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A functorial version of a construction of Hochschild and Mostow for representations of Lie algebras

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Let $\underline{\underline{g}}$ be a Lie algebra, $\underline{\underline{h}}$ a complemented ideal of $\underline{\underline{g}}$, and W an $\underline{\underline{h}}$-module. Hochschild and Mostow have described the construction of a $\underline{\underline{g}}$-module "induced" from W , which is finite-dimensional provided W is finite-dimensional and satisfies a nilpotent action condition. This note describes a modification of their construction which is functorial and a weak adjoint to the restriction functor from $\underline{\underline{g}}$-modules to $\underline{\underline{h}}$-modules.

Throughout this paper we shall suppose that $\underline{\underline{g}}$ is a Lie algebra over a field k , that $\underline{\underline{h}}$ is an ideal of $\underline{\underline{g}}$, and that there is a subalgebra $\underline{\underline{s}}$ of $\underline{\underline{g}}$ such that $\underline{\underline{g}} = \underline{\underline{h}} \oplus \underline{\underline{s}}$. $U\underline{\underline{h}}$, $U\underline{\underline{g}}$ will denote the universal enveloping algebras of $\underline{\underline{h}}$ and $\underline{\underline{g}}$. Clearly, every $g \in \underline{\underline{g}}$ can be written uniquely as $g = h + s$ with $h \in \underline{\underline{h}}$ and $s \in \underline{\underline{s}}$. This allows us to define, (with Hochschild and Mostow [1]), a composition $*$ by

(1) $$g * u = hu + (su-us) \text{ for } g \in \underline{\underline{g}} , \ u \in U\underline{\underline{h}} .$$

It can be shown that $su - us \in U\underline{\underline{h}}$, hence (1) determines a $\underline{\underline{g}}$-module structure on $U\underline{\underline{h}}$.

We shall use mod-$\underline{\underline{h}}$, mod-$\underline{\underline{g}}$ to denote the categories of right $\underline{\underline{h}}$- and $\underline{\underline{g}}$-modules, and F : mod-$\underline{\underline{g}}$ $\to$ mod-$\underline{\underline{h}}$ to denote the restriction functor. Now let $W \in$ mod-$\underline{\underline{h}}$. Then $\hom_k(U\underline{\underline{h}}, W)$ has a $\underline{\underline{g}}$-module structure given by

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$f^g(u) = f(g{*}u)$ for $f \in \hom_k(U\underline{\underline{h}}, W)$, $g \in \underline{g}$, and $u \in U\underline{\underline{h}}$.

Construction of the functor I

Define a map $\hat{j}_W : W \to \hom_k(U\underline{\underline{h}}, W)$ by setting $\hat{j}_W(w)(u) = w.u$ for $w \in W$ and $u \in U\underline{\underline{h}}$. It is easy to check that $\hat{j}_W$ is a $U\underline{\underline{h}}$-monomorphism. We define a $\underline{g}$-submodule IW of $\hom_k(U\underline{\underline{h}}, W)$ by setting $IW = \left(\operatorname{im} \hat{j}_W\right).U\underline{g}$. Now let j_W be $\hat{j}_W$ with codomain restricted to be FIW . Let $W, W' \in$ mod-$\underline{\underline{h}}$ and $\psi \in \hom_{U\underline{\underline{h}}}(W, W')$. We define $I\psi : IW \to IW'$ by $[(I\psi)(f)](u) = (\psi \circ f)(u)$ for $u \in U\underline{\underline{h}}$ and $f \in IW$. We must show $(I\psi)(f) \in IW'$. Since $f \in IW$, f may be written as

$$f = \sum_{i=1}^{n} j_W(w_i).x_i$$

for suitable $w_i \in W$, $x_i \in U\underline{g}$. Then, for $u \in U\underline{\underline{h}}$,

$$\begin{aligned} [(I\psi)(f)](u) &= \left[\psi \circ \sum_{i=1}^{n} j_W(w_i).x_i\right](u) \\ &= \psi\left(\sum_{i=1}^{n} x_i(x_i * u)\right) \\ &= \sum_{i=1}^{n} \psi(w_i).(x_i * u) \\ &= \left[\sum_{i=1}^{n} j_{W'}(\psi(w_i)).x_i\right](u) , \end{aligned}$$

so $(I\psi)(f) = \sum_{i=1}^{n} j_{W'}(\psi(w_i)).x_i \in IW'$. If W'' is another $\underline{\underline{h}}$-module and $\psi' \in \hom_{U\underline{\underline{h}}}(W', W'')$, then $(\psi' \circ \psi) \circ f = \psi' \circ (\psi \circ f)$; it follows that I has the multiplicative property of a functor.

LEMMA 1. *j_W is natural in W .*

Proof. We must show that if $\psi \in \hom_{U\underline{\underline{h}}}(W, W')$ then $FI\psi \circ j_W = j_{W'} \circ \psi$. Suppose $w \in W$ and $u \in U\underline{\underline{h}}$. Then

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$$\big(FI\psi \circ j_W\big)(w)(u) = \big(\psi \circ j_W(w)\big)(u) = \psi(w.u) = \psi(w).u = j_{W'}\big(\psi(w)\big)(u)$$
$$= \big(j_{W'} \circ \psi\big)(w)(u)$$

as required. □

LEMMA 2. *I is a faithful functor.*

Proof. If $I\psi = 0$, then for all $w \in W$, $0 = (I\psi)\big(j_W(w)\big)$; so $0 = \big[(I\psi)\big(j_W(w)\big)\big]\big(1_{U\underline{\underline{h}}}\big) = \psi(w)$. That is, $\psi = 0$. □

THEOREM 3. *The functor $I : \text{mod-}\underline{\underline{h}} \to \text{mod-}\underline{\underline{g}}$, described above, is an injective weak left adjoint to F. That is, for $W \in \text{mod-}\underline{\underline{h}}$ and $V \in \text{mod-}\underline{\underline{g}}$, there is an injection*

$$\theta_{WV} : \hom_{U\underline{\underline{g}}}(IW, V) \to \hom_{U\underline{\underline{h}}}(W, FV)$$

which is natural in W and V.

Proof. For $\phi \in \hom_{U\underline{\underline{g}}}(IW, V)$, we define $\theta_{WV}(\phi) = F\phi \circ j_W$. The naturality of θ_{WV} follows from that of j_W and the definition of θ_{WV}. We must show that θ_{WV} is injective. Suppose that $\phi_1, \phi_2 \in \hom_{U\underline{\underline{g}}}(IW, V)$, and that $F\phi_1 \circ j_W = F\phi_2 \circ j_W$. Then ϕ_1 and ϕ_2 coincide on $\operatorname{im} j_W$. Since ϕ_1, ϕ_2 are $U\underline{\underline{g}}$-homomorphisms, it follows that they must coincide on $\big(\operatorname{im} j_W\big).U\underline{\underline{g}} = IW$. □

THEOREM 4 (compare Hochschild and Mostow [1] and Zassenhaus [2]). *Let $\underline{\underline{g}}$ be a finite-dimensional Lie algebra over a field k of characteristic zero, and let $\underline{\underline{h}}$ be an ideal of $\underline{\underline{g}}$ with complementary subalgebra $\underline{\underline{s}}$. Let W be a finite-dimensional $\underline{\underline{h}}$-module on which $[\underline{\underline{h}}, \underline{\underline{s}}]$ acts nilpotently. Then IW, as defined above, is a finite-dimensional $\underline{\underline{g}}$-module.*

Proof. If $\{0\} = W_0 < W_1 < \ldots < W_n = W$ is a composition series for W, then set $S(W) = \big(W_n/W_{n-1}\big) \oplus \ldots \oplus \big(W_2/W_1\big) \oplus \big(W_1/W_0\big)$. By the Jordan-Hölder theorem, $S(W)$ is determined up to isomorphism. Clearly, a subalgebra of $\underline{\underline{h}}$ acts nilpotently on W if and only if it annihilates $S(W)$. Let us write $d = \dim_k W$, and let $\operatorname{ann}_{U\underline{\underline{h}}}(M)$ denote the annihilator in $U\underline{\underline{h}}$ of an $\underline{\underline{h}}$-module M. Obviously,

$$(2) \qquad \left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d \subseteq \operatorname{ann}_{U\underline{\underline{h}}}(W) \subseteq \operatorname{ann}_{U\underline{\underline{h}}}(S(W)) \ .$$

Since, by hypothesis, $[\underline{\underline{h}}, \underline{\underline{s}}] \subseteq \operatorname{ann}_{U\underline{\underline{h}}}(S(W))$, it follows that $\operatorname{ann}_{U\underline{\underline{h}}}(S(W))$ is a $\underline{\underline{g}}$-submodule of W. Hence $\left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d$ is a $\underline{\underline{g}}$-submodule of $U\underline{\underline{h}}$.

If $f \in \hom_k(U\underline{\underline{h}}, W)$ and $f\left(\operatorname{ann}_{U\underline{\underline{h}}}(W)\right) = \{0\}$, then for all $x \in U\underline{\underline{g}}$,

$$\begin{aligned} f^x\left\{\left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d\right\} &\subseteq f\left\{x\cdot\left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d\right\} \\ &\subseteq f\left(\operatorname{ann}_{U\underline{\underline{h}}}(W)\right) \quad \text{by (2)}, \\ &= \{0\} \ . \end{aligned}$$

Now $\operatorname{im} j_W$ annihilates $\operatorname{ann}_{U\underline{\underline{h}}}(W)$; so $IW = \left(\operatorname{im} j_W\right).U\underline{\underline{g}}$ annihilates $\left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d$. Let us write $J = \left(\operatorname{ann}_{U\underline{\underline{h}}}(S(W))\right)^d$. Then it is easy to see that IW is embedded in $\hom_k(U\underline{\underline{h}}/J, W)$. Since W is finite-dimensional, $\operatorname{ann}_{U\underline{\underline{h}}}(W)$ is of finite codimension in $U\underline{\underline{h}}$. Hence, by (2), $\operatorname{ann}_{U\underline{\underline{h}}}(S(W))$ is of finite codimension in $U\underline{\underline{h}}$. Now we appeal to a result of Zassenhaus [2, page 263], which states that if X, Y are ideals of $U\underline{\underline{h}}$ of finite codimension, then so is XY. We deduce from this that J is of finite codimension in $U\underline{\underline{h}}$, so that $\dim_k \hom_k(U\underline{\underline{h}}/J, W) < \infty$, and so $\dim_k IW < \infty$. □

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