4.2.13}$$

From the conditions (2) and (3) and for $x \in \operatorname{supp} \alpha$ from the relation

$$x \cdot N - y \cdot N > \min_{z \in \operatorname{supp} \alpha} z \cdot N - \max_{z \in \bar{\omega}} z \cdot N$$
$$> -\epsilon - \delta,$$

it follows that for a differential operator D there exist constants C_0 and B such that

$$|D\alpha(S^\vee * \phi_j)(x)| \leqslant C_0 \exp[t_j(x-y)\cdot N](1 + |\xi_j + it_j N|)^B \exp(\sigma t_j)$$
$$\leqslant C_0 \exp\left[\left(\sigma - (\epsilon + \delta) - \frac{B}{m_j}\right) t_j\right].$$

In view of (4.2.10) we obtain that for all j and sufficiently small ϵ, δ

$$\sum_{\nu=1}^{k} \rho_\nu(\psi_j)\cdot ||f_j||_\infty \leqslant M \exp\left[\left(\frac{\epsilon}{2} - \frac{C}{m_j}\right) t_j\right] \tag{4.2.14}$$

where C and M are suitable constants. The second member of (4.2.14) tends to 0 as $j \to \infty$ and so does the first member of (4.2.14).

On the other hand the definition of ϕ_j and f_j implies that

$$\begin{aligned} \langle f_j, \phi_j \rangle &= \exp(\tfrac{1}{3}\epsilon|t_j|) \int F((y-x)\exp(\epsilon|t_j|/3n)) \\ &\qquad \times \exp[i(y-x)(\xi_j + it_j N)]\lambda(x \cdot N - \eta)dx \\ &= \int F(x)\lambda(-x \cdot N \exp(\epsilon t_j/3n) + y \cdot N - \eta) \\ &\qquad \times \exp[\{-ix\cdot(\xi_j + it_j N)\exp(\epsilon t_j/3n)\}]dx. \end{aligned} \tag{4.2.15}$$

However, we have

$$\operatorname{Re}[-ix\cdot(\xi_j + it_j N)\exp(\epsilon t_j/3n)] \leqslant |t_j|\cdot|x\cdot N|\exp(\epsilon t_j/3n)$$
$$\leqslant L\exp(|t_j|/m_j)\exp(\epsilon t_j/3n) = L\exp\left[\left(\frac{\epsilon}{3n} - \frac{1}{m_j}\right)t_j\right]$$

for a suitable constant L, since $C_j \to \infty$ $(j \to \infty)$ and since for large j

$$e^{|t_j|} \geqslant C_j(1 + |\xi_j + it_j N|)^{m_j} \geqslant |t_j|^{m_j}.$$

Thus in the last member of (4.2.15) the exponential factor in the integrand

$$\exp[-ix\cdot(\xi_j + it_j N)(\exp(\epsilon t_j/3n))] \to 1$$

when $j \to \infty$ because of condition (1), and then also

$$\lambda[-x\cdot N\exp(\epsilon t_j/3n) + y\cdot N - \eta] \to \lambda(y\cdot N - \eta) = 1.$$

Making $j \to \infty$ under the integral sign in (4.2.15), we conclude that

$$\lim_{j\to\infty} \langle f_j, \phi_j\rangle = 1$$

since

$$\int F(x)dx = 1.$$

Thus we have proved our theorem.

5. Some hyperbolic operators. In this section we shall study when finite sum distributions of the form

$$S_1 = \sum P_k(D)\mu_k$$

are hyperbolic, where μ_k are measures with compact support. We say that a differential operator $P(D)$ is "strictly stronger" than another differential operator $Q(D)$ (which we denote by $Q \prec\prec P$) if $Q(D) \prec P(D)$ and $\deg Q(\zeta) < \deg P(\zeta)$. Now let us consider the case where

$$S_1 = P(D)\delta + \sum_{k=1}^{l} P_k(D)\mu_k \in \mathscr{E}', \tag{5.1}$$

$$\operatorname{supp}\mu_k \subset \operatorname{supp} S.$$

Hörmander proved in **(6)** that if $\mu_k = \delta$, $P_k \prec\prec P$ for $k = 1, 2, \ldots, l$ and if P is a homogeneous hyperbolic differential operator (with respect to N), then S_1 is hyperbolic (with respect to N). In the following we shall prove a generalization of this theorem.

THEOREM 5.1. *Let $S = S_1 + S_2$ where S_2 is an arbitrary distribution $\in \mathscr{E}'$ with support in $\{x\cdot N > 0\}$. Then the conditions that $P(D)$ be hyperbolic with respect to N and* $\operatorname{supp} S \subset \Gamma$*-cone are together equivalent to the following conditions*:

(1) *S is hyperbolic with respect to N*,

(2) *for some constants C and* t_1, $|t|\cdot|P_k(\xi + itN)| \leqslant C|P(\xi + itN)|$ *when* $t \leqslant t_1$ *and* $\xi \in R^n$ $(k = 1, 2, \ldots, l)$,

(3) $P(N) \neq 0$.

Proof. Let S satisfy the conditions (1), (2), and (3). Then from Theorem 2.1.3, it follows that supp $S \subset \Gamma$-cone. In order to see that $P(D)$ is hyperbolic with respect to N, it suffices to show that there is a real number t_0 such that

$$P(\xi + itN) \neq 0 \quad \text{when} \quad t < t_0 \quad \text{and} \quad \xi \in R^n. \tag{5.2}$$

Since S is hyperbolic with respect to N, in view of Theorem 4.1, for any $\sigma > 0$ there exist positive constants m and C such that

$$|S\hat{\ }(\xi + itN)| \geqslant e^{\sigma t} \tag{5.3}$$

when

$$e^{-t} \geqslant C(1 + |\xi + itN|)^m \tag{5.4}$$

where $t < 0$. Suppose that (5.2) is false. Then we can find two functions ξ_ρ and t_ρ in ρ (>0) such that

$$\begin{aligned} P(\xi_\rho + it_\rho N) &= 0, \\ |\xi_\rho| &= \rho, \\ t_\rho &= a\rho^\mu(1 + o(1)), \end{aligned} \tag{5.5}$$

where $0 < \mu \leqslant 1$ and $a < 0$ **(6)**. Certainly we have, for some positive constants C and A,

$$|\xi_\rho + it_\rho N|^m \leqslant C\rho^m,$$
$$\exp|t_\rho| \geqslant \exp A^{\rho^\mu}.$$

For sufficiently large ρ, (ξ_ρ, t_ρ) satisfy (5.4), so that

$$|S\hat{\ }(\xi_\rho + it_\rho N)| \geqslant \exp \sigma t_\rho. \tag{5.6}$$

We note that

$$S\hat{\ }(\xi_\rho + it_\rho N) = S\hat{\ }_2(\xi_\rho + it_\rho N).$$

This follows immediately from condition (2) and (5.5). Since

$$\operatorname{supp} S_2 \subset \{x\cdot N > 0\},$$

there exists $\epsilon > 0$ such that supp $S_2 \subset \{x \cdot N \geqslant \epsilon\}$ and S_2 is of the form:

$$S_2 = \sum_{|\alpha| \leqslant \nu} D^\alpha f_\alpha,$$

where $f_\alpha \in L^\infty(\omega)$ and ω is a compact set $\subset \{x \cdot N \geqslant \epsilon\}$. Hence we have

$$S\hat{\ }_2(\xi_\rho + it_\rho N) = \sum_{|\alpha| \leqslant \nu} (\xi_\rho + it_\rho N) f\hat{\ }_\alpha(\xi_\rho + it_\rho N).$$

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Since

$$f\hat{}_\alpha(\xi_\rho + it_\rho N) = \int_{x.N \geqslant \epsilon} f_\alpha(x)\exp[-ix\cdot(\xi_\rho + it_\rho N)]dx,$$

we obtain

$$|f\hat{}_\alpha(\xi_\rho + it_\rho N)| \leqslant C_1 ||f_\alpha||_\infty \exp \epsilon t_\rho.$$

Also we have for all α ($|\alpha| \leqslant \nu$)

$$|(\xi_\rho + it_\rho N)^\alpha| \leqslant C_2 \rho^\nu.$$

Combining these inequalities we have for some constant C

$$|S\hat{}_2(\xi_\rho + it_\rho N)| \leqslant C\rho^\nu \exp \epsilon t_\rho. \tag{5.7}$$

Thus (5.6) and (5.7) give, for sufficiently large ρ,

$$\exp \sigma t_\rho \leqslant C\rho^\nu \exp \epsilon t_\rho. \tag{5.8}$$

Here we note that σ can be chosen so that $\sigma < \epsilon$. Hence from (5.8) it follows that

$$\exp[(\epsilon - \sigma)A\rho^\mu] \leqslant C\rho^\nu,$$

which leads to a contradiction when $\rho \to \infty$. Thus (5.2) must be true, and together with the condition (3) this implies that $P(D)$ must be hyperbolic with respect to N.

Conversely, let us assume that $P(D)$ is hyperbolic with respect to N and $\operatorname{supp} S \subset \Gamma$-cone. The argument in the proof of Theorem 2.3.1 applies to $S = S_1 + S_2$, where $\operatorname{supp} S_2 \subset \{x\cdot N > 0\}$, even though S_1 is not a truncation of S. Then it remains only to prove condition (2) and that S_1 is hyperbolic with respect to N. According to (**6**, Lemma 5.5.1), if

$$P(D) \succ Q(D) = Q_m(D) + Q_{m-1}(D) + \ldots,$$

then $P(D) \succ Q_k(D)$ for every k, where P and Q_k are homogeneous polynomials ($\deg Q_k = k$). Hence we have for some constant C_0,

$$|Q_k(\xi)| \leqslant C_0\tilde{P}(\xi).$$

The Taylor expansions of $\tilde{P}$ and Q_k yield

$$|Q_k(\xi - iN)| \leqslant C_1\tilde{P}(\xi) \leqslant C_2\tilde{P}(\xi - iN) \tag{5.9}$$

for suitable constants C_1 and C_2.

On the other hand, we have (P being homogeneous, we may take $t_0 = 0$)

$$|\tilde{P}(\xi - iN)| \leqslant \text{const.}\ |P(\xi - iN)| \tag{5.10}$$

provided that P is hyperbolic with respect to N (**6**). Thus (5.9) and (5.10) imply that

$$|Q_k(\xi - iN)| \leqslant C\,|P(\xi - iN)| \tag{5.11}$$

with some constant C, for every k. In (5.11) we replace ξ by $-\xi/t$. Since both sides of (5.11) are homogeneous polynomials, we obtain that for every k

$$(5.12) \qquad |t|\cdot|Q_k(\xi + itN)| \leqslant C\,|P(\xi + itN)|$$

when $\deg Q_k < \deg \mathrm{P}$ and $t < -1$, which is condition (2). Now we recall that

$$E_P(x) = (2\pi)^{-n}\int \frac{\exp[ix\cdot(\xi + itN)]}{P(\xi + itN)}\,d\xi, \qquad t \leqslant t_1,$$

gives a fundamental solution for $P(D)$, with support in a Γ-cone. In an analogous way we observe that

$$(2\pi)^{-n}\int \frac{\exp[ix\cdot(\xi + itN)]}{P(\xi + itN)^k}\,d\xi$$

defines a fundamental solution for $(P(D))^k$ with support in a Γ-cone. On the other hand, the k-tuple convolution of E_P

$$E_P^{*k} = \overbrace{E_P * \ldots * E_P}^{k}$$

is also a fundamental solution for $(P(D))^k$, and

$$\operatorname{supp} E_P^{*k} \subset k \operatorname{supp} E_P \subset \Gamma\text{-cone}.$$

Hence from Theorem 2.1.2 it follows immediately that

$$E_P^{*k}(x) = (2\pi)^{-n}\int \frac{\exp[ix\cdot(\xi + itN)]}{P(\xi + itN)^k}\,d\xi.$$

Now set $Q = S_1 - P(D)\delta$. Then for $\phi \in C_0^\infty$ we have

$$\langle E_P^{*(k+1)} * Q^{*k}, \phi\rangle = \langle E_P^{*(k+1)}, Q^{\vee *k} * \phi\rangle$$
$$= (2\pi)^{-n}\int \frac{Q^\wedge(\xi + itN)^k}{P(\xi + itN)^{k+1}}\,\tilde{\phi}(\xi + itN)d\xi.$$

Since $|P(\xi + itN)|$ is bounded from below when $t \leqslant t_1$ for some constant t_1, (5.1) and (5.12) give the estimate

$$|\langle E_P^{*(k+1)} * Q^{*k}, \phi\rangle|$$
$$\leqslant (2\pi)^{-n}\sum_{j=1}^{l}\int \frac{|P_j(\xi + itN)|^k}{|P(\xi + itN)|^{k+1}}\,|\mu^\wedge_j(\xi + itN)|\cdot|\tilde{\phi}(\xi + itN)|d\xi$$
$$\leqslant \text{const.}\ |t|^{-k}\sum_{j=1}^{l}\int |\mu^\wedge_j(\xi + itN)|(1 + |\xi|)^{-(n+1)}d\xi,$$

when $t < \min(-1, t_1)$.

However, we have

$$\mu^\wedge_j(\xi + itN) = \int \exp[-ix\cdot(\xi + itN)]d\mu_j(x)$$

and

$$\operatorname{supp} \mu_j \subset \operatorname{supp} S_1 \qquad \text{for } j = 1, 2, \ldots, l.$$

Therefore for some constant $M > 0$,

(5.13) $$|\mu\hat{}_j(\xi + itN)| \leqslant M, \qquad j = 1, 2, \ldots, l.$$

For, let Ω be an open set, containing $\operatorname{supp} S_1$ and with compact closure K. Then there is a constant $a > 0$ such that for all $f \in C_0(K)$

$$|\langle \mu_j, f\rangle| \leqslant a \, ||f||_\infty.$$

Now take $\alpha \in C_0(K)$ so that $0 \leqslant \alpha \leqslant 1$ and $\alpha(x) = 1$ on a neighbourhood of $\operatorname{supp} S_1$. Then we have

$$|\langle \mu_j, f\rangle| = |\langle \mu_j, \alpha f\rangle| \leqslant a \, ||f||_\infty$$

for $f \in C^\infty$. Since for $x \cdot N \geqslant 0$

$$|\alpha(x) \exp[-ix\cdot(\xi + itN)]| \leqslant 1,$$

we obtain (5.13) for $j = 1, 2, \ldots, l$. Thus we have proved that for a constant $C > 0$

$$|\langle E_P{}^{*(k+1)} * Q^{*k}, \phi \rangle | \leqslant C \, |t|^{-k}, \qquad k = 1, 2, \ldots .$$

Hence for each $\phi \in C_0{}^\infty$ the series

$$\sum_{k=0}^{\infty} (-1)^k \langle E_P{}^{*(k+1)} * Q^{*k}, \phi\rangle$$

is convergent when $t < \min(t_1, -1)$, or

$$E_1 = \sum_{k=0}^{\infty} (-1)^k E_P{}^{*(k+1)} * Q^{*k}$$

converges in $\mathscr{D}'$ when $t < \min(t_1, -1)$. Because of (5.1) and our assumption, we obtain $\operatorname{supp} E_1 \subset \Gamma$-cone. Thus we have proved that S_1 is hyperbolic with respect to N. This completes the proof.

6. Structure of fundamental solutions.

6.1. Fundamental solutions for $P(D)\delta + Q$. By H^s (s real) we mean the space of $u \in L^2$ such that

$$(1 + |\xi|^2)^{s/2} \, u\hat{}(\xi) \in L^2$$

with the norm

$$||(1 + |\xi|^2)^{s/2} \, u\hat{}||_2,$$

by which H^s is made a Hilbert space .If $s = -m$ (m a positive integer) it is well known that $u \in H^{-m}$ if and only if there exist $f_\alpha \in L^2$ for $|\alpha| \leqslant m$ such that

$$u = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha.$$

Replacing L^2 by L^2_{loc} in the definition of H^{-m}, we get the space $H^{-m}{}_{\text{loc}}$.

Now we shall discuss the structure of fundamental solutions for the operator

$$S = P(D)\delta + \sum_{j=1}^{l} P_j(D)\mu_j, \tag{6.1.1}$$

where

(1) $P(D)$ is homogeneous and hyperbolic with respect to N;

(2) $D_k P_j(D) \prec P(D)$ for $j = 1, 2, \ldots, l$; $k = 1, 2, \ldots, n$; and

(3) supp $\mu_j \subset \Gamma$-cone for $j = 1, 2, \ldots, l$.

From (2) it follows that

$$(1 + |\xi + itN|)\, |P_j(\xi + itN)| \leqslant C\, |P(\xi + itN)|$$

for some constant C, when $t \leqslant t_1$ for some t_1. For, we can obtain as in the proof of Theorem 5.1 that when $t \leqslant t_1$

$$|\xi_j + itN_j|\, |P_k(\xi + itN)| \leqslant C_1\, |P(\xi + itN)|, \qquad j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, l$$

for suitable constants C_1 and t_1. Hence if we set

$$Q = \sum_{j=1}^{l} P_j(D)\mu_j,$$

then

$$\left| \frac{Q^\wedge(\xi + itN)}{P(\xi + itN)} \right| \leqslant C(1 + |\xi|)^{-1} \tag{6.1.2}$$

with some constant C when $t \leqslant t_1$. Let us define, as usual, the unique fundamental solution for S with support in a Γ-cone by

$$E = E_P * \sum_{k=0}^{\infty} (-1)^k (E_P * Q)^{*k},$$

where E_P is a fundamental solution for $P(D)$ and is given by

$$(2\pi)^{-n} \int \frac{\exp[ix\cdot(\xi + itN)]}{P(\xi + itN)}\, d\xi$$

for an arbitrarily fixed $t \leqslant t_1$. Thus we have for $\phi \in C_0^\infty$

$$\begin{aligned} \langle E_P{}^{*(k+1)} * Q^{*k}, \phi\rangle &= \langle E_P{}^{*(k+1)}, Q^{\vee *k} * \phi\rangle \\ &= (2\pi)^{-n} \int \frac{Q^\vee(\xi + itN)^k}{P(\xi + itN)^{k+1}}\, \tilde{\phi}(\xi + itN)\, d\xi. \end{aligned}$$

Therefore the distribution $E_P{}^{*(k+1)} * Q^{*k}$ is represented by the formula

$$(2\pi)^{-n} e^{|t| x.N} \int \frac{Q^\wedge(\xi + itN)^k}{P(\xi + itN)^{k+1}}\, e^{ix.\xi} d\xi. \tag{6.1.3}$$

In view of (6.1.2) we have for $t \leqslant t_1$

$$\frac{Q^\wedge(\xi + itN)^k}{P(\xi + itN)^{k+1}} \in L^2$$

when $k \geqslant [\frac{1}{2}n] + 1$ since $|P(\xi + itN)| \geqslant \text{const.} > 0$, so that the Fourier inverse transform

$$(2\pi)^{-n} \int \frac{Q^\wedge(\xi + itN)^k}{P(\xi + itN)^{k+1}} e^{ix\cdot\xi} d\xi \in L^2,$$

and the distribution $E_P{}^{*(k+1)} * Q^{*k} \in L^2{}_{\text{loc}}$ when $k \geqslant [\frac{1}{2}n] + 1$. On the other hand if $k < [\frac{1}{2}n] + 1$, we observe that

$$\frac{Q^\wedge(\xi + itN)^k}{P(\xi + itN)^{k+1}} = (1 + |\xi|^2)^{[\frac{1}{4}n]+1} \tilde{u}_{k,n},$$

where $\tilde{u}_{k,n} \in L^2$, and hence that the Fourier inverse transform is

$$(1 - \Delta)^{[\frac{1}{4}n]+1} u_{k,n} \in H^{-[\frac{1}{2}n]-2}$$

$$\text{or } E_P{}^{*(k+1)} * Q^{*k} \in H_{\text{loc}}^{-[\frac{1}{2}n]-2} \qquad \text{when} \qquad k < [\tfrac{1}{2}n] + 1.$$

Defining E_0 and F_k by

$$E_0 = \sum_{k=0}^{[\frac{1}{2}n]} (-1)^k E_P{}^{*(k+1)} * Q^{*k},$$

$$F_k = (-1)^{[\frac{1}{2}n]+k} E_P{}^{*([\frac{1}{2}n]+k+1)} * Q^{*([\frac{1}{2}n]+k)},$$

we obtain

$$E = E_0 + F_1 + F_2 + \ldots .$$

Thus we have proved the following theorem.

THEOREM 6.1.1. *Let $S \in \mathscr{E}'$ be defined as in* (6.1.1.) *Then the unique fundamental solution E for S with support in a Γ-cone is of the form*

$$E = E_0 + F_1 + F_2 + \ldots ,$$

where $E_0 \in H_{\text{loc}}{}^{-[\frac{1}{2}n]-2}$ and $F_k \in L^2{}_{\text{loc}}$ for $k = 1, 2, \ldots$.

COROLLARY 6.1.1. *Let S be the hyperbolic finite difference–differential operator defined in Section 3 with P_0 homogeneous, and let $D_k P_j \prec P_0$ for $j = 1, 2, \ldots, l$; $k = 1, 2, \ldots, n$. Then the fundamental solution E with support in a Γ-cone is of the form*

$$E = E_0 + F_1 + F_2 + \ldots ,$$

where $E_0 \in H_{\text{loc}}{}^{-[\frac{1}{2}n]-2}$ and $F_k \in L^2{}_{\text{loc}}$ with support in the half-space $x\cdot N \geqslant ([\frac{1}{2}n] + k)a_1\cdot N$ for $k = 1, 2, \ldots$. (Here we assume that $a_1\cdot N \leqslant a_k\cdot N$ for $k = 2, 3, \ldots, l$.)

For, supp Q is contained in the half-space $x \cdot N \geqslant a_1 \cdot N$; and then we have

$$\operatorname{supp} F_k \subset ([\tfrac{1}{2}n] + k + 1) \operatorname{supp} E_{P_0} + ([\tfrac{1}{2}n] + k) \operatorname{supp} Q \subset ([\tfrac{1}{2}n] + k) H_{a_1 \cdot N},$$

where we set $H_{a_1 \cdot N} = \{x \cdot N \geqslant a_1 \cdot N\}$.

Example. Let $S = S(x, y, t)$ be given by

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) \delta + \delta_{(a,b,c)}$$

in R^3. Since $\partial^2/\partial t^2 - \Delta$ is hyperbolic with respect to $N = (0, 0, 1)$,

$$E_P(x, y) = (2\pi)^{-3} e^{\sigma\tau} \int \frac{e^{i(x\xi + y\eta)}}{(\tau + i\sigma)^2 - (\xi^2 + \eta^2)} d\xi d\eta$$

for any fixed $\sigma > 0$. Now $P_0(\xi, \eta, \tau) = (\xi^2 + \eta^2) - \tau^2$. Since

$$|P_0(\xi, \eta, \tau + i\sigma)| \geqslant C\tilde{P}_0(\xi, \eta, \tau)$$

for large $|\sigma|$ and since

$$\tilde{P}_0(\xi, \eta, \tau)^2 \geqslant \{1 + (\xi^2 + \eta^2 + \tau^2)^{\frac{1}{2}}\}^2,$$

we have

$$|(\tau + i\sigma)^2 - (\xi^2 + \eta^2)| \geqslant \text{const.}\{1 + (\xi^2 + \eta^2 + \tau^2)^{\frac{1}{2}}\}$$

for large $|\sigma|$. Hence we obtain, for $k \geqslant 1$,

$$F_k = E_{P_0}(x, y, t) * E_{P_0}(x - a, y - b, t - c)^{*k} \in L^2_{\text{loc}}$$

and supp $F_k \subset \{x \cdot N \geqslant kc\}$, so that $E_0 = E_{P_0} \in H_{\text{loc}}^{-2}$.

6.2. Singular support of E. Finally we shall study the singular support of the fundamental solution E, with support in a Γ-cone, for a distribution $S \in \mathscr{E}'$ such that S has a hyperbolic truncation S_1. We denote the singular support of a distribution f by $\operatorname{ss}(f)$, that is, the smallest closed set outside of which f is equal to a C^∞-function.

LEMMA 6.2.1. *Let f and g be in $\mathscr{D}'$ and let one of them have compact support. Then*

$$\operatorname{ss}(f * g) \subset \operatorname{ss}(f) + \operatorname{ss}(g).$$

Proof. First we assume that both f and g are in $\mathscr{E}'$. Take $\alpha, \beta \in C_0^\infty$ so that $\alpha(x) = 1$ in a neighbourhood of $\operatorname{ss}(f)$ and $\beta(x) = 1$ in a neighbourhood of $\operatorname{ss}(g)$. Since

$$f * g = \alpha f * \beta g + h,$$

where $h \in C_0^\infty$, we have

$$\operatorname{ss}(f * g) \subset \operatorname{supp}(\alpha f * \beta g).$$

Let W be a neighbourhood of $O \in R^n$. Let us take another neighbourhood U of $O \in R^n$ so that $2U \subset W$. If we take $\operatorname{supp} \alpha$, $\operatorname{supp} \beta$ so small that

$$\operatorname{supp} \alpha \subset U + \operatorname{ss}(f), \qquad \operatorname{supp} \beta \subset U + \operatorname{ss}(g),$$

then

$$\begin{aligned} \operatorname{supp}(\alpha f * \beta g) &\subset \operatorname{supp} \alpha \cap \operatorname{supp} f + \operatorname{supp} \beta \cap \operatorname{supp} g \\ &\subset 2U + \operatorname{ss}(f) + \operatorname{ss}(g) \\ &\subset W + \operatorname{ss}(f) + \operatorname{ss}(g). \end{aligned}$$

Consequently we have

$$\operatorname{ss}(f * g) \subset \operatorname{ss}(f) + \operatorname{ss}(g).$$

Now we shall pass to the general case where $f \in \mathscr{E}'$, $g \in \mathscr{D}'$. We take a partition of unity $\{\alpha_j\{ \subset C_0^\infty$, and apply the above argument to f and $\alpha_j g$ to obtain

$$\begin{aligned} ss(f * g) &= \operatorname{ss}(\textstyle\sum_j f * \alpha_j g) \\ &\subset \textstyle\bigcup_j \operatorname{ss}(f * \alpha_j g) \\ &\subset \textstyle\bigcup_j (\operatorname{ss}(f) + \operatorname{ss}(\alpha_j g)). \end{aligned}$$

Since $\operatorname{ss}(\alpha_j g) \subset \operatorname{ss}(g) \cap \operatorname{supp} \alpha_j$ we conclude that

$$\operatorname{ss}(f * g) \subset \operatorname{ss}(f) + \operatorname{ss}(g),$$

which proves our lemma.

THEOREM 6.2.1. *Let $S \in \mathscr{E}'$ have a hyperbolic operator S_1 with respect to N as a truncation, and let E be the fundamental solution for S with support in a Γ-cone. Then*

$$\operatorname{ss}(E) \subset \bigcup_{k=0}^{\infty} ((k + 1)\operatorname{ss}(E_1) + k \operatorname{ss}(S_2)),$$

where E_1 is the fundamental solution for S_1 with support in a Γ-cone.

Proof. Put

$$E_{\nu+1} = E_1 * \sum_{k=0}^{\nu} (-1)^k (E_1 * S_2)^{*k},$$

where $S_2 = S - S_1$. Then we obtain $\lim_\nu E_{\nu+1} = E$. From Lemma 6.2.1 it follows that

$$\begin{aligned} \operatorname{ss}(E_{\nu+1}) &= \operatorname{ss}\left(E_1 * \sum_{k=0}^{\nu} (-1)^k (E_1 * S_2)^{*k}\right) \\ &\subset \operatorname{ss}\left(\sum_{k=0}^{\nu} E_1^{*(k+1)} * S_2^{*k}\right) \\ &\subset \bigcup_{k=0}^{\nu} ((k + 1)\operatorname{ss}(E_1) + k \operatorname{ss}(S_2)). \end{aligned}$$

Thus we conclude that

$$\mathrm{ss}(E) \subset \bigcup_{k=0}^{\infty} ((k + 1)\mathrm{ss}(E_1) + k\,\mathrm{ss}(S_2)),$$

which proves our theorem.

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