A CLASS OF MAXIMAL ORDERS INTEGRAL OVER THEIR CENTRES

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1. Introduction. In a recent paper [**1**], Brown, Hajarnavis and MacEacharn have considered non-commutative Noetherian local rings of finite global dimension which are integral over their centres. For such a ring R they have shown:

(i) R is a prime ring whose Krull and global dimensions coincide;

(ii) $R = \bigcap R_p$ where p runs through the set of rank one primes of the centre of R, and each R_p is hereditary;

(iii) the centre of R is a Krull domain.

We shall show that each R_p in (ii) above is in fact a principal right and left ideal ring. We deduce that the above ring R is a maximal order (defined below), and that if R is in addition a PI-ring with centre Z then it is a maximal Z-order in the sense of Fossum [**5**].

Our result covers the case where R is a local Noetherian ring of finite global dimension finitely generated as a module over its centre, which has previously been discussed in [**7**], and indeed our proof is somewhat easier than that given there. However, let D be a division ring which is locally finite dimensional, but not finite dimensional, over its centre. Then the localization of the polynomial ring $D[X_1, \ldots, X_n]$ at the maximal ideal generated by $X_1, \ldots, X_n$ is a local Noetherian ring of global dimension n which is integral, but not finitely generated, over its centre. The reader will find further details in [**1**, 7.1].

Throughout, all rings will be assumed to have an identity, and Noetherian will mean left and right Noetherian. A ring R with Jacobson radical J is called *semilocal* (respectively *local*) if R/J is semisimple (respectively simple) Artinian. For a right R-module M, $M^{\oplus s}$ denotes a direct sum of s copies of M.

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2. The Main Theorem. We shall first briefly define maximal orders (in the sense of Asano). Details may be found in [**6**].

Let R be a ring with a simple Artinian (right and left) quotient ring Q. A subset I of Q is called a *right* R*-ideal* if:

(i) I is a right R-submodule of Q;

(ii) I contains a unit of Q;

(iii) $uI \subset R$ for some unit u of Q.

Left R-ideals and (two-sided) R-ideals are defined in the obvious fashion. Clearly any non-zero ideal of R is an R-ideal. If I is an R-ideal, write

$$O_r(I) = \{q \in Q \mid Iq \subset I\}$$
$$O_l(I) = \{q \in Q \mid qI \subset I\}.$$

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Then R is called a *maximal order* if $O_r(I) = R = O_l(I)$ for each R-ideal I of Q. We note by [**6**, 3.1], R is a maximal order precisely when $O_r(I) = R = O_l(I)$ for each non-zero (ordinary) ideal I of R.

We begin with the following lemma, whose proof may be found in, for example, [**4**, 10.2].

LEMMA 1. *Let R be any ring, J its Jacobson radical, and P and Q finitely generated (f.g.) projective right R-modules. If P/PJ is an R/J-module direct summand of Q/QJ, then P is a direct summand of Q.*

Since a local ring has a unique simple right module (up to isomorphism), it follows that such a ring has a unique f.g. projective indecomposable right module. We shall, however, wish to apply Lemma 1 to certain semilocal localizations of a local ring, and thus require:

LEMMA 2. *Let R be a right Noetherian ring of finite right global dimension, and suppose that R has a unique f.g. projective indecomposable right module P. Let $S = R_{\mathcal{T}}$ be the classical localization of R at a right Ore set $\mathcal{T}$ of regular elements. Suppose that S is semilocal. Then S has a unique f.g. projective indecomposable right module, namely $P \otimes_R S$.*

Proof. Let Q be a f.g. projective indecomposable right S-module. We can write $Q = q_1S + \ldots + q_tS$ with each $q_i \in Q$. Let

$$K = q_1R + \ldots + q_tR,$$

and form an R-projective resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to K \to 0.$$

Each P_i can be chosen finitely generated, and hence is a direct sum of copies of P. Since $K \otimes_R S \cong Q$ and ${}_RS$ is flat, we have an exact sequence of S-modules

$$0 \to P_n \otimes_R S \to \ldots \to P_0 \otimes_R S \to Q \to 0.$$

As Q is S-projective an easy induction on the length of this resolution shows that there are integers k and l such that

$$(P \otimes_R S)^{\oplus k} \oplus Q \cong (P \otimes_R S)^{\oplus l}$$

If J is the Jacobson radical of S, we obtain

$$\frac{(P \otimes_R S)^{\oplus k}}{(P \otimes_R S)^{\oplus k} . J} \oplus \frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus l}}{(P \otimes_R S)^{\oplus l} . J}$$

Comparing the simple modules occurring, we must therefore have

$$\frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus (l-k)}}{(P \otimes_R S)^{\oplus (l-k)} . J}$$

From Lemma 1 and the indecomposability of Q we deduce $Q \cong P \otimes_R S$, as required.

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We fix some notation. For the remainder of the paper, R will be a local Noetherian ring of finite global dimension integral over its centre Z. Further, $\mathfrak{P}$ will denote the set of rank one primes of Z. We can now prove:

PROPOSITION 3. *For each* $\mathfrak{p}\in\mathfrak{P}$, $R_{\mathfrak{p}}$ *is a principal left and right ideal ring.*

Proof. By the result quoted in the introduction, $R_{\mathfrak{p}}$ is certainly a hereditary Noetherian prime ring, and is semilocal by [**3**, 2.2]. Let I be a non-zero right ideal of $R_{\mathfrak{p}}$. We are to prove that I is principal, and so we may assume that I is essential as a right ideal of $R_{\mathfrak{p}}$. By Lemma 2, $R_{\mathfrak{p}}$ has a unique f.g. projective indecomposable right module Q, and so $I\cong Q^{\oplus s}$ for some s. Also, $R\cong Q^{\oplus t}$ for some t. Since the uniform dimensions of I_R and R_R are equal, we have $s=t$ and I is right principal.

We are in a position to obtain our main result.

THEOREM 4. *R is a maximal order.*

Proof. We have $R=\bigcap_{\mathfrak{p}\in\mathfrak{P}} R_{\mathfrak{p}}$ by [**1**, 6.7], and by Proposition 3 each $R_{\mathfrak{p}}$ is a principal left and right ideal ring. If now I is a non-zero ideal of R and q lies in the quotient ring of R,

$$qI\subset I\Rightarrow qIR_{\mathfrak{p}}\subset IR_{\mathfrak{p}} \quad \text{for each} \quad \mathfrak{p}\in\mathfrak{P}\Rightarrow q\in\bigcap_{\mathfrak{p}\in\mathfrak{P}} R_{\mathfrak{p}}$$

since $IR_{\mathfrak{p}}$ is an invertible ideal of $R_{\mathfrak{p}}$. Thus R is a maximal order by [**6**, 3.1].

Theorem 4 fails should the requirement that R be local be weakened to one of semilocality. To see this, let S be the ring of integers localized at 2 and, using the usual notation, put

$$T=\begin{bmatrix} S & 2S \\ S & S\end{bmatrix}.$$

Then T is a semilocal hereditary Noetherian prime ring finitely generated over its centre. However, T is not a maximal order. For if I is the ideal

$$\begin{bmatrix} 2S & 2S \\ S & S\end{bmatrix}$$

of T, and

$$q=\begin{bmatrix} 0 & 0 \\ 1/2 & 0\end{bmatrix}$$

then $qI\subset I$ and q lies in the quotient ring of T, yet $q\notin T$.

We recall the definition of a maximal C-order from [**5**]. Let C be a Krull domain with quotient field K, and Q a finite dimensional central simple K-algebra. A C-*order* is, in the sense of Fossum, a subring T of Q satisfying:

(i) $C\subset T$;

(ii) $K.T=Q$;

(iii) T is integral over C.

A C-order is called *maximal* if it is not properly contained in any C-order in Q.

Suppose that R is, in addition to our previous assumptions, a PI-ring. Proposition 1.5 of [**2**] now guarantees that R is a maximal Z-order.

In particular we note that, by [**6**, 4.2 p. 147], for each $p \in \mathfrak{P}$ there is a unique prime ideal of R lying over p. Each R_p is thus a local ring. Presumably this last statement remains valid without the additional PI hypothesis, but we have been unable to confirm this.

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