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FREE GROUPS IN SUBNORMAL SUBGROUPS AND THE RESIDUAL NILPOTENCE OF THE GROUP OF UNITS OF GROUPS RINGS

BY

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ABSTRACT. Let KG be the group ring of the group G over the field K and $U(KG)$ its unit group. When G is finite we derive conditions which imply that every noncentral subnormal subgroup of $U(KG)$ contains a free group of rank two. We also show that residual nilpotence of $U(KG)$ coincides with nilpotence, this being no longer true if G is infinite.

We can answer partially the following question: when is G sub-normal in $U(KG)$?

1. **Introduction.** Let K be a field, G be a finite group, KG the group ring of G over K and $U(KG)$ its unit group. In [2], the author established necessary and sufficient conditions for $U(KG)$ to contain no free subgroup of rank two. Now we will work in the reverse direction, studying when every subnormal subgroup of $U(KG)$, not contained in the center, has a free subgroup of rank two.

As a corollary, we can give a partial answer to a question posed by Polcino Milies, [7], (2.18): when is G subnormal in $U(KG)$? This is the content of Section 2.

In Section 3, motivated by a paper of Musson and Weiss [6], we study the residual nilpotence of $U(KG)$ and show that when G is finite this coincides with the nilpotence of $U(KG)$. In the infinite case this is no longer true.

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2. **Free groups in subnormal subgroups of the multiplicative group of division rings**

THEOREM 2.1. *Let D be a division ring, finite dimensional over its center Z, and let H be a subnormal subgroup of $D^* = D - \{0\}$, not contained in Z^*. Then H contains a free subgroup of rank two.*

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Proof. Suppose not. Then by [12], Theorems 1 and 2 H contains a normal solvable subgroup L such that H/L is locally finite. Hence by [10], Theorem 14.4.4 $L \subseteq Z^*$.

Now, let N be the norm on D to Z and H' be the commutator subgroup of H. Since every element x of H' is a product of commutators $h_1h_2h_1^{-1}h_2^{-1}$, $h_1, h_2 \in H$ and, for some m $x^m = \lambda \in Z^*$, it follows that $N(x) = 1$ and $1 = N(x^m) = N(x) = N(\lambda) = \lambda^r$, where $r = \dim_Z D$. Therefore H' is a torsion subnormal subgroup of D^*. By [4], Theorem 8 $H' \subseteq Z^*$. Hence H is a solvable subnormal subgroup of D^* and by [10], Theorem 14.4.4 again, $H \subseteq Z^*$, a contradiction.

The theorem above was motivated by the conjecture that the multiplicative group of a division ring contains a free subgroup of rank two [5].

In support to that conjecture we prove:

PROPOSITION 2.2. *Let D be a division ring containing a noncentral torsion element a. Then D^* contains a free subgroup of rank two.*

Proof. Let n be the order of $\langle a \rangle$, the cyclic group generated by a. Since $p(X) = \text{irrat}(a, Z)$ divides $X^n - 1$, every root of $p(X)$ is a power of a. Let

$$\phi : Z(a) \to Z(a)$$
$$\phi(a) = a^r$$

be a nonidentity automorphism of $Z(a)$. By the Noether–Skolem theorem there is a $b \in D$ such that $b^{-1}ab = a^r$.

Since $a \mapsto b^{-1}ab$ is an automorphism of $\langle a \rangle$, (r, n), the greatest common divisor of r and n, is equal to 1. Hence there exists a positive integer m such that $r^m \equiv 1 \pmod n$ and that $b^m a = ab^m$.

Let $A = \{\sum_{i,j} \alpha_{ij} a^i b^j \in D \mid \alpha_{ij} \in Z(b^m)\}$. Then A is a $Z(b^m)$-algebra finitely generated over $Z(b^m)$. By Wedderburn's Theorem, A is a division ring finite dimensional over its center. By Theorem 2.1 $U(A)$ contains a free subgroup of rank two.

Let now D be a division ring or a field. We will denote by $GL_n(D)$ the general linear group, by $SL_n(D)$ the $n \times n$ special linear group and by $GF(p)$, for a rational prime p, the Galois field with p-elements.

LEMMA 2.3. *Suppose that D is different from $GF(2)$ and $GF(3)$ and let N be a noncentral subnormal subgroup of $GL_n(D)$. Then $SL_n(D) \subseteq N$*

Proof. By [11], Theorems II 10.1 and II 10.2, every noncentral subgroup of $GL_n(D)$ normalized by $SL_n(D)$ contains $SL_n(D)$. Now apply induction on the length of the subnormal series.

THEOREM 2.4. *Let K be a field of characteristic 0 and G be a nonabelian finite group. Then every noncentral subnormal subgroup of $U(KG)$ contains a free subgroup of rank two.*

Proof. By Wedderburn's Theorem $KG = \bigoplus_{i=1}^{r} M_{n_i}(D_i)$, the direct sum of full matrix rings over division rings, each one finite dimensional over its center. Let π_i, $1 \le i \le r$, denote the projection of the direct sum onto the ith component. If N is a noncentral subnormal subgroup of $U(KG)$, for some m, $1 \le m \le r$, $\pi_m(N)$ is a noncentral subnormal subgroup of $GL_{n_m}(D_m)$. If $n_m = 1$, by Theorem 2.1 $\pi_m(N)$ contains a free subgroup of rank two. If $n_m > 1$, by Lemma 2.3 $\pi_m(N)$ contains $SL_{n_m}(D_m)$. Now

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

which can be obviously embedded in $SL_{n_m}(D_m)$, generate a free subgroup of rank two, as is well known.

COROLLARY 2.5. *Let K be a field of characteristic O and G be a finite nonabelian group. Then G is not subnormal in $U(KG)$.*

THEOREM 2.5. *Let K be a field of characteristic $p > 0$, G be a finite group and O_p the maximal normal p-subgroup of G. Suppose, moreover, that:*

(i) *K is not algebraic over $GF(p)$ and*
(ii) *G/O_p is nonabelian.*

Then a subnormal subgroup of $U(KG)$ is either solvable or contains a free subgroup of rank two.

Proof. Let $J(KG)$ be the Jacobson radical of KG. Since $J(KG)$ is a nilpotent ideal it follows that $N = 1 + J(KG)$ is a nilpotent normal subgroup of $U(KG)$ and the restriction of the canonical epimorphism

$$\Psi : KG \to KG/J(KG)$$

to $U(KG)$ is a group epimorphism

$$\Psi : U(KG) \to U((KG)/J(KG))$$

whose kernel is N.

We observe initially that the semisimple algebra $KG/J(KG)$ is noncommutative, since we have the embedding $G/\mathbb{O}_p \hookrightarrow U((KG)/J(KG))$.

Next we note that a nonsolvable subnormal subgroup H of $U(KG)$ has a noncentral subnormal image in $U(KG/J(KG))$.

Suppose that $\Psi(H)$ is central. In particular $\Psi(H)$ is abelian and therefore H' is contained in N. Hence HN/N is abelian and since $HN/N \cong H/H \cap N$ it follows that H is solvable, a contradiction.

Finally, let λ be an element of K transcendental over $GF(p)$. By [12], Proposition 3.12, some power of the matrices

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{ and } P\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}P^{-1}, \text{ where } P = \begin{bmatrix} 1+\lambda & \lambda \\ -\lambda & 1-\lambda \end{bmatrix},$$

belonging to $SL_2(K)$ freely generate a free subgroup. Now arguing as in the proof of Theorem 2.4 we get our conclusion.

COROLLARY 2.6. *Let K be a field of characteristic $p>0$, not algebraic over $GF(p)$ and G be a nonsolvable finite group. Then G is not subnormal in $U(KG)$.*

3. **The residual nilpotency of the group of units**

LEMMA 3.1. *Let D be a field or a division ring. Then $GL_n(D)$, $n\geq 2$, is not residually nilpotent.*

Proof. If $n=2$ and $D=GF(2)$ or $GF(3)$ the result is immediate. The remaining case follows from Lemma 2.3.

Let $\mathbb{H}$ be the quaternion algebra over the rational field Q, i.e.,

$$\mathbb{H}=\{x_1+x_2i+x_3j+x_4k \mid i^2=j^2=-1,\ ji=-ij=-k,\ x_i\in Q,\ 1\leq i\leq 4\}$$

and let us denote by $\mathbb{H}^*$ its multiplicative group.

LEMMA 3.2. *$\mathbb{H}^*$ is not residually nilpotent.*

Proof. It is enough to observe that $\mathbb{H}^*$ contains

$$E_{24}=\left\{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \frac{\pm 1 \pm i \pm j \pm k}{2}\right\}$$

the binary tetrahedral group with 24 elements which is not nilpotent.

THEOREM 3.3. *Let K be a field of characteristic p and G be a finite group without p-elements (if $p>0$). Then $U(KG)$ is residually nilpotent if and only if $U(KG)$ is nilpotent.*

Proof. Only the necessity deserves a proof.

We will consider two cases:

(i) $p=0$. We can suppose that $Q\subseteq K$ and, by Wedderburn's Theorem

$$QG=\bigoplus_{i=1}^{r} M_{n_i}(D_i)$$

the direct sum of full matrix rings over division rings. Therefore

$$U(QG)=\prod_{i=1}^{r} GL_{n_i}(D_i)$$

and by Lemma 3.1 $n_i=1$ for $1\leq i\leq r$ and

$$QG=\bigoplus_{i=1}^{r} D_i$$

This implies that every idempotent of QG is central and by [8], 2.6, G is a Hamiltonian group.

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As is well known, $G = A \times E \times K_8$, the direct product of an abelian group A of odd order by a 2-elementary abelian group E by the quaternion group K_8 of order 8.

Since $QK_8 \cong Q \oplus Q \oplus Q \oplus Q \oplus \mathbb{H}$ we conclude that $\mathbb{H}^*$ is residually nilpotent, contradicting Lemma 3.2.

(ii) $p > 0$. We can argue as in (i) above and conclude that G is a Hamiltonian group. Since G has no p-elements it follows that $p \neq 2$.

Let $\mathbb{H}_K = \{x_1 + x_2 i + x_3 j + x_4 k \mid i^2 = j^2 = -1,\ ji = -ij = -k,\ x_i \in K,\ 1 \leq i \leq 4\}$ be the quaternion algebra over K. As $\mathbb{H}_K$ contains the nonabelian finite group $\langle i, j\rangle$, $\mathbb{H}_K$ is not a division algebra. Hence

$$\mathbb{H}_K \cong M_2(K)$$

and $U(KG) \supseteq GL_2(K)$, in contradiction with Lemma 3.1.

COROLLARY 3.4. *Let K be a field of characteristic $p > 0$ and G be a finite group. Then $U(KG)$ is residually nilpotent if and only if $U(KG)$ is nilpotent.*

Proof. Suppose $U(KG)$ residually nilpotent.

Since G is finite G is nilpotent and therefore the direct product of its q-Sylow subgroups $S_q(G)$

$$G = S_p(G) \times \prod_{q \neq p} S_q(G)$$

From Theorem 3.3 we conclude that $\prod_{q \neq p} S_q(G)$ is abelian, and by [1] that $U(KG)$ is nilpotent.

THEOREM 3.5. *Let K be a field of characteristic p, let G be a nontorsion nilpotent group and let T be its torsion subgroup. Suppose that T has no p-elements (if $p > 0$) and that every element of T has prime order. Then $U(KG)$ is residually nilpotent if and only if T is central.*

Proof. If T is central in G by [9], Theorem VI 3.6, $U(KG)$ is nilpotent.

Now, let us suppose $U(KG)$ residually nilpotent.

Let $a, b \in T$. Since $\langle a, b\rangle$ is a finite group, by Theorem 3.3 $\langle a, b\rangle$ is abelian.

We claim, first, that every finite subgroup of T is normal in G. Suppose not. Then there exist a finite subgroup H of T and an element $x \in G$ which does not normalize T. Now, arguing as in [3], Lemma 4, we conclude that $U(KG)$ contains $GL_2(K)$, contradicting Lemma 3.1.

But this implies that every idempotent of KT is central in KG. Since this is not the case, by [9], Lemma VI 3.12, $U(KG)$ contains $GL_m(K)$ for some $m > 1$ a contradiction.

Suppose now that T is not central. We may assume that $G = \langle T, x\rangle$, $|T| < \infty$ and x, $|\langle x\rangle| = \infty$, does not centralize T.

By Wedderburn's Theorem

$$KT = \bigoplus_{i=1}^{r} F_i, \text{ a direct sum of fields.}$$

Hence

$$KG = (KT)_\alpha\langle x\rangle = \left(\bigoplus_{i=1}^{r} F_i\right)_\sigma \langle x\rangle = \bigoplus_{i=1}^{r} (F_i)_\sigma\langle x\rangle,$$

where $\alpha \to \sigma(\alpha) = x\alpha x^{-1}$ is the automorphism of F_i induced by conjugation by x and $(F_i)_\sigma\langle x\rangle$ denotes the skew group ring of $\langle x\rangle$ over F_i, with automorphism σ.

Hence, we can assume that $U(KG)$ contains the nonabelian subgroup $H = \langle \theta, x \mid \theta^p = 1, x\theta x^{-1} = \theta^j\rangle$, where p is a rational prime greater than 2 and $(p, j) = 1$.

Finally, we claim that H is not residually nilpotent.

In fact, by [6], Lemma 4.1, a residually nilpotent finitely generated FC group is nilpotent.

Let m be the order of j in $GF(p)^*$. We have that $Z(H)$, the center of H, is $\langle x^m\rangle$ and that $H/Z(H) = \langle \theta, x \mid \theta^p = x^m = 1, x\theta x^{-1} = \theta^j\rangle$ is nilpotent, a contradiction.

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