Canad. Math. Bull. Vol. **17** (1), 1974

EQUATIONAL COMPACTNESS OF G-SETS

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This paper deals with the notion of equational compactness and related concepts in the special case of G-sets for an arbitrary group G. It provides characterizations of pure extensions, pure-essential extensions, and equational compactness in terms of the stability groups of a G-set, proves the general existence of equationally compact hulls, and gives an explicit description of these. Further, it establishes, among other results, that all G-sets are equationally compact iff all subgroups of the group G are finitely generated, that every equationally compact G-set is a retract of a topologically compact one, and that for free groups G with infinite basis there are homogeneous G-sets which are not equationally compact.

Some helpful advice from Evelyn Nelson, especially concerning the proof in Section 4, is gratefully acknowledged.

1. **Background.** Recall the following familiar notions regarding arbitrary (universal) algebras, relative to some fixed equational class: A *free extension* of an algebra A by a set X (of indeterminates) is an extension E of A generated over A by X such that every map from X into any extension B of A lifts to a (necessarily unique) homomorphism $E\to B$ over A, i.e. mapping A identically. One readily sees that such extensions exist for any given A and X (disjoint from A), and in the following, $A[X]$ will always be a free extension of A by the set X. For a pair (u, v) in $A[X]^2$, a *solution of the equation* $u=v$ *in an extension* B *of* A is a homomorphism $h: A[X]\to B$ over A such that $h(u)=h(v)$.

An extension B of A is called *pure* iff for any finite set $\Sigma\subseteq A[X]^2$ the equations $u=v$, for $(u, v)\in\Sigma$, have a common solution in A whenever they have a common solution in B. A *pure embedding* of A into B is a monomorphism $A\to B$ such that B is a pure extension of the image of A. An algebra A is called *equationally compact* iff for any set $\Sigma\subseteq A[X]^2$, the equations $u=v$, for $(u, v)\in\Sigma$, have a common solution in A whenever this is the case for every finite subset of Σ.

Concerning these concepts one has the following basic results for finitary algebras:

(1) An algebra A is equationally compact iff it is a *retract* of every pure extension B of A, i.e. there exists a homomorphism $B\to A$ over A.

(2) If an algebra A has an equationally compact pure extension then it also has an essentially unique *equationally compact hull*, i.e. an equationally compact pure extension $C\supseteq A$ which is pure-essential in the sense that the identity congruence on

Received by the editors March 10, 1972.

C is the only congruence Θ on C for which the natural map $A \to C/\Theta$ is a pure embedding.

The first of these assertions is due to Weglorz [4], and the second can be obtained by a slight modification of the discussion of relative injectivity outlined in Banaschewski [1], based on the further fact that equational compactness is the same as injectivity relative to pure embeddings.

For any equational class **A** one has, then, the obvious question whether every algebra in **A** has an equationally compact pure extension. This is the case, for instance, in the class of all modules over a ring (Warfield [3]), and will among other things be shown here for the class G**Ens** of all G-sets for any group G.

2. **Pure extensions.** In the following, all algebras $A, B, C, \ldots$ are G-sets, i.e. of the form $(X, (f_s)_{s\in G})$ where the unary operations $f_s: X \to X$ satisfy the conditions $f_{st} = f_s f_t$ and $f_1 = 1_x$ for all $s, t \in G$ and the unit 1 of G. As usual, we let $sx = f_s(x)$.

Note that for any A and the set X the free extension of A by X is $A \amalg \coprod Gx (x \in X)$ where coproducts are merely disjoint unions, and each orbit Gx is a free G-set with x as single basis element, i.e. $Gx \simeq G$, the latter taken as G-set by means of multiplication on the left. To describe the pure extensions of A, the only pairs in $A[X]$ relevant are those of the forms

$$(w, a),\ (sx, y),\ (sz, z)$$

where $w, x, y, z \in X$, $a \in A$, and $s \in G$. Call two finite sets Σ and Σ' of such pairs equivalent iff, for any extension B of A, the equations $u = v$, for $(u, v) \in \Sigma$, have a common solution in B whenever the equations given by Σ' do, and conversely. Now, if Σ contains a pair (s_0x_0, y_0) then the set Σ' obtained from Σ by omitting (s_0x_0, y_0) and first replacing y_0 wherever it occurs in any of the remaining pairs by s_0x_0, and then multiplying, where necessary, by s_0^{-1} (e.g. transition from (sx, s_0x_0) to $(s_0^{-1}sx, x_0)$) to arrive at pairs of the above form, is evidently equivalent to Σ: y_0 no longer occurs in Σ', and given any common solution h of the equations determined by Σ' this can be modified to map y_0 to $s_0(x_0)$, which produces a common solution for the equations given by Σ. Now, as Σ' contains one element less of the form (sx, y) than Σ, it follows that Σ is equivalent to a set of pairs of the form (w, a) and (sz, z), and for the consideration of pure extensions one only has to take sets of the latter type into account.

LEMMA 1. *An extension B of A is pure iff any $s_1, \ldots, s_n \in G$ with common fixed point in B also have a common fixed point in A.*

Proof. The "only if" part is obvious from the definition of purity. For the converse, let the finitely many equations

$$w_i = a_i \quad \text{and} \quad s_jz_j = z_j$$

have a common solution in B. For this, the image of w_i must of course be a_i which already belongs to A; also any z_j is mapped to a common fixed point of some of the

$s_1, s_2, \ldots$, and by mapping z_j to a common fixed point in A of these s_k one obtains a common solution, of the given equations, in A.

It is useful to reformulate this criterion for pure extensions as follows: For any element x of a G-set A, let $S(x)=\{t \mid t \in G, tx=x\}$ be its *stability group*, and let $\mathfrak{S}A$ be the set of all subgroups of all $S(x)$, $x \in A$. Also, for any subgroup H of G, let $\mathfrak{F}H$ be the set of all finitely generated subgroups of H. Now, if $s_1, \ldots, s_n \in G$ have a common fixed point $b \in B$ then $s_1, \ldots, s_n \in S(b)$ and conversely; it follows from this that B is a pure extension of A iff $\mathfrak{F}S(b) \subseteq \mathfrak{S}A$ for all $b \in B$. Finally, let $\mathfrak{T}A$ be the set of all subgroups H of G such that $\mathfrak{F}H \subseteq \mathfrak{S}A$; then one has that *$B$ is a pure extension of A iff $S(b) \in \mathfrak{T}A$ for all $b \in B$.*

Next, we derive a criterion for pure-essential extensions. First, a pure extension B of A is evidently pure-essential iff for any pure extension C of A, any onto homomorphism $f: B \rightarrow C$ over A is an isomorphism.

LEMMA 2. *A pure extension B of A is pure-essential iff* (1) *no two distinct orbits in $B \backslash A$ are isomorphic*, (2) *no $S(b)$, $b \in B \backslash A$, belongs to $\mathfrak{S}A$, and* (3) *each $S(b)$, $b \in B \backslash A$, is maximal in $\mathfrak{T}A$.*

Proof. Let B be pure-essential. Re(1), suppose there are distinct isomorphic orbits Gx and Gy in $B \backslash A$; then let C be the extension of A obtained from B by removing Gy and map $B \rightarrow C$ by mapping all orbits distinct from Gy identically and Gy to Gx by an isomorphism. C is still a pure extension of A, and this homomorphism thus violates the pure-essentialness of B. Similarly, re(2), if $S(b) \in \mathfrak{S}A$, say $S(b) \subseteq S(a)$ for $a \in A$, where $b \in B \backslash A$ then Gb, which is isomorphic to $G/S(b)$, can be mapped homomorphically to $G/S(a) \cong Ga$ via the natural map from $S(b)$-cosets to $S(a)$-cosets, and this produces a homomorphism $B \rightarrow C$ of the excluded kind, C obtained from B by removing Gb. Finally, re(3), if there exists a $b \in B \backslash A$ for which one has an $H \supset S(b)$ in $\mathfrak{T}A$ then one can take C, obtained from B by replacing Gb by G/H, and a homomorphism $B \rightarrow C$ which maps all orbits other than Gb identically, and Gb onto G/H via the natural map $G/S(b) \rightarrow G/H$. Since $H \in \mathfrak{T}A$, C is still a pure extension of A, and thus, again, the pure-essentialness of B is contradicted.

Conversely, if the given three conditions are satisfied, and one has a homomorphism $f: B \rightarrow C$ of the crucial type then, by (2), no orbit of $B \backslash A$ is mapped into A. Also, by (3), every orbit of $B \backslash A$ is mapped one-one, for if Gb is not mapped one-one then, since $S(b) \subseteq S(f(b))$, one has $S(b) \subset S(f(b))$, hence $S(f(b)) \notin \mathfrak{T}A$,and this contradicts the purity of C. Finally, by (1), no two distinct orbits of $B \backslash A$ are mapped to the same orbit of C, and in all this shows f is an isomorphism.

3. **Equational Compactness.** It is clear that a G-set A is a retract of an extension B of A iff every orbit of $B \backslash A$ can be mapped homomorphically into A, or equivalently, iff $S(b) \in \mathfrak{S}A$ for all $b \in B \backslash A$.

Proposition 1. *A G-set A is equationally compact iff $\mathfrak{T}A=\mathfrak{S}A$, i.e. any subgroup H of G for which every finitely generated subgroup is contained in some stability group of A is itself contained in a stability group of A.*

Proof. If A is equationally compact then, for any $H\in\mathfrak{T}A$, $A\amalg G/H$ is a pure extension of A, and hence A is a retract of it, i.e. $H\in\mathfrak{S}A$. This shows $\mathfrak{T}A\subseteq\mathfrak{S}A$, and the other inclusion is obvious. Conversely, if $\mathfrak{T}A=\mathfrak{S}A$ and B is a pure extension then, for each $b\in B\backslash A$, $S(b)\in\mathfrak{T}A$ by purity and thus $S(b)\in\mathfrak{S}A$, which shows, as remarked, that A is a retract of B. The characterization of equational compactness mentioned in Section 1 then completes the proof.

Corollary 1. *Every free G-set is equationally compact.*

Proof. If A is free then $S(a)=\{1\}$ for each $a\in A$, hence A consists just of $\{1\}$, and then the same holds for $\mathfrak{T}A$.

Corollary 2. *For abelian G, all G/H are equationally compact.*

Proof. For $A=G/H$, $\mathfrak{S}A$ is just the set of all subgroups of H, and if all finitely generated subgroups of some subgroup S of G belong to $\mathfrak{S}A$ S is obviously itself a subgroup of H.

Another immediate consequence of Proposition 1 is that every G-set is equationally compact whenever all subgroups of G are finitely generated, for in that case $\mathfrak{T}A$ is evidently equal to $\mathfrak{S}A$, no matter what A is. Actually, one has:

Proposition 2. *All G-sets are equationally compact iff all subgroups of G are finitely generated.*

Proof. We merely have to prove the "only if" part; consider then, a group G such that all G-sets are equationally compact, and assume G has subgroups which are not finitely generated. Since the union of a chain of subgroups of G, each of which is not finitely generated, is itself not finitely generated, let H be a subgroup of G which is not finitely generated and *maximal such*. For the G-set $A=\amalg G/S(S\in\mathfrak{F}H)$, one then has $H\in\mathfrak{T}A=\mathfrak{S}A$, and hence there exist $S\in\mathfrak{F}H$ and $a\in G$ such that $H\subseteq aSa^{-1}$. Now, by the choice of H, one first has the strict inclusions $S\subset H\subset aSa^{-1}\subset aHa^{-1}$, hence aHa^{-1} is finitely generated, and thus H is finitely generated, a contradiction.

Remark. The above argument also lends itself to showing that a union of an up-directed set of equationally compact sub-G-sets need not be equationally compact: Take G above to be abelian; then, for each $S\in\mathfrak{F}H$, $A_S=\amalg G/T(S\supseteq T\in\mathfrak{F}H)$ is equationally compact, and these A_S form an up-directed set of sub-G-sets of A with $A=A_S$, but A is not equationally compact.

Proposition 3. *Every G-set has an equationally compact hull.*

Proof. For any A, consider $B=A\amalg\coprod G/H(H\in\mathfrak{T}A)$. This is, by the reformulation of Lemma 1, a pure extension of A: Every stability group of G/H is conjugate to

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H and $\mathfrak{T}A$ is stable under conjugation since $\mathfrak{S}A$ is. Moreover, $\mathfrak{S}B=\mathfrak{T}A$ since any subgroup of an $H\in\mathfrak{T}A$ agains belongs to $\mathfrak{T}A$ by the definition of $\mathfrak{T}A$. Also, if $\mathfrak{F}H\subseteq\mathfrak{T}A$ for some subgroup H of G then $H\in\mathfrak{T}A$ since $S\in\mathfrak{T}A$ implies $S\in\mathfrak{S}A$ for finitely generated S so that $\mathfrak{F}H\subseteq\mathfrak{T}A$ implies $\mathfrak{F}H\subseteq\mathfrak{S}A$. It follows that $\mathfrak{T}B=\mathfrak{T}A=\mathfrak{S}B$, and thus B is equationally compact. This provides A with an equationally compact pure extension, and the general theory referred to in Section 1 provides the rest.

Actually, though, since we have an explicit characterization of pure-essential extensions, an explicit description of equationally compact hulls can readily be given as follows: Let $\mathfrak{M}$ be any subset of $\mathfrak{T}A$ with the following properties: (1) Each $H\in\mathfrak{M}$ is maximal in $\mathfrak{T}A$, (2) no $H\in\mathfrak{M}$ belongs to $\mathfrak{S}A$, (3) every $S\in\mathfrak{T}A$ maximal in $\mathfrak{T}A$ and not in $\mathfrak{S}A$ is conjugate to exactly one $H\in\mathfrak{M}$. With this one now has:

PROPOSITION 4. *$C=A\amalg\coprod G/H(H\in\mathfrak{M})$ is an equationally compact hull of A.*

Proof. Since $\mathfrak{M}\subseteq\mathfrak{T}A$ C is a pure extension of A, and by Lemma 2 it is a pure-essential extension in view of the properties of $\mathfrak{M}$. Also, $\mathfrak{S}C=\mathfrak{T}A$, for clearly $\mathfrak{S}A\subseteq\mathfrak{S}C\subseteq\mathfrak{T}A$, and if $S\in\mathfrak{T}A$ does not belong to $\mathfrak{S}A$ then it has a conjugate contained in some $H\in\mathfrak{M}$ since $\mathfrak{T}A$ is inductive (partially ordered by inclusion). From the proof of Proposition 3 it now follows that $\mathfrak{T}C=\mathfrak{S}C$, i.e. C is equationally compact.

Given any G-set A, let βA be the G-set whose elements are the ultrafilters on the set underlying A and whose G-action is induced by the action in A. Since the ultrafilters have a natural topology with which they constitute a compact Hausdorff space, and since the $s\in G$ provide continuous maps on this, βA is the underlying G-set of a compact Hausdorff G-space.

The natural map from a set to the set of its ultrafilters which assigns to each point x the fixed ultrafilter at x evidently provides an embedding of G-sets, $\eta:A\rightarrow\beta A$. For this, the following holds:

PROPOSITION 5. *The embedding η is pure, and hence a G-set A is equationally compact iff it is a retract, by η, of βA.*

Proof. We use the criterion for purity given in Lemma 1. For any $s\in G$, we use a device from Wenzel [6] to split the underlying set X of A as follows: If the (s)-orbit 0 of an element $a\in A$, for the subgroup (s) of G generated by s, is infinite let

$$0_1=\{s^{2k}a \mid k\in N\},\qquad 0_2=\{s^{2k+1}a \mid k\in N\},\qquad 0_3=\varnothing;$$

for finite 0 of even cardinality $2m$, let

$$0_1=\{s^{2k}a \mid k=0,\ldots,m-1\},\qquad 0_2=\{s^{2k+1}a \mid k=0,\ldots,m-1\},$$

$$0_3=\varnothing;$$

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finally, for finite 0 of odd cardinality $2m+1>1$, let

$$0_1 = \{s^{2k}a \mid k = 0, \dots, m-1\}, \qquad 0_2 = \{s^{2k+1}a \mid k = 0, \dots, m-1\},$$
$$0_3 = \{2^{2m}a\}.$$

Now take the decomposition $X=X_0 \cup X_1 \cup X_2 \cup X_3$ where X_0 is the set of fixed points of s, and for the (s)-orbits 0 of more than one element X_i is the union of all 0_i. It is then clear that $X_i \cap sX_i=\varnothing$ for $i=1, 2, 3$; hence, for any ultrafilter $\mathfrak{A}$ with $s\mathfrak{A}=\mathfrak{A}$ one has $X_i \notin \mathfrak{A}$ for $i=1, 2, 3$ and thus $X_0 \in \mathfrak{A}$ by the basic properties of ultrafilters. It follows immediately from this that any $s_1, \dots, s_n \in G$ for which $s_i\mathfrak{A}=\mathfrak{A}$ for some ultrafilter $\mathfrak{A}$ have a common fixed point in A. The remainder of the proposition is now an obvious consequence of what has already been stated.

4. **A counterexample.** The second corollary of Proposition 1 states that all homogeneous G-sets are equationally compact for an *abelian* group G. We shall now show that this does not hold for groups in general. To this end, note that a G-set G/H, H a subgroup of G, is equationally compact iff a subgroup K of G belongs to some conjugate sHs^{-1} of H whenever each finite subset belongs to some conjugate of H.

The desired counterexample is then obtained as follows: Let X be any infinite set, and Z a set disjoint from X with a fixed one-one correspondence $F \rightarrowtail z_F$, $z \rightsquigarrow F_z$ between Z and the collection of all finite subsets F of X. Let G be the free group with basis $X \cup Z$, H the subgroup of G generated by the set $T=\bigcup z_F F z_F^{-1}$ (finite $F\subseteq X$), and K the subgroup of G generated by X. Clearly, $F\subseteq z_F^{-1}Hz_F$ for each finite $F\subseteq X$, and therefore every finite subset of K also lies in some conjugate of H. We have to show that there is no $s \in G$ for which $X\subseteq sHs^{-1}$.

Suppose there is such an s; then take any $x \in X$ and let

$$(*) \qquad x = sz_1x_1^{\varepsilon_1}z_1^{-1} \cdots z_nx_n^{\varepsilon_n}z_n^{-1}s^{-1} \qquad (z_ix_iz_i^{-1} \in T, \varepsilon_i = \pm 1)$$

where the product is chosen, among all such products for x, with the least possible n. Note that $x_i=x$ for some i, since otherwise one could substitute all x_i by 1 and would obtain $x=1$, a contradiction. We want to show, moreover, that there is *only one* such i. Let $i_1<i_2<\cdots<i_m$ be the indices i for which $x_i=x$, and write the product in (*) as $a_0x^{\varepsilon_{i_1}}a_1x^{\varepsilon_{i_2}}\cdots a_{m-1}x^{\varepsilon_{i_m}}a_m$. Now, if $m>1$ there must be some $k \in \{1, \dots, m-1\}$ for which $\varepsilon_{i_k}=-\varepsilon_{i_{k+1}}$ and $a_k=1$. To see this we argue via the reduced factorizations of elements in G with respect to the basis $X \cup Y$, assuming no such k exists: the reduced factorization of $a_0x^{\varepsilon_{i_1}}a_1$ clearly has one factor $x^{\pm 1}$; assume, then, that the reduced factorization of $c=a_0x^{\varepsilon_{i_1}}a_1x^{\varepsilon_{i_2}}\cdots a_{t-1}x^{\varepsilon_{i_t}}a_t$, $1\leq t<m-1$, has factors $x^{\pm 1}$ and consider $cx^{\varepsilon_{i_{t+1}}}a_{t+1}$. Here, if $a_t=1$ then $\varepsilon_{i_t}=\varepsilon_{i_{t+1}}$ by assumption, and the reduced factorization of c ends with the factor $x^{\varepsilon_{i_t}}$ so that no cancellation occurs in the product $cx^{\varepsilon_{i_{t+1}}}a_{t+1}$; on the other hand, for $a_t\neq 1$ it is clear there is no such cancellation, and it follows that $a_0x^{\varepsilon_{i_1}}\cdots a_tx^{\varepsilon_{i_{t+1}}}a_{t+1}$ has

$t+1$ factors $x^{\pm 1}$ in its reduced factorization. This proves this assertion for all $t=1, \ldots, m-1$, a contradiction for $t=m-1$.

We now have, for some k,

$$z_i^{-1} z_{i+1} x_{i+1}^{\varepsilon_{i+1}} z_{i+1}^{-1} \cdots z_h x_h^{\varepsilon_h} z_h^{-1} z_{h+1} = 1$$

where $i=i_k$, $h=i_{k+1}-1$, and $\varepsilon_{h+1}=-\varepsilon_i$. Here, if $h=i$ then $z_i=z_{i+1}$, and hence $x_{i+1} \in Z \cup Z^{-1}$, a contradiction; similarly, if $h>i$ then for one of the occurring indices r one has $z_r=z_{r+1}$, $x_r=x_{r+1}$, $\varepsilon_r=-\varepsilon_{r+1}$ so that $z_r x_r^{\varepsilon_r} z_r^{-1} z_{r+1} x_{r+1}^{\varepsilon_{r+1}} z_{r+1}^{-1}=1$ which reduces the number n in (*), contrary to the choice of that product.

This establishes there is only one i such that $x_i=x$, and hence $x=a_0xa_1$. It follows that $a_0=1$, and this shows that $1=suz_i$ for some $u \in H$, so that $sH=z_i^{-1}H$ and hence $sHs^{-1}=z_i^{-1}Hz_i$.

Thus we have obtained that $X \subseteq z^{-1}Hz$ for some $z \in Z$. Now, take $x \in X$ such that $x \notin F_z$, and let $x=z^{-1}z_1x_1^{\varepsilon_1}z_1^{-1} \cdots z_nx_n^{\varepsilon_n}z_n^{-1}z$ $(z_ix_i^{\varepsilon_i}z_i^{-1} \in T,\ \varepsilon_i=\pm 1)$ again be a representation of x as element of $z^{-1}Hz$ with least possible n. In this, $x_i=x$ for all i, for otherwise the product can be shortened by substituting 1 for an x_i, but then, from the argument above, $n=1$ so that $x=z^{-1}z_1xz_1^{-1}a$. Finally, $z_1xz_1^{-1} \in T$ shows that $x \in F_{z_1}$; hence we have $z \neq z_1$ since $x \notin F_z$, but this clearly implies $z^{-1}z_1xz_1^{-1}z \neq x$, a contradiction.

In all this proves:

Proposition 6. *There exists a group G with a subgroup H such that the G-set G/H is not equationally compact.*

5. **Remarks on M-sets.** We conclude with a couple of observations regarding M-sets for *monoids M*. To begin with, equationally compact hulls do not generally exist even though every M-set has equationally compact extensions, the latter being, for instance, a consequence of the fact that M-sets have enough injectives (Berthiaume [2]), but of course also readily obtained with the aid of ultrafilter spaces. A monoid M with an M-set A which does not have an equationally compact hull is the free monoid with two basis elements x and y, A having the set N of natural numbers as its underlying set, with the action of x and y given such that $k \rightsquigarrow (xk, yk)$ maps N one-one onto $N^2-\{(k,k) \mid k \in N\}$. (Weglorz–Wojciechowska [5]).

Another fact which does not carry over from G-sets to M-sets is Corollary 1 of Proposition 1; this is an immediate consequence of the following generalization of a result of Weglorz [4].

Proposition 7. *Any left-cancellative monoid M which is equationally compact as left M-set is a group.*

Proof. We first show that $\cap\, s^nM (n \in N)$ is non-void for any $s \in M$. This is obvious if $s^h=1$ for some $h \neq 0$, since s is then invertible, and hence we only have to consider s such that $s^n \neq 1$ for all $n \neq 0$. Let S be the submonoid of M generated by s, and take A to be the S-set whose elements are the integers and whose S-action is defined by $sm=m+1$. For the S-subset $B\subseteq A$ determined by the natural numbers, one then has that any finitely generated S-subset of A can be mapped homomorphically into B, and this readily implies that the canonical embedding $B\to A\amalg B$ is pure. Now, M is also an equationally compact S-set, and hence the homomorphism $f:B\to M$ by $f(k)=s^k$ extends to a homomorphism $g:A\amalg B\to M$. From the effect of g on A, one sees that $s^n g(-n)=g(0)$ for all $n\in N$, and this shows that $g(0)\in \cap s^n M(n\in N)$.

Next, for any $s\in M$, let (s_λ) be a sequence in M, defined on all ordinals by transfinite induction, such that $s_\lambda M\supseteq s_\mu M$ whenever $\lambda\leq\mu$, $s_0=s$, $s_{\lambda+1}\in \cap s_\lambda^n M(n\in M)$, and $s_\lambda\in\cap s_\mu M(\mu<\lambda)$ for any limit ordinal λ. That the latter intersection is indeed always non-void results from equational compactness and the fact that the $s_\mu M$, $\mu<\lambda$, are totally ordered by inclusion: Any finite subset of the set of equations $s_\mu x_\mu=x$, $\mu<\lambda$, has a solution, provided by any element of $s_\mu M$ for the largest occurring μ, hence the whole set has a solution. It now follows that $s_\lambda=s_\mu$ for some $\lambda>\mu$. This implies that $s_\lambda\in s_{\mu+1}M=\cap s_\mu^n M(n\in N)$, hence $s_\lambda\in s_\mu^2 M=s_\lambda^2 M$ so that $1\in s_\lambda M$, thus also $1\in sM$, i.e. s has a right inverse. Since M is left-cancellative, this shows M is a group.

Since an equationally compact monoid M is also equationally compact as left M-set, it follows from Proposition 5 that any left-cancellative equationally compact monoid is a group, which was proved in Weglorz [4].

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