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ON A CONTINUUM PERCOLATION MODEL

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Abstract

Consider particles placed in space by a Poisson process. Pairs of particles are bonded together, independently of other pairs, with a probability that depends on their separation, leading to the formation of clusters of particles. We prove the existence of a non-trivial critical intensity at which percolation occurs (that is, an infinite cluster forms). We then prove the continuity of the cluster density, or free energy. Also, we derive a formula for the probability that an arbitrary Poisson particle lies in a cluster consisting of k particles (or equivalently, a formula for the density of such clusters), and show that at high Poisson intensity, the probability that an arbitrary Poisson particle is isolated, given that it lies in a finite cluster, approaches 1.

POISSON PROCESS; CLUSTER DENSITY; LARGE DEVIATIONS AT HIGH DENSITY

1. Introduction

In this paper we consider the following percolation model on Euclidean space $\mathbb{R}^d$, $d \geqq 2$ (with the Euclidean norm $|\cdot|$). This model was introduced by Gilbert (1961). Let $g(x)$, $x \in \mathbb{R}^d$, be a measurable function taking values in $[0, 1]$, such that

$$g(-x) = g(x), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

$$0 < \int_{\mathbb{R}^d} g(x)\, dx < \infty. \tag{1.2}$$

Let $\mathcal{P}$ be a homogeneous Poisson process on $\mathbb{R}^d$ with rate ρ (so the expected number of points of $\mathcal{P}$ in any region is ρ times the Lebesgue measure ('volume') of that region). Each realization of $\mathcal{P}$ may be viewed as a random subset $\{X_1, X_2, X_3, \cdots\}$ of $\mathbb{R}^d$. Think of particles being placed at $X_1, X_2, X_3, \cdots$ by $\mathcal{P}$. Given a realization of $\mathcal{P}$, for each pair (X_i, X_j) of particles of $\mathcal{P}$, form a 'bond' between X_i and X_j with probability $g(X_i - X_j)$, independently of all other pairs of points of $\mathcal{P}$. Let the connected components of the resulting infinite random graph be called *clusters*.

There are several reasons for studying this model (see Kesten (1987), Section 2.4). A version of the motivating model of Gilbert (1961) (see also Kesten (1987), Section 2.4) is to set $d = 2$ and X_i to be the location of a communications station. Two stations separated by a vector x may pass messages directly between each other

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with probability $g(x)$. The clusters of stations which can communicate with one another may be of interest.

Another motivating model is to consider the spread of disease in a forest, in which trees are scattered randomly over $\mathbb{R}^2$ by a Poisson process, and disease communicates between two trees with a probability depending on their separation. Other percolation models in polymerization and statistical mechanics also fit naturally into a continuous context.

Generally, in these physical models the point process used consists of a large finite number of points independently uniformly distributed over some large set in $\mathbb{R}^d$. It is more convenient to consider an infinite Poisson process, which is a good approximation to this finite point process, away from the boundary of the set.

A special case of our model has $g(x) = I_{\{|x| \leqq r_0\}}$, where $I_{\{\ \}}$ denotes indicator function, that is $g(x) = 1$ if $|x| \leqq r_0$, 0 if $|x| > r_0$. In this case all the randomness of the model is from the Poisson process. This model is equivalent to the placing of balls of radius $r_0/2$ around each particle X_i, and examining the connected components (i.e. clusters) of the union of these balls. Such a model (the 'Poisson blob' model) can be found in Section 10.5 of Grimmett (1989), and references therein. One generalization of the Poisson blob model is to replace the balls of fixed size by random shapes (the 'Boolean model'). See Hall (1985), (1986), (1988), and Stoyan et al. (1987). The model here is an alternative generalization.

In this paper we derive three facts about this model. First, we show that the critical intensity ρ_1 of the Poisson process, at which the mean cluster size becomes infinite, and the critical intensity ρ_2 at which an infinite cluster appears, are non-trivial in the sense that $0 < \rho_i < \infty$ $(i = 1, 2)$. This is one difference between our model and the Boolean model, in which ρ_1 may be 0 (Hall (1985)).

Second, we look at the cluster density (number of clusters per unit volume) and show it is a well-defined continuous function of the parameter ρ. This is known as the 'free energy' in the physics literature. Our results are the continuum analog of the results of Grimmett (1976).

Third, we show that under some extra conditions on g, one of which is that g have bounded support, when ρ becomes large, the expressions for the density of finite clusters, and for the proportion of the particles not in an infinite cluster, are dominated by terms from the isolated particles. In other words, when ρ is large, 'most of the finite clusters are 1-clusters' (a 1-cluster is one that consists of a single Poisson unit). The argument of this result can be extended to some cases of the Boolean model.

2. The cluster at the origin

Suppose now that we add a point $X_0 = 0$ to the Poisson process $\{X_1, X_2, X_3, \cdots\} = \mathcal{P}$. The resulting point process, given by $\mathcal{P} \cup \{0\} = \{X_0, X_1, X_2, \cdots\}$, is a Poisson process 'conditioned to have a point at 0', in the sense of Palm measures (see for example Daley and Vere-Jones (1988), and references therein). Indeed, if we condition $\mathcal{P}$ to have a point in a small neighborhood of 0, the conditional distribution of the points of $\mathcal{P}$ outside that neighborhood is still a Poisson process. In the terminology of Hall ((1985), (1986)), we are considering an 'arbitrary point of the Poisson process', which is without loss of generality assumed to lie at 0.

Given a realization of $\mathcal{P}$, form bonds between points of $\mathcal{P} \cup \{0\}$ by the same rule as before. That is, form a bond between X_i and X_i $(0 \leqq i < j < \infty)$ with probability $g(X_i - X_j)$, independently of all other pairs (X_i, X_j).

Denote by $C(0)$ the 'cluster at the origin', that is $C(0)$ is the union of $\{0\}$ and the set of X_i for which there exists $n \geqq 1$ and $i(0), i(1), \cdots, i(n)$ with $i(0) = 0$, $i(n) = i$ and $X_{i(k)}$ bonded to $X_{i(k+1)}$, $0 \leqq k < n$. Denote by $\#[C(0)]$ the number of points (including 0) in $C(0)$, so that $\#[C(0)]$ is a random variable taking values in $\{\infty, 1, 2, 3, \cdots\}$. Let $q_k(\rho)$ denote the probability, when $\mathcal{P}$ has intensity ρ, that $C(0)$ consists of k points. That is,

$$q_k(\rho) = P(\#[C(0)] = k), \qquad k = 1, 2, 3, \cdots.$$

Also, for any $N \in \mathbb{R}$, define the function $f_N(\rho)$ by

$$f_N(\rho) = \sum_{k=1}^{\infty} k^N q_k(\rho).$$

Functions of the form $f_N(\rho)$ are of great interest in this model. For example, $f_0(\rho) = P[\#[C(0)] < \infty]$, so that $f_0(\rho)$ is the proportion of the particles of $\mathcal{P}$, at intensity ρ, which do not lie in an infinite cluster. We shall see that $f_0(\rho)$ is non-increasing in ρ and the probability that an infinite cluster exists is 0 if $f_0(\rho) = 1$, 1 if $f_0(\rho) < 1$. These facts correspond to basic results in lattice percolation (Grimmett (1989), Section 1.4).

As another example,

$$\rho f_{-1}(\rho) = \rho \sum_{k=1}^{\infty} (q_k(\rho)/k)$$

is the *cluster density*; that is, the number of clusters per unit volume in a sense to be made precise below. This quantity, also known as the 'free energy', is sometimes useful (see, for example, Sykes and Essam (1964), and Aizenman et al. (1987)).

More generally, $f_N(\rho)$ is the Nth moment of the size of the cluster at an arbitrary point of $\mathcal{P}$, discounting infinite clusters. That is,

$$f_N(\rho) = E[(\#[C(0)])^N I_{\{\#[C(0)] < \infty\}}],$$

where $I_{\{\ \}}$ denotes indicator function.

3. Definitions and notation

Suppose we are given a measurable function $g : \mathbb{R}^d \to [0, 1]$, satisfying (1.1) and (1.2). On a probability space $(\Omega, \mathcal{F}, P_\rho)$, with corresponding expectation E_ρ, set up

$\mathscr{P}$, a homogeneous Poisson process with rate ρ on $\mathbb{R}^d$, and a set of Bernoulli (zero–one) random variables $(D_{\{x,y\}}, \{x, y\} \in \mathscr{A})$, where the indexing set $\mathscr{A}$ consists all unordered pairs $\{x, y\}$ of distinct elements of $\mathbb{R}^d$. Using the Kolmogorov existence theorem, arrange that for each $\{x, y\} \in \mathscr{A}$, $P[D_{\{x,y\}} = 1] = g(x - y)$, and that the $D_{\{x,y\}}$ are independent of $\mathscr{P}$ and of one another.

Given a realization of $(\mathscr{P}, (D_{\{x,y\}}, \{x, y\} \in \mathscr{A}))$, write $\mathscr{P}$ as a random set: $\mathscr{P} = \{X_1, X_2, X_3, \cdots\} \subset \mathbb{R}^d$. Define an undirected graph $\mathscr{G}$ with vertices at the points of $\mathscr{P}$, by including the edge (X_i, X_j) if and only if $D_{\{X_i,X_j\}} = 1$. The connected components of $\mathscr{G}$ are the *clusters* associated with this realization of $(\mathscr{P}, (D_{\{x,y\}}))$.

We can obtain the cluster at the origin $C(0)$ on the same probability space by extending $\mathscr{G}$ to a graph $\mathscr{G}_0$, obtained by adding a vertex at 0 to $\mathscr{P}$ and including the edge $(0, X_j)$ in $\mathscr{G}_0$ if and only if $D_{\{0,X_j\}} = 1$. Define the cluster at the origin $C(0)$ to be the component of $\mathscr{G}_0$ which includes 0 (or more precisely, the set of vertices of $\mathscr{G}_0$ lying in that component). Write $\#[C(0)]$ for the cardinality of $C(0)$.

We introduce some more notation. Suppose $C = \{x_1, x_2, \cdots, x_k\}$ is a finite set of points in $\mathbb{R}^d$, and x_0 is another point of $\mathbb{R}^d$. Suppose a random graph G is formed on the points of $C \cup \{x_0\}$ by closing the edge (x_i, x_j) with probability $g(x_i - x_j)$ independently of all other edges. Let $g_1(x_0; C)$ denote the probability that x_0 is not isolated in this random graph. That is,

$$g_1(x_0; C) = 1 - \prod_{j=1}^{k} (1 - g(x_0 - x_j)).$$

Also, define $g_2(x_0, x_1, \cdots, x_k)$ to be the probability that the random graph G is connected. That is,

$$g_2(x_0, x_1, \cdots, x_k) = \sum_{G_1} \Pi' g(x_i - x_j) \Pi''(1 - g(x_i - x_j)),$$

where the summation $\sum$ is over all connected graphs G_1 on $\{0, 1, 2, \cdots, k\}$, the product Π' is over all edges (i, j) $(1 \leqq i < j \leqq k)$ which are in G_1, and the product Π'' is over all edges (i, j) $(1 \leqq i < j \leqq k)$ which are not in G_1.

Finally, if A is a (Lebesgue) measurable set in $\mathbb{R}^d$ let $|A|$ denote its Lebesgue measure. If also $x \in \mathbb{R}^d$, let $x + A$ denote the translated set $\{x + y : y \in A\}$.

4. Statement of results

By way of practice, consider the points of $\mathscr{P}$ which are bonded to 0. By proposition 3.8 of Resnick (1987), for example, the random set of points in $\mathbb{R}^d \times \{0, 1\}$, given by $\{(X_i, D_{\{0,X_i\}}), i \geqq 1\}$, is a homogeneous Poisson process on $\mathbb{R}^d \times \{0, 1\}$ for which the expected number of points in $A \times \{1\}$ is $\rho \int_A g(x)\, dx$. So the set of points of $\mathscr{P}$ which are bonded to 0 forms a non-homogeneous Poisson process on $\mathbb{R}^d$ with intensity measure $\rho g(x)\,dx$. In particular,

$$q_1(\rho) = P_\rho[\#(C(0)) = 1] = \exp\left(-\rho\int_{\mathbb{R}^d} g(x)\,dx\right). \tag{4.1}$$

The next result extends this to a formula for all $q_j(\rho)$.

Proposition 1. (a) *For any measurable* $A \subset \mathbb{R}^d$ *and* $k \in \{1, 2, 3, \cdots\}$,

$$\begin{aligned} P_\rho[\#(C(0)) = k \quad \text{and} \quad C(0) \subset A] \\ = \frac{\rho^{k-1}}{(k-1)!}\int_A \cdots \int_A g_2(0, x_1, \cdots, x_{k-1}) \\ \times \exp\left\{-\rho\int_{\mathbb{R}^d} g_1(y; \{0, x_1, x_2, \cdots, x_{k-1}\})\,dy\right\} dx_1 \cdots dx_{k-1}. \end{aligned} \tag{4.2}$$

(b) *If* $\mathbf{A}$ *is a measurable set in* $(\mathbb{R}^d)^{k-1}$ *which is permutation invariant, then*

$$\begin{aligned} P_\rho[\#(C(0)) = k \text{ and } C(0) = \{0, x_1, x_2, \cdots, x_{k-1}\} \text{ for some } (x_1, \cdots, x_{k-1}) \in \mathbf{A}] \\ = \frac{\rho^{k-1}}{(k-1)!}\int_{\mathbf{A}} g_2(0, x_1, \cdots, x_{k-1}) \\ \times \exp\left\{-\rho\int_{\mathbb{R}^d} g_1(y; \{0, x_1, \cdots, x_{k-1}\})\,dy\right\} d(x_1, \cdots, x_{k-1}). \end{aligned} \tag{4.3}$$

By setting $A = \mathbb{R}^d$ in (4.2), we obtain an expression for $q_k(\rho)$. In the special case where $g(x) = I_{\{|x| \leqq r_0\}}$, this expression reduces to

$$q_k(\rho) = (\rho^{k-1}/(k-1)!)\int \cdots \int \exp(-\rho V(0, x_1, \cdots, x_{k-1}))\,dx_1 \cdots dx_{k-1}, \tag{4.4}$$

where the integral is over all $x_1, \cdots, x_{k-1}$ such that the union of the radius $(r_0/2)$ balls centered at $0, x_1, \cdots, x_{k-1}$ is connected, and $V(0, x_1, \cdots, x_{k-1})$ is the volume of the union of the balls of radius r_0 centered at $0, x_1, \cdots, x_{k-1}$.

Proposition 1 shows why the integrability condition (1.2) on $g(\cdot)$ is needed for the model to be non-trivial: if it fails, then with probability 1 there are no finite clusters.

The next three results are continuum reworkings of well-known facts on the critical phenomena of lattice percolation.

Proposition 2. *The function* $1 - f_0(\rho)$ (*which equals* $P_\rho[\#(C(0)) = \infty]$) *is non-decreasing in* ρ. *Also,* $E_\rho[\#(C(0))]$ *is non-decreasing in* ρ.

Proposition 3. *The probability* (*under* P_ρ) *that there exists an infinite cluster is* 0 *if* $f_0(\rho) = 1$, 1 *if* $f_0(\rho) < 1$.

Theorem 1. *There exist critical intensities* ρ_1 *and* ρ_2 *with* $0 < \rho_1 \leqq \rho_2 < \infty$, *such that*

$$E_\rho[\#(C(0))] < \infty, \qquad \rho < \rho_1, \qquad E_\rho[\#(C(0))] = \infty, \qquad \rho > \rho_1,$$

and

$$P_\rho[\text{an infinite cluster of } \mathcal{G} \text{ exists}] = 0, \qquad \rho < \rho_2;$$

$$P_\rho[\text{an infinite cluster of } \mathcal{G} \text{ exists}] = 1, \qquad \rho > \rho_2.$$

Zuev and Sidorenko (1985a) (see also Grimmett (1989), Section 10.5) proved that $\rho_1 = \rho_2$ in the Poisson blob model (that is, when $g(x) = I_{\{|x| \leqq r_0\}}$).

The next result says that as a function of ρ, the 'cluster density' is a well-defined continuous function of ρ. Let $k_1(n), k_2(n), \cdots, k_d(n)$ be sequences of positive numbers, each of which converges (independently) to ∞ as $n \to \infty$. Let $B = B(n)$ denote the box $(-k_1(n), k_1(n)) \times (-k_2(n), k_2(n)) \times \cdots \times (-k_d(n), k_d(n))$. Let $K(B)$ denote the number of clusters contained in the box B. For a cluster which is partially in B you may take its contribution to $K(B)$ to be either 0 or as many components as it splits into when particles not in B are removed, or any number in between; the following theorem is true in all these cases.

Theorem 2. (a) $[K(B(n))/|B(n)|] \underset{n\to\infty}{\longrightarrow} \rho f_{-1}(\rho)$ *almost surely and in* α*th mean, for all* $\alpha \geqq 1$.

(b) *The function* $f_{-1}(\rho)$ *is continuous in* ρ.

Thus it is reasonable to speak of $\rho f_{-1}(\rho)$ as the 'cluster density', and this density is continuous in ρ.

The motivation for the next result is the fact that actual calculations of the value of $f_N(\rho)$ (for example when N is -1 or 0) for given values of ρ, are sometimes of physical interest. The definition of $f_N(\rho)$ by the series

$$f_N(\rho) = \sum_1^\infty k^N q_k(\rho) \tag{4.5}$$

suggests that we approximate to $f_N(\rho)$ by the sum of the first few terms in the series (4.5). By (4.1), the first term is given explicitly by $q_1(\rho) = \exp\{-\rho \int g(y)\, dy\}$, but in subsequent terms, the expression given by (4.2) for $q_k(\rho)$ rapidly becomes very complicated.

If we sum a finite number of terms in the series (4.5), we obtain a lower bound for $f_N(\rho)$; for example, $f_N(\rho) \geqq q_1(\rho)$. In the case $N = -1$, we can obtain an upper bound for the cluster density by the argument of Mack (1954), (1956); combining these upper and lower bounds, we have, for all ρ,

$$\exp\left\{-\rho \int_{\mathbb{R}^d} g(y)\, dy\right\} \leqq f_{-1}(\rho) \leqq \exp\left\{-(\rho/2) \int_{\mathbb{R}^d} g(y)\, dy\right\}. \tag{4.6}$$

The next result shows that at least under certain conditions on g, when ρ is large the lower bound in (4.6) is much sharper than the upper bound, and in fact the first term of the series (4.5) for $f_N(\rho)$ dominates (a similar statement applies when ρ is small; see Theorem 3.1 of Hall (1986), in the case of the Boolean model when ρ is small).

Let us say a function $g:\mathbb{R}^d \to [0, 1]$ *encloses zero* if there exists a continuous function $h:\mathbb{S}^{d-1} \to (0, \infty)$, where $\mathbb{S}^{d-1}$ denotes the sphere of all unit vectors in $\mathbb{R}^d$, such that:

(i) $h(e) = h(-e)$, $e \in \mathbb{S}^{d-1}$,

(ii) $g(x) = 0$ for almost all $x \in \mathbb{R}^d$ of the form $x = \lambda e$ where $\lambda > 2h(e)$, $e \in \mathbb{S}^{d-1}$; and

(iii) for some $\varepsilon > 0$, $g(x) > \varepsilon$ for all $x \in \mathbb{R}^d$ of the form $x = \lambda e$, where $|\lambda - h(e)| < \varepsilon$, $e \in \mathbb{S}^{d-1}$. In other words, g is bounded away from zero in some open neighborhood of the surface $\{h(e)e : e \in \mathbb{S}^{d-1}\}$.

Condition (ii) ensures that g has bounded support if it encloses zero. The condition that g encloses zero is not very elegant, but it covers many natural cases. For example, if $g(x)$ is a decreasing function of $|x|$ (or of some other norm on $\mathbb{R}^d$) with bounded support, it encloses zero. Also, if g is the indicator function of some annulus, it encloses zero.

Theorem 3. *Suppose g encloses zero. Then for all* $N \in \mathbb{R}$,

$$(f_N(\rho)/q_1(\rho)) \to 1 \quad \text{as} \quad \rho \to \infty. \tag{4.7}$$

We have been unable to prove (4.7) when g has unbounded support, but we can at once obtain the following.

Corollary. Let $N \leqq 0$. *If* $g(x)$ *depends only on* $|x|$ *and is a decreasing function of* $|x|$ *then*

$$(\log f_N(\rho)/\log q_1(\rho)) \to 1 \quad \text{as} \quad \rho \to \infty. \tag{4.8}$$

In the case $N = 0$, we can restate (4.8) as a large deviations principle for the probability that the cluster at 0 is finite:

$$\log P_\rho[\#(C(0)) < \infty] \Big/ \left(\rho \int_{\mathbb{R}^d} g(x)\, dx\right) \xrightarrow[\rho \to \infty]{} 1.$$

As motivation for the proof of Theorem 3, consider first the case $g(x) = I_{\{|x| \leqq r_0\}}$. At high density the exponential term in (4.4) makes it very beneficial, in terms of maximizing probabilities, to make the volume $V(0, x_1, \cdots, x_{k-1})$ of the region which must be empty for $\{0, x_1, \cdots, x_{k-1}\}$ to be a finite cluster, as small as possible. This is done by setting $k = 0$.

5. Proof of Proposition 1

(a) It suffices to prove (4.2) for bounded Borel $A \subset \mathbb{R}^d$. Let $E(k, A)$ denote the event that $\#[C(0)] = k$ and $C(0) \subset A$. To find its probability, condition on the number of points of $\mathcal{P}$ in A. Conditional on there being r points of $\mathcal{P}$ in A, where

$r \geqq k-1$, we may denote the positions of these points as $X_1, X_2, \cdots, X_r$, where $X_1, \cdots, X_r$ are independent and uniformly distributed over A. The set of points other than 0 in $C(0)$ is then equally likely to be any size $k-1$ subset of $\{X_1, \cdots, X_r\}$. So

$$P_\rho[E(k, A) \mid \mathcal{P} \text{ places } r \text{ points in } A]$$

(5.1)
$$= \binom{r}{k-1} |A|^{-r} \int_A dx_1 \cdots \int_A dx_r Q[C(0) = \{0, x_1, \cdots, x_{k-1}\}],$$

where $Q[\ ]$ denotes probability when points at $0, x_1, \cdots, x_r$ are superimposed on a rate ρ Poisson process on $\mathbb{R}^d \backslash A$, and the points of the resulting point process are bonded by the usual rule (points at x and y are bonded with probability $g(x-y)$, independently of all other pairs of points).

But the probability that a point at y is bonded to at least one of $\{0, x_1, x_2, \cdots, x_{k-1}\}$ is $g_1(y; \{0, x_1, \cdots, x_{k-1}\})$ by definition. By the same argument as in the proof of (4.1), the probability that no point of a homogeneous rate ρ Poisson process on $\mathbb{R}^d \backslash A$ is bonded to any of $0, x_1, \cdots, x_{k-1}$ is

$$\exp\left[-\rho \int_{\mathbb{R}^d \backslash A} g_1(y; \{0, x_1, \cdots, x_{k-1}\})\, dy\right].$$

Hence

$$Q[C(0) = \{0, x_1, \cdots, x_{k-1}\}] = g_2(0, x_1, \cdots, x_{k-1})$$
$$\times \exp\left[-\rho \int_{\mathbb{R}^d \backslash A} g_1(y; \{0, x_1, \cdots, x_{k-1}\})\, dy\right] \prod_{j=k}^{r} (1 - g_1(x_j; \{0, x_1, \cdots, x_{k-1}\})).$$

Substituting in (5.1), we have

$$P_\rho[E(k, A) \cap \{\mathcal{P} \text{ places } r \text{ points in } A\}] = \exp(-\rho\, |A|)\rho^r[(k-1)!\,(r-(k-1))!]^{-1}$$
$$\times \int_A dx_1 \cdots \int_A dx_{k-1} g_2(0, x_1, \cdots, x_{k-1}) \exp\left[-\rho \int_{\mathbb{R}^d \backslash A} g_1(y; \{0, x_1, \cdots, x_{k-1}\})\, dy\right]$$
$$\times \left(\int_A [1 - g_1(z; \{0, x_1, \cdots, x_{k-1}\})]\, dz\right)^{r-(k-1)}.$$

Summing this last expression over $r \geqq k-1$, we obtain

$$P[E(r, A)] = \exp(-\rho\, |A|)(\rho^{k-1}/(k-1)!)$$
$$\times \int_A dx_1 \cdots \int_A dx_{k-1} \Big[g_2(0, x_1, \cdots, x_{k-1}) \exp\left\{-\rho \int_{\mathbb{R}^d \backslash A} g_1(y; \{0, x_1, \cdots, x_{k-1}\})\, dy\right\}$$
$$\times \exp\left\{\rho \int_A (1 - g_1(z; \{0, x_1, \cdots, x_{k-1}\}))\, dz\right\}\Big]$$

which reduces to the desired result.

(b) By (a), the result holds for all sets $\mathbf{A}$ of the form $\mathbf{A} = (A \times A \times \cdots \times A)$, where A is a Borel set in $\mathbb{R}^d$. Unfortunately, such sets do not generate the sigma-field of all permutation-invariant Borel sets in $(\mathbb{R}^d)^{k-1}$. However, this sigma-field *is* generated by the sets $\mathbf{A}$ of the form

$$\mathbf{A} = \{(x_1, \cdots, x_{k-1}) : \text{exactly } j_m \text{ out of } \{x_1, \cdots, x_{k-1}\} \text{ lie in } A_m,\ 1 \leqq m \leqq M\}$$

where $M \leqq k-1$, $j_m \geqq 1$ $(1 \leqq m \leqq M)$, $\sum_{m=1}^{M} j_m = k-1$, and A_m $(1 \leqq m \leqq M)$ are mutually exclusive, bounded Borel sets in $\mathbb{R}^d$.

For sets $\mathbf{A}$ of this form, it is possible to prove (4.3) by conditioning on the number of points placed by $\mathcal{P}$ in $A_1, A_2, \cdots, A_m$. The argument is similar to the proof of (a) above.

6. Proofs on critical probabilities

Proof of Proposition 2. Let $C_\rho(0)$ denote the cluster at the origin when the Poisson process $\mathcal{P}$ has intensity ρ. It suffices to show that whenever $\rho' < \rho$, the random variable $\#[C_\rho(0)]$ stochastically dominates the random variable $\#[C_{\rho'}(0)]$ (that is, $P_\rho[\#(C(0)) \leqq x] \leqq P_{\rho'}[\#(C(0)) \leqq x]$, for all real x).

We can couple $C_\rho(0)$ and $C_{\rho'}(0)$; place them on a single probability space by superimposing an independent Poisson process $\mathcal{P}''$ of intensity $\rho - \rho'$ on a Poisson process $\mathcal{P}'$ of intensity ρ', to obtain a Poisson process $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$, of intensity ρ. To obtain $C_{\rho'}(0)$, use only points of $\mathcal{P}'$; to obtain $C_\rho(0)$, use all points of $\mathcal{P}$, using the same collection of Bernoulli random variables $D_{\{x,y\}}$ in each construction. In this coupling, every point of $C_{\rho'}(0)$ lies in $C_\rho(0)$, which implies the desired stochastic domination.

Proof of Proposition 3. Let B be any cube in $\mathbb{R}^d$. Let N_B be the number of particles of $\mathcal{P}$ which lie in B and are part of an infinite cluster. Then

$$E_\rho[N_B] = \rho\,|B|\,(1 - f_0(\rho)). \tag{6.1}$$

To see this, condition on the number of particles of $\mathcal{P}$ in B.

Hence, if $f_0(\rho) = 1$, then $N = 0$ almost surely. Since B is an arbitrary cube, there are almost surely no infinite clusters in this case.

On the other hand, if $f_0(\rho) < 1$, then if we set $B(n) = [-n, n]^d$, then by an ergodic theorem (Dunford and Schwartz (1958), Theorem VIII.6.9),

$$N_{B(n)}/(2n)^d \underset{n \to \infty}{\longrightarrow} \rho(1 - f_0(\rho)) \text{ almost surely,}$$

which implies the existence of an infinite cluster almost surely (there exists a more elementary proof of this fact using the Kolmogorov zero–one law for sequences of independent sigma-fields).

Proof of Theorem 1. Let $\rho_1 = \sup\{\rho : E_\rho[\#(C(0))] < \infty\}$ and

$$\rho_2 = \sup\{\rho : P_\rho[\#(C(0)) < \infty] = 1\} = \sup\{\rho : f_0(\rho) = 1\}.$$

Clearly $\rho_1 \leqq \rho_2$. In view of Propositions 2 and 3, it suffices to show that $\rho_1 > 0$ and $\rho_2 < \infty$.

To prove that $\rho_1 > 0$, use an adaptation of the 'method of generations' (Zuev and Sidorenko (1985b), Section 4.1; also Gilbert (1961), Section 2). Here, the first generation consists of particles bonded directly to 0. The $(k+1)$th generation consists of particles bonded to particles from the kth generation but to no particles of earlier generations. This method shows that

$$E_\rho[\#(C(0))] \leqq \sum_{k=0}^{\infty} \left[\rho \int_{\mathbb{R}^d} g(x)\, dx\right]^k.$$

In particular, if $\rho \int g(x)\, dx < 1$, then $E_\rho[\#(C(0))] < \infty$. Thus,

$$\rho_1 \geqq \left[\int_{\mathbb{R}^d} g(x)\, dx\right]^{-1} > 0. \tag{6.2}$$

It remains to show that $\rho_2 < \infty$ (this is immediate from the case $N = 0$ of Theorem 3 or of its corollary, if g satisfies the extra hypotheses of those results).

Take linearly independent x_1 and x_2 such that $g(x_i) > 0$ and x_i lies in the Lebesgue density set of g (see Dunford and Schwartz (1958), Theorem III.12.8), $i = 1, 2$. Then there exists δ so small that for all cubes B of side at most δ with $x_i \in B$,

$$\int_{\mathbb{R}^d} g(y)\, dy/|B| > (\tfrac{1}{2}) g(x_i) \quad (i = 1, 2). \tag{6.3}$$

Let B_δ denote the cube $[-\delta, \delta]^d$. We may assume that δ is so small that all the sets of form $(nx_1 + mx_2 + B_\delta)$, $n, m \in \mathbb{Z}$, are mutually exclusive.

If $x \in B_\delta$ and $\{X_1, X_2, X_3, \cdots, X_N\}$ is a homogeneous Poisson process with rate $\rho/4$ on the cube $x_i + B_\delta$ (or on $-x_i + B_\delta$) $(i = 1$ or $2)$, then the probability that $D_{\{x, X_j\}} = 0$, $1 \leqq j \leqq N$, (i.e. that x is not bonded to any of the particles $X_1, X_2, \cdots, X_N$) is

$$\begin{aligned} &\exp\left\{(-\rho/4) \int_{x_i + B_\delta} g(y - x)\, dy\right\} \\ &\quad = \exp\left\{(-\rho/4) \int_{x + x_i + B_\delta} g(y)\, dy\right\} \leqq \exp\{(-\rho/8)\, \delta^d g(x_i)\}, \end{aligned} \tag{6.4}$$

where the inequality is from (6.3).

One way to construct the Poisson process $\mathcal{P}$ is as follows. For each edge e of $\mathbb{Z}^2$, say between (n, m) and (n', m') (where $|(n - n')| + |(m - m')| = 1$), place independent Poisson processes with rate $\rho/4$, $\mathcal{P}_{e,n,m}$ and $\mathcal{P}_{e,n',m'}$, on the sets $(nx_1 + mx_2 + B_\delta)$ and $(n'x_1 + m'x_2 + B_\delta)$, respectively, independently of all the other edges of $\mathbb{Z}^2$. The superposition (that is, the union) of all the Poisson processes given in this way is a rate ρ Poisson process on the union of all sets in $\mathbb{R}^d$ of the form $nx_1 + mx_2 + B_\delta$,

$n, m \in \mathbb{Z}$. If we superimpose a Poisson process with rate ρ on the complement of this set, we end up with $\mathcal{P}$ a Poisson process with rate ρ on the whole of $\mathbb{R}^d$.

Given a realization of $(\mathcal{P}, (D_{\{x,y\}}))$, constructed in this way, we can construct a discrete cluster $D(0)$ (that is, a subset of $\mathbb{Z}^2$) by the following algorithm.

Step 1. Place 0 in $D(0)$. Define $X_{0,0}$ to be the origin in $\mathbb{R}^d$.

Step 2. (ii) Consider some edge e of $\mathbb{Z}^2$, between (n, m) and (n', m'), such that $(n, m) \in D(0)$, $(n', m') \notin D(0)$ and e has not been considered before. If no such e exists, stop.

Step 3. If there exists a point of the Poisson process $\mathcal{P}_{e,n',m'}$ which is bonded to $X_{n,m}$, then choose such a point, denote it $X_{n',m'}$, and add (n', m') to the set $D(0)$.

Step 4. Return to Step 2.

If $C(0)$ is finite, then so is $D(0)$ and the algorithm terminates. But at Step 3 of the algorithm, the probability of adding a new point of $\mathbb{Z}^2$ to $D(0)$ exceeds $1-\exp(-c\rho)$, c being a finite constant, by (6.4). The critical probability for bond percolation in $\mathbb{Z}^2$ lies strictly below 1 (see for example Grimmett (1989), Theorem 1.10), so that for large ρ, the probability that the set $D(0)$ (and hence $C(0)$) is infinite exceeds zero.

7. The cluster density: proof of Theorem 2

(a) In the statement of this theorem, $B(n)$ is a box in $\mathbb{R}^d$, centered at 0, whose side lengths are $2k_1(n)$, $2k_2(n), \cdots, 2k_d(n)$, which converge independently to infinity as $n \to \infty$. Assume for now that $k_1(n), \cdots, k_d(n)$ are all integers.

First, we follow Section 4.1 of Grimmett (1989); see also Lemma 2.1 of Aizenman et al. (1987). For any set $B \subset \mathbb{R}^d$ define $\bar{K}(B)$ by

$$\bar{K}(B) = \sum_{X_i \in B} [\#(C(X_i))]^{-1}.$$

Then

$$E_\rho[\bar{K}(B)] = \rho\,|B|\,f_{-1}(\rho); \tag{7.1}$$

as in (6.1), this can be proved by conditioning on the number of particles of $\mathcal{P}$ in B. By an ergodic theorem (Dunford and Schwartz (1958), Theorem VIII.6.9), as $n \to \infty$

$$\bar{K}(B(n))/|B(n)| \to \rho f_{-1}(\rho) \tag{7.2}$$

a.s. and in L^α, for all $\alpha \geqq 1$. Also,

$$|K(B(n)) - \bar{K}(B(n))| \leqq 2N(n), \tag{7.3}$$

where $N(n)$ is the number of particles X_i of $\mathcal{P}$ for which $X_i \in B(n)$, and X_i is bonded to some particle X_j of $\mathcal{P}$ lying in $\mathbb{R}^d \backslash B(n)$. It remains to estimate $N(n)$.

Let $\varepsilon > 0$. Denote by $(1-\varepsilon)B(n)$ the box $\{(1-\varepsilon)x : x \in B(n)\}$. Let π denote the (random) counting measure on $\mathbb{R}^d$ associated with $\mathcal{P}$. By the ergodic theorem already referred to, as $n \to \infty$, $\pi(B(n))/|B(n)| \to \rho$ and $\pi[(1-\varepsilon)B(n)]/|(1-\varepsilon)B(n)| \to \rho$ so that

$$\pi[B(n)\backslash(1-\varepsilon)B(n)]/|B(n)| \to \rho(1-(1-\varepsilon)^d) \tag{7.4}$$

a.s. and in L^α, $\alpha \geqq 1$.

Now choose $r_0 > 0$ to be so big that

$$\int_{|x| \geqq r_0} g(x)\,dx < \varepsilon.$$

For large n, each point of $(1-\varepsilon)B(n)$ lies a distance of at least r_0 from all points of $\mathbb{R}^d \backslash B(n)$. Hence the number of points of $\mathcal{P}$ which lie in $(1-\varepsilon)B(n)$ and are bonded to at least one point of $\mathbb{R}^d \backslash B(n)$ is at most $M(n)$, where we define $M(n)$ to be the number of points X_i of $\mathcal{P}$ lying in $B(n)$ for which X_i is bonded to some point X_j of $\mathcal{P}$ with $|X_i - X_j| \geqq r_0$. By conditioning on the number of particles $\mathcal{P}$ places in $B(n)$, we find that

$$E(M(n)) = \rho^2\,|B(n)| \int_{|x| \geqq r_0} g(x)\,dx.$$

So by the same ergodic theorem as before, as $n \to \infty$

$$M(n)/|B(n)| \xrightarrow[L^\alpha]{\text{a.s.}} \rho^2 \int_{|x| \geqq r_0} g(x)\,dx < \rho^2 \varepsilon. \tag{7.5}$$

Since for large n,

$$0 \leqq N(n) \leqq \pi[B(n)\backslash(1-\varepsilon)B(n)] + M(n)$$

and ε is arbitrary, (7.4) and (7.5) imply that

$$N(n)/|B(n)| \to 0 \tag{7.6}$$

a.s. and in $L^\alpha(\alpha \geqq 1)$; together with (7.2) and (7.3) this implies the desired result. Finally, we can remove the restriction that $k_1(n), k_2(n), \cdots, k_d(n)$ be integers by approximating to them by integers and using similar arguments to the above.

(b) By (7.1) and (7.3),

$$|(E_\rho[K(B(n))]/|B(n)|) - \rho f_{-1}(\rho)| \leqq 2E_\rho[N(n)]/|B(n)|, \tag{7.7}$$

where $N(n)$ is as in (7.3). By conditioning on the number of points of $\mathcal{P}$ lying in $B(n)$, or otherwise, we have

$$\begin{aligned} E_\rho[N(n)] &= \rho \int_{B(n)} \left[1 - \exp\left\{-\rho \int_{\mathbb{R}^d \backslash B(n)} g(y-x)\,dy\right\}\right] dx \\ &\leqq \rho^2 \int_{B(n)} \left(\int_{\mathbb{R}^d \backslash B(n)} g(y-x)\,dy\right) dx, \end{aligned}$$

so that $(E_\rho[N(n)]/|B(n)|)\to 0$ locally uniformly in ρ as $n\to\infty$. Hence by (7.7), for continuity of $\rho f_{-1}(\rho)$ in ρ it suffices to prove that for any fixed box B, $E_\rho[K(B)]$ is continuous in ρ. We prove this in the case where $K(B)$ is the number of clusters obtained when all points of $\mathcal{P}$ not lying in B are removed.

Let $u(r)$ denote the expected number of clusters formed when r particles are placed independently in B according to a uniform distribution. Clearly, $0\leqq u(r)\leqq r$. Then conditioning on the number of points $\mathcal{P}$ places in B, we have

$$E_\rho[K(B)]=\sum_{r=0}^{\infty}(e^{-\rho|B|}(\rho\,|B|)^r/r!)u(r).$$

Each term in the series is continuous in ρ, and the tail of the series converges to 0 locally uniformly in ρ. Hence, $E_\rho[K(B)]$ is continuous in ρ.

8. Large density: proof of Theorem 3

Since $f_N(\rho)$ is an increasing function of N, it suffices to prove the theorem when N is a positive integer. An important tool in this proof is the discretization of $\mathbb{R}^d$. This technique has been used before, for example in Zuev and Sidorenko (1985a,b).

Let $\delta>0$ and let $\delta\mathbb{Z}^d$ be the lattice of points in $\mathbb{R}^d$ of the form $(\delta n_1, \delta n_2, \cdots, \delta n_d)$, where $n_1, n_2, \cdots, n_d$ are integers. Let $F_\delta:\mathbb{R}^d\to\delta\mathbb{Z}^d$ denote the many-to-one mapping which sends each point of $\mathbb{R}^d$ to the closest point to it of $\delta\mathbb{Z}^d$. The mapping F_δ is well defined except on a Lebesgue-null set which we may ignore. Then the image of $C(0)$ under F_δ is a random subset of $\delta\mathbb{Z}^d$, which we denote by $S(0)$. Clearly $0\in S(0)$ and $\#(S(0))\leqq\#(C(0))$, where $\#(\ )$ denotes cardinality. If $\#(C(0))=1$, then $S(0)=\{0\}$. We shall prove the following three lemmas which when combined prove the theorem. The first one does not require that g enclose zero, but the other two do.

Lemma 1. *If δ is small enough, then as $\rho\to\infty$,*

$$\sum_{k=1}^{\infty}k^N P_\rho[\#(C(0))=k \text{ and } S(0)=\{0\}]/q_1(\rho)\to 1.$$

Lemma 2. *Suppose g encloses zero. If δ is small enough, then for some $m_0>0$, as $\rho\to\infty$,*

$$\sum_{m=m_0}^{\infty}\sum_{k=2}^{\infty}k^N P_\rho[\#(C(0))=k \text{ and } \#(S(0))=m]/q_1(\rho)\to 0.$$

Lemma 3. *Suppose g encloses zero. If δ is small enough, then for each fixed m, as $\rho\to\infty$*

$$\sum_{k=1}^{\infty}k^N P_\rho[\#(C(0))=k \text{ and } \#(S(0))=m)]/q_1(\rho)\to 0.$$

In the proofs which follow, c, c_1, c_2 and so on denote positive finite constants which may change from line to line. We shall prove Lemmas 2 and 3 in the case $d=2$; the modifications to $d \geqq 3$ are not hard. To prove Lemma 1 we require the following.

Lemma 4. *If* $g:\mathbb{R}^d \to [0,\infty)$ *satisfies* $0<\int g(x)\,dx<\infty$, *then*

$$\liminf_{|h|\downarrow 0} |h|^{-1}\int_{\mathbb{R}^d} |g(x+h)-g(x)|\,dx>0.$$

Proof. We prove this only in the case when g has bounded support. Consider the vector integral

$$\int (g(x-h)-g(x))x\,dx=\int g(x-h)x\,dx-\int g(x)x\,dx$$
$$=h\int g(x)\,dx.$$

Here, all integrals are over $\mathbb{R}^d$. Taking Euclidean norms, we have $\int |x|\,|g(x-h)-g(x)|\,dx \geqq |h|\int g(x)\,dx$. Take r_1 so that $g(x)=0$ for $|x|>r_1$. For $|h|\leqq 1$, both $g(x-h)$ and $g(x)$ are zero for $|x|>1+r_1$. Hence

$$(1+r_1)\int |g(x-h)-g(x)|\,dx \geqq |h|\int g(x)\,dx,\quad |h|\leqq 1.$$

This implies the desired result.

Proof of Lemma 1. Let A_δ denote the d-dimensional cube $(-\delta/2,\delta/2)^d$. Set $q_k^\delta(\rho)=P_\rho[S(0)=\{0\}$ and $\#(C(0))=k]$. We are required to prove that for any given $N>0$,

$$\sum_{k=2}^{\infty} k^N q_k^\delta(\rho)/q_1(\rho)\to 0 \quad \text{as} \quad \rho\to\infty. \tag{8.1}$$

By Proposition 1 and the formula (4.1) for $q_1(\rho)$,

$$\begin{aligned} q_k^\delta(\rho)/q_1(\rho) &= (\rho^{k-1}/(k-1)!)\int_{A_\delta}\cdots\int_{A_\delta} g_2(0,x_1,\cdots,x_{k-1}) \\ &\quad\times \exp\Big(-\rho\int_{\mathbb{R}^d} g_1(y;\{0,x_1,\cdots,x_{k-1}\})\,dy\Big)\Big/\exp\Big(-\rho\int g(y)\,dy\Big)\,dx_1\cdots dx_{k-1} \\ &\leqq (\rho^{k-1}/(k-1)!)\int_{A_\delta}\cdots\int_{A_\delta}\exp\Big(-\rho\int_{\mathbb{R}^d}[g_1(y;\{0,x_1,\cdots,x_{k-1}\})-g(y)]\,dy\Big) \\ &\quad\times dx_1\cdots dx_{k-1}. \end{aligned} \tag{8.2}$$

Define the set $\Delta(k,\delta)\subset(\mathbb{R}^d)^{k-1}$ by

$$\Delta(k,\delta)=\{(x_1,\cdots,x_{k-1})\in(A_\delta)^{k-1}:|x_1|\geqq|x_i|,\ 2\leqq i\leqq k-1\}.$$

The integrand on the right-hand side of (8.2) is symmetric in $x_1, \cdots, x_{k-1}$; hence,

$$q_k^\delta(\rho)/q_1(\rho) \leqq (\rho^{k-1}/(k-2)!) \int_{\Delta(k,\delta)} \cdots \int \exp\left(-\rho \int_{\mathbb{R}^d} \{g_1(y; \{0, x_1, \cdots, x_{k-1}\}) - g(y)\}\, dy\right) dx_{k-1} \cdots dx_1.$$

By definition, the integrand inside the braces equals the probability that if particles are placed at $0, x_1, x_2, \cdots, x_{k-1}$ and y by $\mathcal{P}$, the particle at y is bonded directly to at least one of $\{x_1, \cdots, x_{k-1}\}$ but not to 0. Thus

$$g_1(y; \{0, x_1, \cdots, x_{k-1}\}) - g(y) \geqq g(y - x_1)(1 - g(y)) \geqq (g(y - x_1) - g(y))_+$$

where $(\cdot)_+$ denotes positive part (that is, $u_+ = \max(u, 0)$). By Lemma 4 there is a constant c such that for all $x_1 \in A_\delta$ we have

$$\int_{\mathbb{R}^d} (g(y - x_1) - g(y))_+\, dy = (\tfrac{1}{2}) \int_{\mathbb{R}^d} |g(y - x_1) - g(y)|\, dy \geqq c\, |x_1|.$$

Hence,

$$\begin{aligned} q_k^\delta(\rho)/q_1(\rho) &\leqq (\rho^{k-1}/(k-2)!) \int_{\Delta(k,\delta)} \cdots \int \exp(-\rho c\, |x_1|)\, dx_{k-1} \cdots dx_1 \\ &= (\rho^{k-1}/(k-2)!) \int_{A_\delta} \exp(-\rho c\, |x_1|)(c_1\, |x_1|^d)^{k-2}\, dx_1 \end{aligned}$$

where c_1 is the volume of the unit ball in $\mathbb{R}^d$. Noting that $r \leqq d\delta$ on A_δ, we have

$$\begin{aligned} q_k^\delta(\rho)/q_1(\rho) &\leqq c_2 \int_0^{d\delta} (\rho^{k-1}(c_1)^{k-2}/(k-2)!) e^{-\rho c r} r^{d(k-2)} r^{d-1}\, dr \\ &= c_2 \int_0^{d\delta} (\rho^{k-1}(c_1)^{k-2}/(k-2)!) e^{-\rho c r} r^{d(k-1)-1}\, dr, \end{aligned} \tag{8.3}$$

so that each term in the series $\sum_{k=2}^\infty k^N q_k^\delta(\rho)/q_1(\rho)$ converges to zero as $\rho \to \infty$.

Also, there exists a constant c such that for $k \geqq (N+2)$, $(k^N/(k-2)!) \leqq c/(k-2-N)!$. Hence, by (8.3)

$$\begin{aligned} \sum_{k=N+2}^\infty k^N q_k^\delta(\rho)/q_1(\rho) &\leqq c_2 \rho^{N+1} \int_0^{d\delta} \Big\{ e^{-\rho c r} r^{d(N+1)-1} \\ &\qquad \times \sum_{k=N+2}^\infty (\rho c_1)^{k-2-N} r^{d(k-2-N)}/(k-2-N)! \Big\}\, dr \\ &= c_2 \rho^{N+1} \int_0^{d\delta} \exp(\rho c_1 r^d - \rho c r) r^{d(N+1)-1}\, dr. \end{aligned} \tag{8.4}$$

Taking δ to be so small that for $r<d\delta$, $c_1r^d \leqq cr/2$, we have by (8.4) that

$$\sum_{k=N+2}^{\infty} k^N q_k^{\delta}(\rho)/q_1(\rho) \leqq c_2\rho^{N+1}\int_0^{d\delta} \exp(-\rho cr/2) r^{d(N+1)-1}\,dr$$

$$\leqq c_2\rho^{-(d-1)(N+1)}\int_0^{\infty} \exp(-sc/2) s^{d(N+1)-1}\,ds,$$

where we substituted $s=\rho r$. Thus,

$$\sum_{k=2}^{\infty} k^N q_k^{\delta}(\rho)/q_1(\rho) = \left(\sum_{k=2}^{N+1} + \sum_{k=N+2}^{\infty}\right) k^N q_k^{\delta}(\rho)/q_1(\rho)$$

This completes the proof of (8.1), and hence of Lemma 1.

Proof of Lemma 2. Assume $d=2$. Let $h(e)$, $e\in\mathbb{S}^{d-1}$, be as in the description immediately preceding the statement of Theorem 3 of when the function g encloses zero. Denote by A_h the open set $\{\lambda e: 0\leqq\lambda<h(e),\ e\in\mathbb{S}^{d-1}\}$. Take $\delta>0$ to be so small that $2\delta\leqq h(e)$ for all $e\in\mathbb{S}^{d-1}$ and, for x on the boundary of A_h and $|y|<3\delta$, $g(x+y)\geqq\delta$. Such a δ can be found by compactness arguments.

Given a finite set C of points in $\mathbb{R}^2$, define the set $W(C)\subset\mathbb{R}^2$ as follows. Let $W_0(C)$ be the union of all sets of the form $x+A_h$, $x\in C$. Let $W(C)$ be the union of all the squares of the form $[n\delta,(n+1)\delta]\times[m\delta,(m+1)\delta]$, n, $m\in\mathbb{Z}$, which intersect $W_0(C)$.

Set $W=W(C(0))$. Then W is a connected subset of $\mathbb{R}^2$. Indeed, by the definition of h, with probability 1 for any X_i and X_j in $\mathcal{P}$ between which a direct bond forms, the sets X_i+A_h and X_j+A_h overlap. The exterior boundary ∂W of W is a closed curve composed of edges of the lattice $\delta\mathbb{Z}^2$, which we endow with its natural graph structure. Let $\#(\partial W)$ denote the number of edges of $\delta\mathbb{Z}^2$ comprising ∂W, so that the length of ∂W is $\delta\#(\partial W)$.

The curve ∂W is completely determined by $C(0)$. We examine the probability that ∂W is a particular closed curve, γ say, composed of edges of $\delta\mathbb{Z}^2$. Let $\bar{\gamma}$ denote the set of points in $\mathbb{R}^2$ which lie a Euclidean distance at most δ from some point of γ. Let int $(\bar{\gamma})$ denote the set of points of $\mathbb{R}^2$ which lie in the interior of γ but not in $\bar{\gamma}$.

If $\partial W=\gamma$, then no point of $C(0)$ lies in $\bar{\gamma}$. For if $y\in\bar{\gamma}$ and $\gamma=\partial W$, then there exists a point of the boundary of W close to y, and hence a point not in $W_0(C(0))$ which is close to y. To be precise, there is a point z with $|z-y|\leqq 2\delta$, for which $z\notin W_0(C(0))$. But for all such z, by the construction of $W_0(C(0))$, z is distant at least $\inf\{h(e): e\in\mathbb{S}^{d-1}\}$ from the nearest point of $C(0)$; hence, by our choice of small δ, y cannot be in $C(0)$.

Let $C_1(0)$ denote the cluster including 0, obtained when we remove all bonds of $\mathcal{G}_0$ except those between points in int $(\bar{\gamma})$. Let $E_{k,\gamma}$ denote the event that $C_1(0)$ consists of k particles and the disposition of these particles is such that the curve $\partial W(C_1(0))$ equals γ. Let $E'_{k,\gamma}$ denote the event that no particle of $\mathcal{P}$ in $\bar{\gamma}$ is attached to any particle of $C_1(0)$. Given that $E_{k,\gamma}$ occurs, the conditional probability of $E'_{k,\gamma}$ is $\exp\{-\rho\int_{\bar\gamma} g_1(y; C_1(0))\,dy\}$ (note that the Poisson process of points in $\bar\gamma$ is independent of the Poisson process of points in $\operatorname{int}(\bar\gamma)$). But for each $y \in \bar\gamma$, if $E_{k,\gamma}$ occurs, by the construction of $W(C_1(0))$ there exists z on the boundary of $W_0(C_1(0))$ for which $|z-y| < 3\delta$. Hence, again by our choice of small δ, $g_1(y; C_1(0)) \geqq \delta$. Also, $\bar\gamma$ contains the side $\delta/2$ squares centered at the center of each edge of γ, which are disjoint. Hence, the area of $\bar\gamma$ is at least $(\delta^2/4)\#(\gamma)$. Hence,

$$P[E'_{k,\gamma} \mid E_{k,\gamma}] \leqq \exp\{-\rho(\delta^3/4)\#(\gamma)\} = \exp(-c\rho\#(\gamma)),$$

where c is independent of γ. Thus,

$$\begin{aligned} P_\rho[\#(C(0)) = k \text{ and } \partial W = \gamma] &\leqq P_\rho[E_{k,\gamma} \cap E'_{k,\gamma}] \\ &\leqq \exp(-c\rho\#(\gamma))P_\rho(E_{k,\gamma}). \end{aligned}$$

Now $\#(C_1(0)) \leqq X+1$, where X is the number of points of $\mathcal{P}$ (in $C_1(0)$ or not) lying in $\operatorname{int}(\bar\gamma)$, a Poisson random variable with parameter $\rho\,|\operatorname{int}(\bar\gamma)|$. Since the area enclosed by γ is at most a constant (independent of γ) times the square of its length, for $N \geqq 0$

$$\begin{aligned} \sum_{k=1}^{\infty} k^N P_\rho[\#(C(0)) = k \text{ and } \partial W = \gamma] &\leqq \exp(-c\rho\#(\gamma)) \sum_{k=1}^{\infty} k^N P_\rho[E_{k,\gamma}] \\ &\leqq \exp(-c\rho\#(\gamma))E_\rho(X+1)^N \leqq c_1 \exp(-c_2\rho\#(\gamma))(\rho[\#(\gamma)]^2)^N. \end{aligned}$$

Also, the lattice set $S(0)$, defined earlier to be the image of $C(0)$ under the many-to-one mapping F_δ, is contained in the interior of γ; hence, $\#[S(0)]$ is at most a constant times $(\#(\gamma))^2$, so

$$\begin{aligned} &\sum_{k\geqq 1} k^N P_\rho[\#(C(0)) = k \text{ and } \#(S(0)) \geqq m_0] \\ &\leqq \sum_{k\geqq 1} k^N P_\rho[\#(C(0)) = k \text{ and } \#(\gamma) \geqq c m_0^{\frac{1}{2}}] \\ &\leqq \sum_{m \geqq c m_0^{\frac{1}{2}}} \; \sum_{\gamma:\#(\gamma)=m} c_1 \exp(-c_2\rho m)\rho^N m^{2N}. \end{aligned}$$

By a Peierls argument (that is, one based on enumeration; see Grimmett (1989), p. 17), the number of closed curves γ along edges of δZ^2, enclosing 0, for which $\#(\gamma) = m$, is at most c^m, for some constant c. So

$$\begin{aligned} &\sum_{k\geqq 1} k^N P_\rho[\#C(0) = k \text{ and } \#(S(0)) \geqq m_0]/q_1(\rho) \\ &\leqq \sum_{m \geqq c m_0^{\frac{1}{2}}} c_1 \exp(c_2 m + c_3\rho - c_4 m\rho)\rho^N m^{2N}. \end{aligned} \tag{8.5}$$

If we take m_0 to be big enough, then for $m \geqq c m_0^{\frac{1}{2}}$, $c_4 m \geqq 2c_3$; if also $\rho \geqq 4c_2/c_4$, the right side of (8.5) is at most

$$\sum_{m \geqq m_0^{\frac{1}{2}}} c_1 \exp(c_2 m - (\tfrac{1}{2}) c_4 m\rho)\rho^N m^{2N} \leqq \sum_{m \geqq m_0^{\frac{1}{2}}} c_1 \exp(-(\tfrac{1}{4}) c_4 m\rho)\rho^N m^{2N}.$$

This last series is convergent; each term converges to 0 as $\rho \to \infty$, and for ρ large, each term is decreasing in ρ. Hence, the sum of the series tends to 0 as $\rho \to \infty$.

Proof of Lemma 3. Assume $d = 2$. Since g has bounded support, there are only finitely many possible configurations for $S(0)$, of cardinality m. So it suffices to show that for any finite subset η of the lattice $\delta\mathbb{Z}^2$,

$$\sum_{k=1}^{\infty} k^N P_\rho[\#[C(0)] = k \text{ and } S(0) = \eta]/q_1(\rho) \to 0 \quad \text{as} \quad \rho \to \infty. \tag{8.6}$$

Let $C_\eta(0)$ be the cluster including 0 obtained when we remove all points of $\mathscr{P}$ not lying in $F_\delta^{-1}(\eta)$. Let $E_{\eta,k}$ be the event that $C_\eta(0)$ has k points, and for each point ξ of η, at least one point of $C_\eta(0)$ lies in $F_\delta^{-1}(\xi)$. So $E_{\eta,k}$ lies in the sigma-field $\mathscr{F}_\eta$ generated by the points of $\mathscr{P}$ in $F_\delta^{-1}(\eta)$ and the bonds between them. Let H_η be the event that there is no point of $\mathscr{P}$ in $\mathbb{R}^2 \backslash F_\delta^{-1}(\eta)$ which is bonded to any point of $C_\eta(0)$.

The event $[\#(C(0)) = k$ and $S(0) = \eta]$ equals the event $E_{\eta,k} \cap H_\eta$. The conditional probability of H_η given $\mathscr{F}_\eta$ is equal to

$$\exp\left\{-\int_{\mathbb{R}^d \backslash F_\delta^{-1}(\eta)} g_1(y; C_\eta(0))\, dy\right\}.$$

We wish to estimate this conditional probability when $E_{\eta,k}$ occurs.

Write points in $\mathbb{R}^2$ in Cartesian form as (x^1, x^2). Write random variables in $\mathbb{R}^2$ similarly as (X^1, X^2). Define the *width* of a set $A \subset \mathbb{R}^2$ to be $\sup\{|x^1 - y^1| : (x^1, x^2) \in A, (y^1, y^2) \in A\}$, and its *height* to be $\sup\{|x^2 - y^2| : (x^1, x^2) \in A, (y^1, y^2) \in A\}$.

Let $\delta_1 > 0$ be small enough for the conclusion of Lemma 1 to hold. We shall take δ to be much smaller than δ_1. If both the width and the height of $F_\delta^{-1}(\eta)$ are less than $\delta_1/2$, then $C(0)$ is contained in a side δ_1 square centered at 0. In this case, Lemma 1 gives us the desired result (8.6); so we may assume without loss of generality that the width of η exceeds $\delta_1/2$. Define the numbers x^1_{left}, x^1_{right}, and x^2_{top} (all of which are of the form $n\delta$, $n \in Z$) as follows:

$$x^1_{\text{left}} = \inf\{x^1 : (x^1, x^2) \in \eta\}, \qquad x^1_{\text{right}} = \sup\{x^1 : (x^1, x^2) \in \eta\},$$
$$x^2_{\text{top}} = \sup\{x^2 : (x^1, x^2) \in \eta\}.$$

Also, define the sets A_{left}, A_{right} and A_{top} in $\mathbb{R}^2$ as follows:

$$A_{\text{left}} = \{(x^1, x^2) : x^1 < x^1_{\text{left}} - \delta/2\}, \qquad A_{\text{right}} = \{(x^1, x^2) : x^1 > x^1_{\text{right}} + \delta/2\},$$
$$A_{\text{top}} = \{(x^1, x^2) : x^2 > x^2_{\text{top}} + \delta/2 \text{ and } x^1_{\text{left}} - \delta/2 \leqq x^1 \leqq x^1_{\text{right}} + \delta/2\}.$$

Then A_{left}, A_{right} and A_{top} are disjoint regions of $\mathbb{R}^d \backslash F_\delta^{-1}(\eta)$.

From the definition of the event $E_{\eta,k}$, if $E_{\eta,k}$ occurs, then at least one point, denoted by $X_j = (X_j^1, X_j^2)$ say, of $C_\eta(0)$, satisfies $|X_j^1 - x_{\text{left}}^1| \leqq \delta/2$. So if $E_{\eta,k}$ occurs we have

$$\int_{A_{\text{left}}} g_1(y; C_\eta(0))\, dy \geqq \int_{A_{\text{left}}} g(y - X_j)\, dy \geqq \int_{(-\infty,-\delta)\times(-\infty,\infty)} g(y)\, dy,$$

and similarly for A_{right}; so

$$\int_{A_{\text{left}} \cup A_{\text{right}}} g_1(y; C_\eta(0))\, dy \geqq \int_{\mathbb{R}^2} g(y)\, dy - \int_{(-\delta,\delta)\times(-\infty,\infty)} g(y)\, dy. \tag{8.7}$$

Also, if $E_{\eta,k}$ occurs there exists at least one point, $X_i = (X_i^1, X_i^2)$ say, of $C_\eta(0)$ such that $|x_{\text{top}}^2 - X_i^2| \leqq \delta/2$. So

$$\int_{A_{\text{top}}} g_1(y; C_\eta(0))\, dy \geqq \int_{A_{\text{top}}} g(y - X_i)\, dy. \tag{8.8}$$

Define the sets A_+ and A_- in $\mathbb{R}^2$ by

$$A_+ = (0, \delta_1) \times (\delta, \infty), \qquad A_- = (-\delta_1, 0) \times (\delta, \infty).$$

The condition that g encloses 0 ensures that

$$\min\left\{\int_{(0,\delta_1/2)\times(0,\infty)} g(x)\, dx, \int_{(-\delta_1/2,0)\times(0,\infty)} g(x)\, dx\right\} > 0,$$

so that for small enough $\delta > 0$,

$$\min\left\{\int_{A_+} g(x)\, dx, \int_{A_-} g(x)\, dx\right\} \geqq \int_{(-\delta,\delta)\times(-\infty,\infty)} g(x)\, dx + c \tag{8.9}$$

for some $c > 0$.

Since we are assuming that the width of η is at least δ_1, it follows that at least one of the sets $X_i + A_+$ and $X_i + A_-$ is contained in A_{top}. Hence, by (8.8),

$$\int_{A_{\text{top}}} g_1(y; C_\eta(0))\, dy \geqq \min\left\{\int_{A_+} g(x)\, dx, \int_{A_-} g(x)\, dx\right\}. \tag{8.10}$$

Combining (8.7), (8.9) and (8.10) we have

$$\int_{A_{\text{left}} \cup A_{\text{right}} \cup A_{\text{top}}} g_1(y; C_\eta(0))\, dy\}) \geqq \int_{\mathbb{R}^2} g(y)\, dy + c. \tag{8.11}$$

Given that $E_{\eta,k}$ occurs, the conditional probability of H_η is at most

$$\exp\left\{-\rho \int_{A_{\text{left}} \cup A_{\text{right}} \cup A_{\text{top}}} g_1(y; C_\eta(0))\, dy\right\}.$$

Hence,

$$\sum_{k=1}^{\infty} k^N P_\rho[\#(C(0)) = k \text{ and } S(0) = \eta]/q_1(\rho)$$

(8.12)
$$= \sum_{k=1}^{\infty} k^N P_\rho[E_{\eta,k} \cap H_\eta]/\exp\left\{-\rho \int_{\mathbb{R}^2} g(y)\,dy\right\}$$
$$\leqq \left(\sum_{k=1}^{\infty} k^N P_\rho(E_{\eta,k})\right) \exp(-\rho c),$$

where the last inequality follows from (8.11). If $N \geqq 0$, the first factor in the right-hand side of (8.12) is at most

$$E_\rho(\text{number of points of } \mathcal{P} \cup \{0\} \text{ in } F_\delta^{-1}(\eta))^N$$
$$= E((X+1)^N) \text{ (where } X \text{ is Poisson } (\rho\delta^2\#(\eta))) \leqq c_1\rho^N.$$

Thus, for small enough (fixed) δ, for each η the right-hand side of (8.12) converges to 0 as $\rho \to \infty$, as desired.

Proof of Corollary. Let $C_r(0)$ be the cluster at the origin obtained when all bonds of length greater than r are removed. Then, for $N \leqq 0$,

$$(\#[C(0)])^N I_{\{C_r(0) \text{ is finite}\}} \leqq (\#[C_r(0)])^N I_{\{C_r(0) \text{ is finite}\}},$$

so that, taking expectations and applying Theorem 3,

$$f_N(\rho) \leqq \sum_1^{\infty} k^N P_\rho[\#C_r(0) = k] \sim \exp\left(-\rho \int_{|x| \leqq r} g(x)\,dx\right) \quad \text{as} \quad \rho \to \infty.$$

Hence,

(8.13)
$$\limsup_{\rho \to \infty} (f_N(\rho)/q_1(\rho))^{1/\rho} \leqq \exp\left(\int_{|x|>r} g(x)\,dx\right).$$

Since $f_N(\rho) \geqq q_1(\rho)$ and we may let $r \to \infty$ in (8.13), $\lim_{\rho \to \infty} (f_N(\rho)/q_1(\rho))^{1/\rho} = 1$. Taking logarithms,

$$\rho^{-1}(\log f_N(\rho) - \log q_1(\rho)) \to 0 \quad \text{as} \quad \rho \to \infty,$$

and dividing through by $\rho^{-1} \log q_1(\rho)$, which equals $\int g(x)\,dx$, we obtain

$$\frac{\log f_N(\rho)}{\log q_1(\rho)} \to 1 \quad \text{as} \quad \rho \to \infty.$$

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