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A MAXIMUM PRINCIPLE FOR BOUNDED HARMONIC FUNCTIONS ON RIEMANNIAN SPACES

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Harmonic functions with certain boundedness properties on a given open Riemann surface R attain their maxima and minima on the harmonic boundary Δ_R of R. The significance of such maximum principles lies in the fact that the classification theory of Riemann surfaces related to harmonic functions reduces to a study of topological properties of Δ_R (cf. [**11**; **8**; **3**; **12**].

For the corresponding problem in higher dimensions we shall first show that the complement of Δ_R with respect to the Royden boundary Γ_R of a Riemannian N-space R is harmonically negligible: given any non-empty compact subset E of $\Gamma_R - \Delta_R$ there exists an Evans superharmonic function v, i.e., a positive continuous function on $R^* = R \cup \Gamma_R$, superharmonic on R, with $v \equiv 0$ on Δ_R, $v \equiv \infty$ on E, and with a finite Dirichlet integral over R. As a consequence we then establish the following maximum principle: every bounded harmonic function on a subregion G of R attains its supremum and infimum on the set $(\bar{G} \cap \Delta_R) \cup \partial G$, where ∂G is the relative boundary of G, and $\bar{G}$ stands for the closure of G in Royden's compactification of R. For similar consequences of the existence of v to Dirichlet-finite harmonic functions on Riemannian spaces we refer the reader to [**7**].

The difficulty in the existence proof of an Evans superharmonic function lies in that, in contrast with the case of dimension 2, we cannot form a "double" of a subregion of a Riemannian space and the maximum principle for Dirichlet-finite harmonic functions is *a priori* not at our disposal.

We start by recalling in § 1 fundamentals of the function spaces $\tilde{\mathbf{M}}(R)$, $\mathbf{M}(R)$, $\tilde{\mathbf{M}}_\Delta(R)$, and $\mathbf{M}_\Delta(R)$. In § 2 we prove the duality theorems of $\tilde{\mathbf{M}}_\Delta(R)$ and $\mathbf{M}_\Delta(R)$. The existence of an Evans superharmonic function, and the maximum principle for bounded harmonic functions are established in § 3.

1. Let R be a Riemannian N-space, i.e., a separable, connected, orientable, C^∞ N-manifold with a positive definite metric tensor (g_{ij}). We designate by (g^{ij}) the inverse matrix and by g the determinant of (g_{ij}).

We consider the vector lattice $\tilde{\mathbf{M}}(R)$ which consists of all continuous real-valued functions f on R with finite Dirichlet integrals

$$D_R(f) = \int_R g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} dV,$$

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where $dV = \sqrt{g}dx^1 \wedge \ldots \wedge dx^N$ and partial derivatives of f with respect to the coordinate variables are assumed to exist almost everywhere on each parametric neighbourhood in R. The subclass $\mathbf{M}(R)$ of bounded functions in $\tilde{\mathbf{M}}(R)$ is, by definition, Royden's algebra of R.

By means of $\mathbf{M}(R)$ we can construct Royden's compactification R^* of R, defined by the following properties:

(i) R^* is a compact Hausdorff space,

(ii) R is an open dense subspace of R^*,

(iii) every function in $\mathbf{M}(R)$ has a continuous extension to R^*,

(iv) $\mathbf{M}(R)$ separates points in R^*.

In fact, $\mathbf{M}(R)$ separates closed sets in R^* and every function in $\tilde{\mathbf{M}}(R)$ has a continuous extension to R^*. The set $\Gamma_R = R^* - R$ is called the Royden boundary of R.

We shall use two modes of convergence in $\tilde{\mathbf{M}}(R)$:

(i) $f = \text{CD-lim}_n f_n$ on R if $\{f_n\}$ converges to f uniformly on compact subsets of R, and $D_R(f_n - f) \to 0$ as $n \to \infty$,

(ii) $f = \text{BD-lim}_n f_n$ on R if $f = \text{CD-lim}_n f_n$ on R and $\{f_n\}$ is uniformly bounded on R.

It is known that $\mathbf{M}(R)$ is BD-complete and $\tilde{\mathbf{M}}(R)$ is CD-complete. The class $\mathbf{M}_0(R)$ of functions in $\mathbf{M}(R)$ with compact supports in R forms an ideal of $\mathbf{M}(R)$. We denote its BD-closure in $\mathbf{M}(R)$ and CD-closure in $\tilde{\mathbf{M}}(R)$ by $\mathbf{M}_\Delta(R)$ and $\tilde{\mathbf{M}}_\Delta(R)$, respectively.

Let $\mathcal{O}_G$ be the class of Riemannian spaces on which there exist no Green's functions. It is well known that if $R \in \mathcal{O}_G$, then the class HD(R) of Dirichlet-finite harmonic functions consists of constants [**13**]. In this case it is understood that HD$(R) = \{0\}$.

For a detailed discussion of topics in § 1 we refer the reader to [**1**; **9**; **10**; **12**].

2. First we establish a topological property of $\mathbf{M}_\Delta(R)$.

PROPOSITION 1. *The potential subalgebra* $\mathbf{M}_\Delta(R)$ *is complete in the* BD-*topology.*

Proof. Let $\{f_n\}_1^\infty$ be a BD-Cauchy sequence in $\mathbf{M}_\Delta(R)$. Since $\mathbf{M}(R)$ is complete in the BD-topology, there exists a function $f \in \mathbf{M}(R)$ such that $f = \text{BD-lim}_n f_n$ on R. By the definition of $\mathbf{M}_\Delta(R)$, there exist sequences $\{f_{nm}\}_{m=1}^\infty$ in $\mathbf{M}_0(R)$ such that $f_n = \text{BD-lim}_m f_{nm}$ on R, $n = 1, 2, \ldots$. Taking subsequences if necessary we may assume that

$$D_R(f_n - f_{nm}) < 1/n^2$$

for all $m \geqq 1$ and each n.

Let $\{R_n\}_1^\infty$ be a regular exhaustion of R. Since $\lim_m \sup_{R_n}|f_n - f_{nm}| = 0$ we may again assume that

$$\sup_{R_n} |f_n - f_{nm}| < 1/n$$

for all m and n.

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Now it suffices to show that $f = \text{BD-lim}_n f_{nn}$ on R since $f_{nn} \in \mathbf{M}_0(R)$. Clearly, $\{f_{nn}\}$ is uniformly bounded on R. Since

$$D_R(f - f_{nn})^{\frac{1}{2}} \leqq D_R(f - f_n)^{\frac{1}{2}} + D_R(f_n - f_{nn})^{\frac{1}{2}}$$
$$< D_R(f - f_n)^{\frac{1}{2}} + 1/n,$$

we conclude that $f = \text{D-lim}_n f_{nn}$ on R. To prove the uniform convergence of $\{f_{nn}\}$ to f on compact sets, take a compact subset F of R. Choose n so large that $F \subset R_n$. Then for $m \geqq n$,

$$\sup_F |f - f_{mm}| \leqq \sup_{R_n} |f - f_{mm}|$$
$$\leqq \sup_{R_n} |f - f_m| + \sup_{R_n} |f_m - f_{mm}|$$
$$< \sup_{R_n} |f - f_m| + \frac{1}{m}$$

and $f = \text{C-lim}_n f_{nn}$ on R. This completes the proof.

In view of the CD-completeness of $\tilde{\mathbf{M}}(R)$, we can similarly prove the following result.

PROPOSITION 2. *The sublattice* $\tilde{\mathbf{M}}_\Delta(R)$ *of the vector lattice* $\tilde{\mathbf{M}}(R)$ *is complete in the* CD-*topology.*

The harmonic boundary Δ_R of R is, by definition, a compact subset of the Royden boundary $\Gamma_R = R^* - R$ and consists of the common zeros of all functions in $\mathbf{M}_\Delta(R)$. Conversely, the potential subalgebra $\mathbf{M}_\Delta(R)$ of $\mathbf{M}(R)$ and the sublattice $\tilde{\mathbf{M}}_\Delta(R)$ of $\tilde{\mathbf{M}}(R)$ can be characterized in terms of Δ_R (cf. [**12**]).

PROPOSITION 3. *The following duality relations are valid:*
(a) $\mathbf{M}_\Delta(R) = \{f \in \mathbf{M}(R) \mid f \equiv 0$ *on* $\Delta_R\}$,
(b) $\tilde{\mathbf{M}}_\Delta(R) = \{f \in \tilde{\mathbf{M}}(R) \mid f \equiv 0$ *on* $\Delta_R\}$.

For the proof we first show the following.

LEMMA. *If* $R \notin \mathcal{O}_G$, *then every* $f \in \mathbf{M}(R)$ *can be decomposed uniquely in the form* $f = u + g$ *on* R *for some* $u \in \text{HBD}(R) = \{u \in \text{HD}(R) \mid \sup_R |u| < \infty\}$ *and* $g \in \mathbf{M}_\Delta(R)$. *In particular,* $u \geqq 0$ *on* R *whenever* $f \geqq 0$ *on* R *and*

$$D_R(f) = D_R(u) + D_R(g).$$

Proof. Let $\{R_n\}_1^\infty$ be a regular exhaustion of R. Consider the sequence $\{u_n\}_1^\infty$ of continuous functions u_n on R such that

$$u_n \in \text{H}(R_n) \quad \text{and} \quad u_n \equiv f \qquad \text{on } R - R_n$$

for each $n \geqq 1$. Then $u_n \in \mathbf{M}(R)$ by Dirichlet's principle. Hence by Green's formula,

$$D_R(u_{n+p} - u_n, u_{n+p}) = D_{R_{n+p}}(u_{n+p} - u_n, u_{n+p})$$
$$= \int_{\partial R_{n+p}} (u_{n+p} - u_n) *du_{n+p} = 0.$$

Thus $D_R(u_{n+p} - u_n) = D_R(u_n) - D_R(u_{n+p})$ and $\{u_n\}_1^\infty$ is D-Cauchy on R. In view of $\sup_R |u_n| \leqq \sup_R |f| < \infty$ for all n we may assume that $\{u_n\}_1^\infty$ is BD-Cauchy on R. Let $u = \text{BD-lim}_n u_n$ on R. Clearly $u \in \text{HBD}(R)$ (cf. [**13**]).

Set $g_n = f - u_n$ on R. Since $\{g_n\}_1^\infty$ is a BD-Cauchy sequence on R and $g_n \in \mathbf{M}_0(R)$, $g = \text{BD-lim}_n g_n$ exists on R and belongs to $\mathbf{M}_\Delta(R)$. Thus we obtain a decomposition $f = u + g$ on R with $u \in \text{HBD}(R)$ and $g \in \mathbf{M}_\Delta(R)$. Since g_n has a compact support in R, Green's formula yields $D_R(g_n, u) = 0$ and $D_R(u + g_n) = D_R(u) + D_R(g_n)$. On letting $n \to \infty$ we obtain

$$D_R(f) = D_R(u) + D_R(g).$$

Clearly $u \geqq 0$ on R whenever $f \geqq 0$ on R.

It remains to prove that the decomposition is unique. Suppose that $f = u' + g'$ is another decomposition with the required properties. Then $v \equiv u - u' = g' - g \in \text{HBD}(R) \cap \mathbf{M}_\Delta(R)$. Take $v_m \in \mathbf{M}_0(R)$ such that

$$v = \text{BD-lim}_m v_m$$

on R. Then

$$D_R(v) = \lim_{m\to\infty} D_R(v_m, v) = 0$$

since $v \in \text{HBD}(R)$ and $v_m \in \mathbf{M}_0(R)$. In view of $v \equiv 0$ on Δ_R we conclude that $u \equiv u'$ and $g \equiv g'$ on R, as desired.

Proof of Proposition 3. (a) By the definition of Δ_R, every $f \in \mathbf{M}_\Delta(R)$ vanishes identically on Δ_R. Thus it suffices to show that every $f \in \mathbf{M}(R)$ with $f \equiv 0$ on Δ_R belongs to $\mathbf{M}_\Delta(R)$.

Let $f = u + g$ be the decomposition in the above lemma. We shall prove that $u \equiv 0$ on R. Then $f = g \in \mathbf{M}_\Delta(R)$. Clearly $u = f - g \equiv 0$ on Δ_R. For any $\epsilon > 0$ set

$$E = \{q \in R^* | \, u(q) \geqq \epsilon\}.$$

Then $\Delta_R \cap E = \emptyset$. For each $q \in E$ choose a function $f_q \in \mathbf{M}_\Delta(R)$ such that $f_q(q) > 1$. Then $g_q \equiv f_q \cup 0 \in \mathbf{M}_\Delta(R)$, $g_q(q) > 1$, and $g_q \geqq 0$ on R. Since E is a compact subset of R^*, there exists a finite subset $\{q_1, \ldots, q_n\}$ of E such that

$$E \subset \bigcup_{i=1}^n \{q \in R^* | \, g_{q_i}(q) > 1\}.$$

Therefore $g \equiv \sum_{i=1}^n g_{q_i} \in \mathbf{M}_\Delta(R)$ and $g > 1$ on E. Set $M = \sup_R |u|$. Clearly $u - \epsilon - Mg \in \mathbf{M}(R)$ and $u - \epsilon - Mg < 0$ on R. Since

$$u - \epsilon - Mg = (u - \epsilon) + (-Mg)$$

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is the decomposition of $u - \epsilon - Mg$ in the above lemma, $u - \epsilon \leqq 0$ on R. On letting $\epsilon \to 0$ we obtain $u \leqq 0$ on R. The same argument for $-u$ yields $u \equiv 0$ on R.

(b) Let $f \in \tilde{\mathbf{M}}_\Delta(R)$. There exists a sequence $\{f_n\}_1^\infty$ in $\mathbf{M}_0(R)$ such that

$$f = \text{CD-}\lim_{n\to\infty} f_n$$

on R. Clearly $(f_n \cap \alpha) \cup (-\alpha) \in \mathbf{M}_0(R)$ for $\alpha > 0$ and

$$(f \cap \alpha) \cup (-\alpha) = \text{BD-}\lim_{n\to\infty} (f_n \cap \alpha) \cup (-\alpha)$$

on R. Hence $(f \cap \alpha) \cup (-\alpha)$ is in $\mathbf{M}_\Delta(R)$ and vanishes identically on Δ_R by (a). Since $\alpha > 0$, $f \equiv 0$ on Δ_R.

Conversely, let $f \in \tilde{\mathbf{M}}(R)$ be such that $f \equiv 0$ on Δ_R. For each $n \geqq 1$ set $f_n = (f \cap n) \cup (-n)$ on R. Then $f_n \in \mathbf{M}_\Delta(R)$ by (a) and

$$f = \text{CD-}\lim_{n\to\infty} f_n$$

on R. Since $f_n \in \tilde{\mathbf{M}}_\Delta(R)$ and $\tilde{\mathbf{M}}_\Delta(R)$ is CD-complete, f is in $\tilde{\mathbf{M}}_\Delta(R)$, as desired.

3. We are ready to establish the existence of an Evans superharmonic function. It will in turn be essential for the proof of the maximum principle for bounded harmonic functions.

We claim the following result (cf. [**2**]).

THEOREM 1. *Let E be an arbitrary non-empty compact subset of $\Gamma_R - \Delta_R$. Then there exists a positive continuous superharmonic function v on R such that $v \equiv 0$ on Δ_R, $v \equiv \infty$ on E, and v is a Dirichlet finite Tonelli function on R.*

Proof. Let K be a distinguished compact subset of R^*, i.e., a subset of R^* with $K = (\overline{K \cap R})$ and with a smooth $\partial(K \cap R)$, such that $K \cap \Delta_R = \emptyset$ and K contains E in the interior.

Choose $f \in \mathbf{M}(R)$ with the properties $f \equiv 1$ on K, $f \equiv 0$ on Δ_R, and $0 \leqq f \leqq 1$ on R^*. For a regular exhaustion $\{R_n\}_1^\infty$ of R set $K_n = K - R_n$. Consider the continuous functions u_{nm} on R such that

$$u_{nm} \in \mathrm{H}(R_m - K_n) \quad \text{and} \quad u_{nm} \equiv f \quad \text{on } R - (R_m - K_n)$$

for $m, n \geqq 1$. Since

$$\begin{aligned} D_R(u_{n,m+p} - u_{nm}, u_{n,m+p}) &= D_{R_{m+p}-K_n}(u_{n,m+p} - u_{nm}, u_{n,m+p}) \\ &= \int_{\partial(R_{m+p}-K_n)} (u_{n,m+p} - u_{nm}) *du_{n,m+p} = 0, \end{aligned}$$

$D_R(u_{n,m+p} - u_{nm}) = D_R(u_{nm}) - D_R(u_{n,m+p})$ and the sequences $\{u_{nm}\}_{m=1}^\infty$ are D-Cauchy on R for all $n \geqq 1$. Clearly $0 < \sup_R u_{nm} \leqq \sup_R f = 1$ for all $m, n \geqq 1$. Hence we may assume that the sequences $\{u_{nm}\}_{m=1}^\infty$ are BD-Cauchy on R. Set $u_n = \text{BD-}\lim_m u_{nm}$ on R. Then $u_n \in \mathrm{HBD}(R - K_n)$ and $u_n \equiv f$

on K_n. Since $g_{nm} = f - u_{nm} \in \mathbf{M}_0(R)$ and the sequences $\{g_{nm}\}_{m=1}^\infty$ are BD-Cauchy on R, $g_n = \text{BD-lim}_m\, g_{nm}$ exists on R and $g_n \in \mathbf{M}_\Delta(R)$. By virtue of $u_n = f - g_n$ we have

$$u_n \in \text{HBD}(R - K_n), \qquad u_n \equiv f$$

on $K_n \cup \Delta_R$ and

$$0 \leqq u_n \leqq \sup_R f = 1$$

on R.

On the other hand, $D_R(u_{n+p,m} - u_{nm}) = D_R(u_{nm}) - D_R(u_{n+p,m})$ as before. Thus the sequences $\{u_{nm}\}_{n=1}^\infty$ are D-Cauchy on R for all $m \geqq 1$. In view of $D_R(u_{n+p} - u_n)^{\frac{1}{2}} \leqq D_R(u_{n+p} - u_{n+p,m})^{\frac{1}{2}} + D_R(u_{n+p,m} - u_{nm})^{\frac{1}{2}} + D_R(u_{nm} - u_n)^{\frac{1}{2}}$, we obtain, on letting $m \to \infty$,

$$D_R(u_{n+p} - u_n)^{\frac{1}{2}} \leqq \liminf_m D_R(u_{n+p,m} - u_{nm})^{\frac{1}{2}}.$$

Since $\{u_{nm}\}_{n=1}^\infty$ is D-Cauchy on R for each $m \geqq 1$, it is easily seen from the above inequality that the sequence $\{u_n\}$ is D-Cauchy on R. The family $\{u_n\}$ being uniformly bounded on R we may again assume that $u = \text{BD-lim}_n\, u_n$ exists on R. Clearly $u \in \text{HBD}(R)$. We know that $g_n \in \mathbf{M}_\Delta(R)$. Also $f \in \mathbf{M}_\Delta(R)$ by Proposition 3 (a). Thus $u_n = f - g_n \in \mathbf{M}_\Delta(R)$ and

$$u = \text{BD-lim}_n\, u_n \in \mathbf{M}_\Delta(R)$$

by Proposition 1. Hence $u \in \text{HBD}(R) \cap \mathbf{M}_\Delta(R)$ and $u \equiv 0$ on R by the lemma in § 2. Thus we have a sequence $\{u_n\}_1^\infty$ of positive superharmonic functions on R such that $u_n \in \text{HD}(R - K_n) \cap \mathbf{M}_\Delta(R)$, $u_n \equiv 1$ on K_n, and $\text{BD-lim}_n\, u_n = 0$ on R.

Let $p_0 \in R$ be a fixed point and choose a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ such that

$$u_{n_k}(p_0) \leqq 2^{-k} \qquad \text{and} \qquad D_R(u_{n_k}) \leqq 2^{-k}.$$

Set

$$v_m = \sum_{k=1}^m u_{n_k} \qquad \text{and} \qquad v = \sum_{k=1}^\infty u_{n_k}$$

on R. Since $\{v_m\}$ is D-Cauchy on R and $\{v_m(p_0)\}$ is convergent, we obtain

$$v = \text{CD-lim}_{m\to\infty}\, v_m$$

on R. Here each v_m is a positive continuous function on R with a finite Dirichlet integral and therefore the CD-limit v has the same properties.

On $K_{n_k} (\supset E)$, $v \geqq v_k \equiv k$ for all $k \geqq 1$, and therefore $v \equiv \infty$ on E. The relation $v|_{\Delta_R} \equiv 0$ follows immediately from Propositions 2 and 3 (b) in view of $v_m \in \mathbf{M}_\Delta(R) \subset \tilde{\mathbf{M}}_\Delta(R)$.

It remains to show that v is superharmonic on R. Let p be an arbitrary point of R and V a parametric ball about p in R. There exists a Green's function $q_V(x, p)$ on V with singularity p (cf. [**5**; **4**]). For each $m \geqq 1$,

$$v(p) \geqq v_m(p) \geqq -\int_{\partial V} v_m(x) *dq_V(x, p),$$

the second inequality being implied by the superharmonicity of v_m. On letting $m \to \infty$ we obtain

$$v(p) \geqq -\int_{\partial V} v(x) *dq_V(x, p)$$

by Lebesgue's convergence theorem. This completes the proof.

Every non-constant superharmonic function on a subregion G of R attains its infimum on the boundary $\bar{G} - G$ of G. It is natural to ask whether we can replace the boundary $\bar{G} - G$ of G by a subset such that all superharmonic functions on G assume their infima on that subset. We shall establish the following generalization of a result of Constantinescu [**2**].

THEOREM 2. *Suppose that v is a superharmonic (subharmonic) function on a subregion G of a Riemannian space such that v is bounded from below (above) on G.*

If v has the property

$$\liminf_{p \in G, p \to q} v(p) \geqq m \qquad \left(\limsup_{p \in G, p \to q} v(p) \leqq M\right)$$

for all $q \in (\bar{G} \cap \Delta_R) \cup \partial G$, then $v \geqq m$ $(v \leqq M)$ on G.

Proof. Since v is superharmonic if and only if $-v$ is subharmonic, it suffices to consider the case in which v is superharmonic on G.

We may assume that $m > -\infty$. For $n \geqq 1$ we consider the set

$$E_n = \left\{ q \in \bar{G} - G \,\middle|\, \liminf_{p \in G, p \to q} v(p) \leqq m - \frac{1}{n} \right\}.$$

Clearly E_n is a closed set in R^*, contained in $\Gamma_R - \Delta_R$ by our assumption. If $E_n = \emptyset$ for all $n \geqq 1$, then the assertion follows from the lemma. Since $E_n \subset E_{n+1}$ for each n, it is sufficient to consider the case $E_n \neq \emptyset$ for all n.

By Theorem 1 for $E = E_n$ there exists a positive continuous superharmonic function v_n on R such that $v_n \equiv 0$ on Δ_R, $v_n \equiv \infty$ on E_n, and v_n has a finite Dirichlet integral over R. For an arbitrary $\epsilon > 0$, $v + \epsilon v_n$ is a superharmonic function on G and bounded from below. Since

$$\liminf_{p \in G, p \to q} (v + \epsilon v_n)(p) \geqq \liminf_{p \in G, p \to q} v(p) + \liminf_{p \in G, p \to q} \epsilon v_n(p) > m - \frac{1}{n}$$

for all $q \in \bar{G} - G$ and $n \geqq 1$, we have

$$v + \epsilon v_n > m - 1/n$$

on G. On letting $\epsilon \to 0$ and then $n \to \infty$ we obtain the desired result.

Observe that u is harmonic on G if and only if u is simultaneously superharmonic and subharmonic on G. Thus as a direct consequence of the above theorem we have the following result (cf. [**8**]).

COROLLARY 1. *Let G be a subregion of a Riemannian space R and u a bounded harmonic function on G such that*

$$m \leqq \liminf_{p \in G, p \to q} u(p) \leqq \limsup_{p \in G, p \to q} u(p) \leqq M$$

for all $q \in (\bar{G} \cap \Delta_R) \cup \partial G$. Then $m \leqq u \leqq M$ on G.

References

1. J. Chang, *Royden's compactification of Riemannian spaces*, Doctoral dissertation, University of California, Los Angeles, 1968.

2. C. Constantinescu, *Dirichletsche Abbildungen*, Nagoya Math. J. *20* (1962), 75–89.

3. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Band 32 (Springer-Verlag, Berlin, 1963).

4. G. F. D. Duff, *Partial differential equations* (Univ. Toronto Press, Toronto, Ontario, 1956).

5. W. Feller, *Über die Lösungen der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Math. Ann. *102* (1930), 633–649.

6. Y. K. Kwon, *Integral representations of harmonic functions on Riemannian spaces*, Doctoral dissertation, University of California, Los Angeles, 1969.

7. Y. K. Kwon and L. Sario, *A maximum principle for Dirichlet-finite harmonic functions on Riemannian spaces*, Can. J. Math. *22* (1970), 855–862.

8. M. Nakai, *A measure on the harmonic boundary of a Riemann surface*, Nagoya Math. J. *17* (1960), 181–218.

9. M. Nakai and L. Sario, *Classification and deformation of Riemannian spaces*, Math. Scand. *20* (1967), 193–208.

10. B. Rodin and L. Sario, *Principal functions* (Van Nostrand, Princeton, N.J., 1968).

11. H. Royden, *On the ideal boundary of a Riemann surface*, Ann. of Math. (2) *30* (1953), 107–109.

12. L. Sario and M. Nakai, *Classification theory of Riemann surfaces* (Springer-Verlag, New York, 1970).

13. L. Sario, M. Schiffer, and M. Glasner, *The span and principal functions in Riemannian spaces*, J. Analyse Math. *15* (1965), 115–134.

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