An algebraic derivation of the distribution of rectangular coordinates

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Let x_{ir} $(i = 1, \ldots, q;\ r = 1, \ldots, m;\ q \leqq m)$ be random variables which have an elementary probability law $p(x_{11}, \ldots, x_{qm})$. Let

$$s_{ij} = \sum_{r=1}^{m} x_{ir} x_{jr}.$$

The fundamental assumption is that $p(x_{11}, \ldots, x_{qm})$ is explicitly a function of the set of s_{ij} alone, so that

$$p(x_{11}, \ldots, x_{qm}) = f(s_{11}, s_{12}, \ldots, s_{qq}). \tag{1}$$

The $\frac{1}{2}q(q+1)$ functions t_{ij} $(i \leqq j)$, defined by the equation

$$S = TT', \tag{2}$$

where

$$S = \begin{vmatrix} s_{11} & \ldots & s_{1q} \\ \ldots & \ldots & \ldots \\ s_{q1} & \ldots & s_{qq} \end{vmatrix}, \quad T = \begin{matrix} t_{11} & 0 & \ldots & 0 \\ t_{12} & t_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ t_{1q} & t_{2q} & \ldots & t_{qq} \end{matrix}, \quad t_{ii} \geqq 0,$$

and T' is the transposed matrix of T, are a generalisation of the rectangular coordinates of multivariate normal samples defined and studied by Mahalanobis and others in a joint paper[1].

We have, directly from (2),

$$s_{ij} = t_{1i} t_{1j} + t_{2i} t_{2j} + \ldots + t_{ii} t_{ij} \tag{3}$$

To express the t_{ij} in terms of the s_{ij}, we notice from (2) that, for $i \leqq j$,

$$\begin{vmatrix} s_{11} & \ldots & s_{1,i-1} & s_{1j} \\ \ldots & \ldots & \ldots & \ldots \\ s_{i-1,1} & \ldots & s_{i-1,i-1} & s_{i-1,j} \\ s_{i1} & \ldots & s_{i,i-1} & s_{ij} \end{vmatrix} = \begin{vmatrix} t_{11} & 0 & \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ t_{1,i-1} & t_{2,i-1} & \ldots t_{i-1,i-1} & 0 \\ t_{1i} & t_{2i} & \ldots t_{i-1,i} & t_{ii} \end{vmatrix} \begin{vmatrix} t_{11} & \ldots & t_{1,i-1} & t_{1j} \\ 0 & \ldots & t_{2,i-1} & t_{2j} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & t_{i-1,i-1} & t_{i-1,j} \\ 0 & \ldots & 0 & t_{ij} \end{vmatrix}.$$

[1] Mahalanobis, Bose and Roy, *Sankhya*, 3 (1937), 1-40. This paper will be referred to as (*M*).

whence, taking determinants,

$$\begin{vmatrix} s_{11} & \cdots & s_{1,i-1} & s_{1i} \\ \cdots & \cdots & \cdots & \cdots \\ s_{i-1,1} & \cdots & s_{i-1,i-1} & s_{i-1,j} \\ s_{i1} & \cdots & s_{i,i-1} & s_{ij} \end{vmatrix} = t_{11}^2 \cdots t_{i-1,i-1}^2 \, t_{ii} t_{ij}. \tag{4}$$

Setting $i = j$ in (4) we have

$$\begin{vmatrix} s_{11} & \cdots & s_{1i} \\ \cdots & \cdots & \cdots \\ s_{i1} & \cdots & s_{ii} \end{vmatrix} = t_{11}^2 \cdots t_{ii}^2, \tag{5}$$

whence

$$t_{ii} = \begin{vmatrix} s_{11} & \cdots & s_{1i} \\ \cdots & \cdots & \cdots \\ s_{i1} & \cdots & s_{ii} \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} s_{11} & \cdots & s_{1,i-1} \\ \cdots & \cdots & \cdots \\ s_{i-1,1} & \cdots & s_{i-1,i-1} \end{vmatrix}^{-\frac{1}{2}}. \tag{6}$$

Dividing both sides of (4) by the corresponding sides of (5) and using (6) we obtain

$$t_{ij} = \begin{vmatrix} s_{11} & \cdots & s_{1,i-1} & s_{1j} \\ \cdots & \cdots & \cdots & \cdots \\ s_{i-1,1} & \cdots & s_{i-1,i-1} & s_{i-1,j} \\ s_{i1} & \cdots & s_{i,i-1} & s_{ij} \end{vmatrix} \left| \begin{vmatrix} s_{11} & \cdots & s_{1i} \\ \cdots & \cdots & \cdots \\ s_{i1} & \cdots & s_{ii} \end{vmatrix}^{-\frac{1}{2}} \begin{vmatrix} s_{11} & \cdots & s_{1,i-1} \\ \cdots & \cdots & \cdots \\ s_{i-1,1} & \cdots & s_{i-1,i-1} \end{vmatrix}^{-\frac{1}{2}} \right.. \tag{7}$$

(3) and (7) are the relations connecting the rectangular coordinates t_{ij} and the product moments s_{ij} and have been derived in (M).

In the case where the elementary probability law $p(x_{11}, \ldots, x_{qm})$ is a normal one, a geometrical demonstration has been given in (M) of the distribution of the t_{ij}. Here we shall give a purely algebraic derivation of the distribution, assuming that (1) holds true.

THEOREM. *If* (1) *is true, the elementary probability law of the rectangular coordinates t_{ij} defined by* (2) *is*

$$2^q \pi^{\frac{1}{2}mq - \frac{1}{4}q(q-1)} \{\prod_{i=1}^{q} \Gamma(\tfrac{1}{2}(m - i + 1))\}^{-1} (\prod_{i=1}^{q} t_{ii}^{m-i}) f(s_{11}, s_{12}, \ldots, s_{qq}), \tag{8}$$

where the arguments of f are to be regarded as the functions (3) *of the t_{ij}.*

Proof. By virtue of (1) we may express the elementary probability as

$$f(s_{11}, s_{12}, \ldots, s_{qq})\, dx_{11} \ldots dx_{qm}.$$

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Let us write more fully as follows:

$$\int \begin{bmatrix} s_{11} & & & \\ s_{12} & s_{22} & & \\ \cdots & \cdots & \cdots & \\ s_{1q} & s_{2q} & \cdots & s_{qq} \end{bmatrix} dx_{11} \ldots dx_{qm}. \tag{9}$$

Let the sets of variables $(x_{21}, \ldots, x_{2m}), \ldots, (x_{q1}, \ldots, x_{qm})$ be subjected to the same linear transformation below whose coefficients are functions of $x_{11}, \ldots, x_{1m}$:

$$\left.\begin{array}{l} y_{i1} = (\sum_{r=1}^{m} x_{1r}^2)^{-\frac{1}{2}} \sum_{r=1}^{m} x_{1r}\, x_{ir} \\ y_{i2} = \sum_{r=1}^{m} c_{2r}\, x_{ir} \\ \cdots\cdots\cdots\cdots \\ y_{im} = \sum_{r=1}^{m} c_{mr}\, x_{ir} \end{array}\right\}, \quad (i = 2, \ldots, q), \tag{10}$$

where the c's are so determined as to make (10) an orthogonal transformation. As the Jacobian is 1, we get the result

$$\int \begin{bmatrix} \sum_{r=1}^{m} x_{1r}^2 & & & \\ y_{21} (\sum_{r=1}^{m} x_{1r}^2)^{\frac{1}{2}} & \sum_{r=1}^{m} y_{2r}^2 & & \\ \cdots & \cdots & \cdots & \\ y_{q1} (\sum_{r=1}^{m} x_{1r}^2)^{\frac{1}{2}} & \sum_{r=1}^{m} y_{2r}\, y_{qr} & \cdots & \sum_{r=1}^{m} y_{qr}^2 \end{bmatrix} dx_{11} \ldots dx_{1m}\, dy_{21} \ldots dy_{qm}.$$

If for $x_{11}, \ldots, x_{1m}$ we substitute spherical coordinates, viz. the radius vector, t_{11}, and $m - 1$ angles, $\theta_{12}, \ldots, \theta_{1m}$, we obtain

$$t_{11}^{m-1} \{ \prod_{a=3}^{m} (\cos \theta_{1a})^{a-2} \} \int \begin{bmatrix} t_{11}^2 & & & \\ t_{11}\, y_{21} & \sum_{r=1}^{m} y_{2r}^2 & & \\ \cdots & \cdots & \cdots & \\ t_{11}\, y_{q1} & \sum_{r=1}^{m} y_{2r}\, y_{qr} & \cdots & \sum_{r=1}^{m} y_{qr}^2 \end{bmatrix} dt_{11}\, d\theta_{12} \ldots d\theta_{1m}\, dy_{21} \ldots dy_{qm}.$$

Writing t_{1i} for y_{i1} $(i = 2, \ldots q)$ and

$$s'_{ij} = \sum_{r=2}^{m} y_{ir}\;\; y_{jr} \;\; (i, j = 2, \ldots, q),$$

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we get

$$t_{11}^{m-1}\{\prod_{a=3}^{m}(\cos\theta_{1a})^{a-2}\}\times$$

$$f\begin{bmatrix} t_{11}^2 & & & \\ t_{11}t_{12} & t_{12}^2+s'_{22} & & \\ \cdots & \cdots & \cdots & \cdots \\ t_{11}t_{1q} & t_{12}t_{1q}+s'_{2q} & \cdots\cdots & t_{1q}^2+s'_{qq}\end{bmatrix} dt_{11}\,..\,dt_{1q}\,d\theta_{12}\,..\,d\theta_{1m}\,dy_{22}\,..\,dy_{qm}. \quad (11)$$

It is seen that, as far as the y-variables are concerned, we have the same situation that only the product moments s'_{ij} figure in the elementary probability law. Hence the same procedure which carries (9) to (11) may be repeated. In doing so we introduce the following variables: $t_{22}, t_{23}, \ldots, t_{2q}$; $\theta_{23}, \ldots, \theta_{2q}$; $z_{33}, z_{34}, \ldots, z_{qm}$, to replace the y's, and write down the elementary probability:

$$t_{11}^{m-1}t_{22}^{m-2}\{\prod_{a=3}^{m}(\cos\theta_{1a})^{a-2}\}\{\prod_{a=4}^{m}(\cos\theta_{2a})^{a-3}\}\times$$

$$f\begin{bmatrix} t_{11}^2 & & & \\ t_{11}t_{12} & t_{12}^2+t_{22}^2 & & \\ t_{11}t_{13} & t_{12}t_{13}+t_{22}t_{23} & t_{13}^2+t_{23}^2+s''_{33} & \\ \cdots & \cdots & \cdots & \cdots \\ t_{11}t_{1q} & t_{12}t_{1q}+t_{22}t_{2q} & t_{13}t_{1q}+t_{23}t_{2q}+s''_{3q} & \cdots t_{1q}^2+t_{2q}^2+s''_{qq}\end{bmatrix}\times$$

$$dt_{11}\ldots dt_{1q}\,dt_{22}\ldots dt_{2q}\,d\theta_{12}\ldots d\theta_{1m}\,d\theta_{23}\ldots d\theta_{2m}\,dz_{33}\ldots dz_{qm},$$

where

$$s''_{ij}=\sum_{r=3}^{m}z_{ir}z_{jr}\ (i,j=3,\ldots,q).$$

Proceeding in this manner we finally obtain

$$(\prod_{i=1}t_{ii}^{m-i})\{\prod_{i=1}^{q}\prod_{a=i+2}^{m}(\cos\theta_{ia})^{a-i-1}\}\times$$

$$f\begin{bmatrix} t_{11}^2 & & \\ t_{11}t_{12} & t_{12}^2+t_{22}^2 & \\ \cdots & \cdots & \cdots \\ t_{11}t_{1q} & t_{12}t_{1q}+t_{22}t_{2q} & \cdots t_{1q}^2+t_{2q}^2+\cdots+t_{qq}^2\end{bmatrix}\times$$

$$d\theta_{12}\ldots d\theta_{q+1,m}\,dt_{11}\,dt_{12}\ldots dt_{qq}. \quad (12)$$

Now the functions t_{ij} introduced in the proof are precisely the rectangular co-ordinates defined by (2) or (3). For, in each step of transformation from (9) to (12) the change in the arguments of f is

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effected by direct substitution. Equating the arguments of f in (9) and (12) we get

$$s_{ij} = t_{1i}t_{1j} + t_{2i}t_{2j} + \dots + t_{ii}t_{ij},$$

which gives (3).

If from (12) the θ's are integrated over the following domain:

$$-\pi \leqq \theta_{ia} \leqq \pi \qquad (a = i + 1;\ i = 1, \dots, q),$$
$$-\tfrac{1}{2}\pi \leqq \theta_{ia} \leqq \tfrac{1}{2}\pi \quad (a = i + 2, \dots, m;\ i = 1, \dots, q),$$

the result is the elementary probability law (8). The proof is therefore complete.

One more step leads to the distribution of the product moments s_{ij}, as is done in (M). The transformation is given by (7) and the reciprocal transformation by (3), which has the Jacobian

$$2^q t_{11}^q t_{22}^{q-1} \dots t_{qq}.$$

Dividing (8) by this Jacobian and then carrying out the substitution we get the following elementary probability law of the s_{ij}:

$$\pi^{\frac{1}{2}mq-\frac{1}{4}q(q-1)}\{\prod_{i=1}^{q}\Gamma(\tfrac{1}{2}(m-i+1))\}^{-1}\begin{vmatrix} s_{11} & \dots & s_{1q} \\ \dots & \dots & \dots \\ s_{q1} & \dots & s_{qq}\end{vmatrix}^{\frac{1}{2}(m-q+1)} f(s_{11}, s_{12}, \dots, s_{qq}). \quad (13)$$

The multiplier of $f(s_{11}, s_{12}, \dots, s_{qq})$ in (13) is well known[1].

The result (12) brings out the following important fact: If $x_{11}, x_{12}, \dots, x_{qm}$ are all the observational data and if only the rectangular coordinates t_{ij} or the product moments s_{ij} are utilised for statistical purposes, the part of the observational data thus thrown away may be regarded as a set of angles which are distributed independently of the t_{ij} or s_{ij} and whose elementary probability law does not involve any of the unknown parameters that may figure in the elementary probability law of the x's.

[1] *Cf.* Wishart and Bartlett, *Proc. Camb. Phil. Soc.*, 29 (1933), 271-6.

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