

SUBDIFFERENTIALS ARE LOCALLY MAXIMAL MONOTONE

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In a recent paper, Fitzpatrick and Phelps introduced a new class of operators on a Banach space, the *locally maximal monotone* operators, and showed that this kind of operator can be approximated by a sequence of nicer maximal monotone operators. We give here an affirmative answer to a question posed in this paper: is the subdifferential of a proper convex lower semicontinuous function necessarily locally maximal monotone? Since a locally maximal operator is maximal monotone, our result represents a strengthening of Rockafellar's maximal monotonicity theorem.

In a recent paper, Fitzpatrick and Phelps [2] introduced a new class of operators on a Banach space, the *locally maximal monotone* operators, and showed that this kind of operator can be approximated by a sequence of nicer maximal monotone operators. It is clear from its definition that a locally maximal monotone operator is maximal monotone. On the other hand, Fitzpatrick and Phelps pointed out in [2, Theorem 3.5] that the range of a locally maximal monotone operator has a convex closure, and hence that an example of Gossez (see [3]), shows that a maximal monotone operator is not necessarily locally maximal monotone. [2] left open the question whether the subdifferential of a proper convex lower semicontinuous function is necessarily locally maximal monotone. In this note, we shall give an affirmative answer to this question. It follows from the above comments that our result represents a strengthening of Rockafellar's maximal monotonicity theorem.

Lemmas 2–4 will be proved in greater generality than is actually required for the Main Theorem. Specifically, they will be phrased in terms of a sublinear functional T satisfying only (0.1). (This greater generality does not seem to require any extra work; on the contrary, it probably allows us to simplify the notion somewhat). For the Main Theorem, we shall apply these Lemmas in the case where T is given by a simple specific formula (see Lemma 5).

A word is in order about two proofs that we have given recently of the maximal monotonicity theorem. The first (see [4]) relies on the formula for the subdifferential of a sum, while the second (see [5]) relies on manipulating sublinear functionals or

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extended sublinear functionals (that is to say, sublinear functionals that can possibly take the value $+\infty$). The advantage of the second method is that it is a more or less mechanical procedure, which appears in the present paper in line (2.4), where the sublinear functional is $(1 + \epsilon)KT$, and the extended sublinear functional is $d^+\phi(z)(\cdot)$. We have also applied this procedure to obtain a number of other results on the existence of subtangents to proper convex lower semicontinuous functions. These include other generalisations of the maximal monotonicity theorem. (See [6].) There is a feature in this paper that does not appear in [4], [5] or [6]: the existence theorem (Lemma 2) uses $(q, h) \in E \times \mathbb{R}$ such that $\phi(q) > h > \inf \phi(E)$ and the new feature (see Lemma 3) is the pushing down of h to the lower end of this range in order to decrease $K(q, h, \phi, T)$. The only place in this paper where we use a result from [4], [5] or [6] is in line (0.3).

DEFINITION of the statement that $\partial\psi$ is locally maximal monotone.

With the notation explained below, this means that if

$$q \in E, a \in E' \setminus \partial\psi(q)$$

and U is a convex open subset of E'

such that $U \ni a$ and $U \cap R(\partial\psi) \neq \emptyset$

then $\exists(z, b) \in \partial\psi$ such that $b \in U$ and $\langle z - q, b - a \rangle < 0$.

WHAT WE SHALL PROVE.

In fact, we shall prove the following result.

MAIN THEOREM. Let $q \in E, a \in E' \setminus \partial\psi(q)$ and $e \in R(\partial\psi)$. Then

$$[a, e] \cap \overline{\{b: \exists z \in E \text{ such that } b \in \partial\psi(z) \text{ and } \langle z - q, b - a \rangle < 0\}} \neq \emptyset.$$

The Main Theorem formally implies the following result, which in turn formally implies that $\partial\psi$ is locally maximal monotone. The author is grateful to Professor S. Fitzpatrick for pointing out that statement in the Main Theorem is in fact *equivalent* to the statement that $\partial\psi$ is locally maximal monotone.

COROLLARY. Let $q \in E, a \in E' \setminus \partial\psi(q)$ and C be a convex subset of E' such that $C \ni a$ and $C \cap R(\partial\psi) \neq \emptyset$. Then

$$C \cap \overline{\{b: \exists z \in E \text{ such that } b \in \partial\psi(z) \text{ and } \langle z - q, b - a \rangle < 0\}} \neq \emptyset.$$

OTHER NOTATION. We suppose throughout that E is a real Banach space and $\phi, \psi: E \rightarrow \mathbb{R} \cup \{\infty\}$ are proper, convex and lower semicontinuous.

Let $0 < m \leq M < \infty$. Let $T: E \rightarrow \mathbb{R}$ be a sublinear functional such that

$$(0.1) \quad m \|\cdot\| \leq T \leq M \|\cdot\| \text{ on } E.$$

Write $\tau = m/M$. τ measures the “degree of asymmetry” of T . We note for future reference that

$$(0.2) \quad \tau T \leq m \|\cdot\| \text{ on } E.$$

Define $T_q: E \rightarrow \mathbb{R}$ by

$$T_q(y) := T(q - y) \quad (y \in E).$$

T_q is a convex, continuous function on E . If $q \in E$ and $\phi(q) > h > \inf \phi(E)$, we write

$$K(q, h, \phi, T) := \sup_{\phi(y) < h} \frac{h - \phi(y)}{T(q - y)} = \sup_{\phi(y) < h} \frac{h - \phi(y)}{T_q(y)}.$$

It was proved in [4, Lemma 2.2(a), p.130] and [5, Théorème 2(a)] that

$$(0.3) \quad 0 < K(q, h, \phi, \|\cdot\|) < \infty.$$

If $z \in E$, the *subdifferential*, $\partial\phi(z)$, of ϕ at z is defined by

$$\{c: c \in E', \forall y \in E, \phi(z) - \phi(y) \leq \langle z - y, c \rangle\},$$

where E' is the adjoint of E . We write $\partial\phi$ for $\{(z, c): z \in E, c \in \partial\phi(z)\} \subseteq E \times E'$ and $R(\partial\phi)$ for $\{d: \exists w \in E \text{ such that } d \in \partial\phi(w)\} \subseteq E'$.

If $z \in \text{dom } \phi$ and $v \in E$ then the *directional derivative* of ψ at z in the direction v is defined by

$$d^+\psi(z)(v) := \lim_{\theta \rightarrow 0^+} \frac{\psi(z + \theta v) - \psi(z)}{\theta}.$$

Since the limit in the above expression can be replaced by an infimum, it follows that

$$(0.4) \quad \psi(z + v) \geq d^+\psi(z)(v) + \psi(z).$$

LEMMA 1. Let $\alpha, \beta > 0$, $y \in \text{dom } \psi$ and $\psi(y) < \inf \psi(E) + \alpha\beta$. Then

$$\exists z \in \text{dom } \psi \text{ such that } \|z - y\| < \alpha \text{ and, } \forall v \in E, d^+\psi(z)(v) \geq -\beta \| -v \|.$$

PROOF: We choose $\gamma < \alpha$ so that $\psi(y) < \inf \psi(E) + \gamma\beta$. From [1, Theorem 1, p.444] with $\varepsilon := \gamma\beta$ and $d(u, v) := \|u - v\|/\gamma$, $\exists z \in \text{dom } \psi$ such that

$$\|z - y\|/\gamma \leq 1 \text{ and, } \forall w \in E, \psi(w) \geq \psi(z) - \gamma\beta \|z - w\|/\gamma.$$

The required result follows from the definition of directional derivative by putting $w := z + \theta v$ and letting $\theta \rightarrow 0^+$. \square

LEMMA 2. *Let $q \in E$ and $\phi(q) > h > \inf \phi(E)$, $K := K(q, h, \phi, T)$ and $\epsilon \in (0, 1)$. Then:*

- (a) $0 < K < \infty$.
- (b) $\exists y \in E$ such that

$$y \neq q \text{ and } (\phi + KT_q)(y) < \inf (\phi + KT_q)(E) + \tau\epsilon mK \|q - y\|.$$

- (c) $\exists(z, c) \in \partial\phi$ such that

$$\langle q - z, c \rangle \geq (1 - \epsilon)KT_q(z) > 0 \text{ and } c \leq (1 + \epsilon)KT \text{ on } E.$$

PROOF: (a) This follows from (0.1) and (0.3).

(b) Let $y \in E$. If $\phi(y) \geq h$ then $h \leq (\phi + KT_q)(y)$. If $\phi(y) < h$ then

$$\frac{h - \phi(y)}{T_q(y)} \leq K,$$

from which, again, $h \leq (\phi + KT_q)(y)$. Taking the infimum over y ,

$$(2.1) \quad h \leq \inf (\phi + KT_q)(E).$$

Since $K(1 - \epsilon\tau^2) < K$, from the definition of K , $\exists y \in E$ such that

$$\phi(y) < h \text{ and } \frac{h - \phi(y)}{T_q(y)} > K(1 - \epsilon\tau^2),$$

from which $y \neq q$ and $K(1 - \epsilon\tau^2)T_q(y) + \phi(y) < h$,

that is $(\phi + KT_q)(y) < h + \epsilon K\tau^2 T_q(y)$.

(b) follows by combining this with (0.2) and (2.1).

(c) From (b) and Lemma 1 with $\psi := \phi + KT_q$, $a := \|q - y\|$ and $\beta := \tau\epsilon mK$, $\exists z \in \text{dom}(\phi + KT_q) = \text{dom} \phi$ such that $\|z - y\| < \|q - y\|$ (thus $z \neq q$, from which $T_q(z) > 0$) and,

$$(2.2) \quad \forall v \in E, d^+(\phi + KT_q)(z)(v) \geq -\epsilon K\tau m \| -v \| \geq -\epsilon K\tau T(-v).$$

We first prove that,

$$(2.3) \quad \forall v \in E, d^+\phi(z)(v) + (1 + \epsilon)KT(q - z - v) \geq (1 - \epsilon)KT_q(z).$$

To this end, let $v \in E$. Then since $\tau \leq 1$,

$$d^+\phi(z)(v) + (1 + \epsilon)KT(q - z - v) \geq d^+\phi(z)(v) + KT_q(z + v) + \epsilon K\tau T_q(z + v),$$

from (0.4) (applied to $\psi := KT_q$),

$$\geq d^+ \phi(z)(v) + d^+ KT_q(z)(v) + KT_q(z) + \epsilon K\tau T_q(z + v),$$

from (2.2) (since $d^+ \phi(z)(v) + d^+ KT_q(z)(v) = d^+(\phi + KT_q)(z)(v)$),

$$\geq -\epsilon K\tau T(-v) + KT_q(z) + \epsilon K\tau T_q(z + v) = -\epsilon K\tau T(-v) + KT_q(z) + \epsilon K\tau T(q - z - v)$$

since T is sublinear,

$$\geq KT_q(z) - \epsilon K\tau T(z - q) \geq (1 - \epsilon)KT_q(z),$$

since, from (0.1) and (0.2), $\tau T(z - q) \leq m \|z - q\| = m \|q - z\| \leq T(q - z) = T_q(z)$. This completes the proof of (2.3). We define a function S on E by

$$(2.4) \quad S(w) := \inf\{d^+ \phi(z)(v) + (1 + \epsilon)KT(w - v) : v \in E\}.$$

From (2.3), $S(q - z) \geq (1 - \epsilon)KT_q(z) > 0$. S is real-valued and sublinear. From the one-dimensional Hahn-Banach theorem, there is a linear functional c on E such that $c \leq S$ on E and

$$\langle q - z, c \rangle = S(q - z) \geq (1 - \epsilon)KT_q(z) > 0.$$

Since $c \leq S$ on E , $c \leq (1 + \epsilon)KT$ and $c \leq d^+ \phi(z)$ on E . From (0.1), $c \in E'$, hence $c \in \partial\phi(z)$. This completes the proof of (c). □

LEMMA 3. *Let $q \in E$ and $\phi(q) > \inf \phi(E)$. Suppose that $N > 0$, $d \in R(\partial\phi)$ and $d \leq NT$ on E . Then there is h such that $\phi(q) > h > \inf \phi(E)$ and $K(q, h, \phi, T) \leq N$.*

PROOF: We first suppose that $\inf \phi(E) = -\infty$. Let $w \in E$ be such that $d \in \partial\phi(w)$ and write $h := \phi(w) + \langle q - w, d \rangle - 1$. Then $\phi(q) > \phi(q) - 1 \geq h > -\infty = \inf \phi(E)$. Further, $\forall y \in E$,

$$\phi(y) + NT_q(y) \geq \phi(y) + \langle q - y, d \rangle \geq \phi(w) + \langle y - w, d \rangle + \langle q - y, d \rangle = h + 1 \geq h,$$

from which the result follows. If, on the other hand, $\inf \phi(E) > -\infty$, we first choose p such that $\phi(q) > p > \inf \phi(E)$, then $\delta > 0$ so that ($\|q - y\| < \delta \Rightarrow \phi(y) > p$) and, finally, h so that

$$\min\{\inf \phi(E) + m\delta N, p\} > h > \inf \phi(E).$$

If $\phi(y) < h$ then $\phi(y) < p$, hence $\|q - y\| \geq \delta$, from which $T_q(y) \geq m\delta$. The result follows since $h - \phi(y) \leq m\delta N$. □

LEMMA 4. *Let $q \in E$ and $\phi(q) > \inf \phi(E)$. Suppose that $0 < N < 1$, $d \in R(\partial\phi)$ and $d \leq NT$ on E . Then $\exists(z, c) \in \partial\phi$ such that $\langle q - z, c \rangle > 0$ and $c \leq T$ on E .*

PROOF: We choose h as in Lemma 3. Since $K(q, h, \phi, T) \leq N < 1$, we can choose $\epsilon \in (0, 1)$ so that $(1 + \epsilon)K(q, h, \phi, T) \leq 1$. The result follows from Lemma 2(c). □

LEMMA 5. *Let $q \in E$ with $\phi(q) > \inf \phi(E)$, $d \in R(\partial\phi)$ and $\eta > 0$. Then $\exists(z, c) \in \partial\phi$ and $\lambda \in [0, 1]$ such that $\langle q - z, c \rangle > 0$ and $\|c - \lambda d\| \leq \eta$.*

PROOF: We define $T: E \rightarrow \mathbb{R}$ by

$$T(x) := \eta \|x\| + \max\{0, \langle x, d \rangle\}.$$

Then T is sublinear, and (0.1) is satisfied with $m := \eta$ and $M := \eta + \|d\|$. We choose N so that

$$\frac{\|d\|}{\eta + \|d\|} < N < 1.$$

Then $d \leq NT$ on E . We find $(z, c) \in \partial\phi$ as in Lemma 4. Let A be the weak* compact convex subset $\{e + \lambda d: e \in E', \|e\| \leq \eta, \lambda \in [0, 1]\}$. Then,

$$\forall x \in E, T(x) = \max\langle x, A \rangle.$$

Since $c \leq T$ on E , from the bipolar theorem, $c \in A$. This gives the required result. \square

LEMMA 6. *Let $q \in E$ with $\phi(q) > \inf \phi(E)$ and $d \in R(\partial\phi)$. Then*

$$[0, d] \cap \overline{\{c: \exists z \in E \text{ such that } c \in \partial\phi(z) \text{ and } \langle q - z, c \rangle > 0\}} \neq \emptyset.$$

($[0, d]$ stands for the line segment joining 0 and d .)

PROOF: From Lemma 5, $\forall n \geq 1, \exists(z_n, c_n) \in \partial\phi$ and $\lambda_n \in [0, 1]$ such that $\langle q - z_n, c_n \rangle > 0$ and $\|c_n - \lambda_n d\| \leq 1/n$. Let λ be a limit point of $\{\lambda_n\}_{n \geq 1}$. Then λd is in the intersection in the statement of the lemma. \square

MAIN THEOREM. *Let $q \in E, a \in E' \setminus \partial\psi(q)$ and $e \in R(\partial\psi)$. Then*

$$[a, e] \cap \overline{\{b: \exists z \in E \text{ such that } b \in \partial\psi(z) \text{ and } \langle z - q, b - a \rangle < 0\}} \neq \emptyset.$$

PROOF: Let $\phi := \psi - a$. Then $0 \notin \partial\phi(q)$, hence $\phi(q) > \inf \phi(E)$. Let $d := e - a$. Then $d \in R(\partial\phi)$. If f is in the intersection of Lemma 6 then $a + f$ is in the intersection in the statement of the Theorem. \square

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