ONE-DIMENSIONAL CELL COMPLEXES WITH HOMEOTOPY GROUP EQUAL TO ZERO

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1. Introduction. Let K denote a connected finite 1-dimensional cell complex (**1**, p. 95), $G(K)$ its group of homeomorphisms, and $D(K)$ the group of homeomorphisms of K which are isotopic to the identity. The group $\mathfrak{H}(K) \equiv G(K)/D(K)$ is a topological invariant of K and is called *the homeotopy group of* K (**4**). K may be thought of as a linear graph (connected finite 1-dimensional simplicial complex) extended to admit loops and multiple edges and $\mathfrak{H}(K)$ as the topological analogue of the automorphism group $A(L)$, (the permutations of vertices which preserve edge incidence relations) of a linear graph L. From this point of view, questions pertaining to linear graphs and their automorphism groups may be considered for cell complexes and their homeotopy groups. It is to be noted that even in the special case where K is a linear graph, $A(K)$ is not necessarily isomorphic to $\mathfrak{H}(K)$. This is clear since the vertices in K of degree 2 play a role in the computation of $A(K)$ but do not in the computation of $\mathfrak{H}(K)$. However, if K is a linear graph without vertices of degree 2, then $A(K) \approx \mathfrak{H}(K)$.

In this paper we obtain a theorem on the structure and existence of 1-dimensional cell complexes K having $\mathfrak{H}(K) = 0$, i.e. every homeomorphism of K is isotopic to the identity.

Let $a_0(K)$ and $a_1(K)$ denote the number of 0-cells and 1-cells, respectively, which appear in K, and let $N(K) \equiv a_1(K) - a_0(K) + 1$ denote the nullity of K.

THEOREM. *If* $\mathfrak{H}(K) = 0$, *then*

(1.1) $$a_0(K) \geqslant 7, \qquad a_1(K) \geqslant 10, \qquad \textit{and } N(K) \geqslant 2.$$

(1.2) *Furthermore, there exist linear graphs*, K, *without vertices of degree* 2, *such that* $\mathfrak{H}(K) = 0$ *and*

(i) *K has a_0 vertices for all $a_0 \geqslant 7$,*

(ii) *K has a_1 edges for all $a_1 \geqslant 10$, and*

(iii) *K has nullity N for all $N \geqslant 2$.*

Remark 1. Conditions (i), (ii), and (iii) are not necessarily satisfied simultaneously.

Remark 2. I. N. Kagno (**2**, p. 859, footnote) has given an example of a linear graph K with 6 vertices and $A(K) = 0$. But, this graph has vertices of degree 2 and $\mathfrak{H}(K) \neq 0$.

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2. Replacement A. *Associated with each* 1*-dimensional cell complex* L *which is not homeomorphic to a* 1*-sphere there is a unique cell complex* L_2 *which has the following properties:*

(i) L_2 *is homeomorphic to* L, *and*

(ii) L_2 *has no vertices of degree* 2.

Proof. If v denotes a 0-cell of L having degree 2, then v is an end point of exactly two 1-cells (u, v) and (v, w) of L. By replacing these two 1-cells by a single 1-cell (u, w), we obtain a cell complex L' which is clearly homeomorphic to L and which has one 0-cell of degree 2 less than L. We note that this replacement fails to yield a cell complex only in the case where v is the 0-cell of a 1-sphere having exactly one 0-cell in its decomposition. Since L has a finite number of cells, the above indicated operation applied a finite number of times yields a cell complex L_2 having properties (i) and (ii).

The uniqueness of the complex L_2 follows from the fact that, if U and V are homeomorphic 1-dimensional cell complexes both not having vertices of degree 2, then U and V are *isomorphic as cell complexes*, i.e. there is a one-to-one correspondence between their 0-cells (vertex sets) such that corresponding 0-cells are joined by k 1-cells (edges) in one if and only if they are joined by k 1-cells in the other. This assertion follows from the fact that the 0-cells of degree not equal to 2 are topological invariants of 1-dimensional cell complexes. Specifically, if f is a homeomorphism of U onto V, then the restriction of f to the set of 0-cells (1-cells) of U is readily seen to establish a one-to-one correspondence with the set of 0-cells (1-cells) of V and this correspondence does preserve incidence relations between corresponding 0-cells. Note that isomorphic 1-dimensional cell complexes, with no restriction on the degrees of their 0-cells, are homeomorphic.

Now, let W be any cell complex with no vertices of degree 2 which is homeomorphic to L. Since W is homeomorphic to L_2 we have, by the preceding remarks, W is isomorphic to L_2. Thus, L_2 is unique.

3. A lemma. We note the following fact.

LEMMA. K *is a* 1*-dimensional cell complex such that* $\mathfrak{H}(K) = 0$ *if and only if* K *is homeomorphic to a linear graph* K_2 *which has no vertices of degree* 2 *and* $A(K_2) = 0$.

Proof. If $\mathfrak{H}(K) = 0$, then, since the homeotopy group of a 1-sphere contains two elements, K is not homeomorphic to a 1-sphere. Let K_2 denote the cell complex obtained from K by Replacement A (cf. §2). K_2 can fail to be a linear graph in exactly two ways: (1) K_2 contains a loop (v, v), or (2) K_2 contains a simple circuit, i.e. two 1-cells $(u, v)_1$ and $(u, v)_2$ having the same 0-cells as end points. Either of these cases implies the existence of a homeomorphism which is not isotopic to the identity. Thus K_2 is a linear graph homeomorphic to K. Since K_2 has no vertices of degree 2, $A(K_2) \approx \mathfrak{H}(K_2)$. But, $\mathfrak{H}(K_2) \approx \mathfrak{H}(K)$. Therefore $A(K_2) = 0$.

Conversely, $0 = A(K_2) \approx \mathfrak{H}(K_2) \approx \mathfrak{H}(K)$.

4. Replacement B. In view of the preceding lemma, K is henceforth assumed to be a linear graph having no vertices of degree 2 and $A(K) = \mathfrak{H}(K) = 0$.

By a *free edge* (u, v) at the vertex u we mean an edge (u, v) such that the vertex v has degree 1.

Associated with K there is a unique connected subgraph K_1 of K which has the following properties:

(i) *K_1 has no vertices of degree* 1, *and*

(ii) *K can be reconstructed from K_1 by adding free edges at vertices of K_1 one at a time.*

Proof. Let F denote the set of closed free edges of K and V the set of vertices having degree not equal to 1 of those edges which are in F. We shall show that $K_1 \equiv (K - F) \cup V$ is the unique graph having properties (i) and (ii). Note that K_1 consists precisely of those edges of K both of whose vertices have degree (in K) greater than 2.

That K_1 is a connected subgraph of K with no vertices of degree 1 is clear. With respect to (ii), we note that any ordering of the elements of F defines a reconstruction of K from K_1, i.e. just replace the free edges one at a time in the given order.

We now show that, if W is a connected subgraph of K having properties (i) and (ii), then W is the subgraph K_1.

Let (u, v) be an edge of W. Since W has no vertices of degree 1 and K has no vertices of degree 2, every vertex of W must have degree (in K) greater than 2. Thus, (u, v) is an edge of K_1. Conversely, let (s, t) be an edge of K_1. Since (s, t) is not a free edge, (s, t) must be an edge in W. For, if this were not the case, it would be impossible to reconstruct K from W as indicated in (ii). Therefore, W is equal to K_1.

Remark. It is clear that the nullity of a graph is not changed when a free edge is adjoined at a vertex of the given graph. Thus, $N(K_1) = N(K)$.

5. Proof of part (1.1) of the theorem. We first show that, if $N(K) < 2$, then $\mathfrak{H}(K) \neq 0$.

If $N(K) = 0$, then K is a tree. If K is a closed 1-cell, then $\mathfrak{H}(K)$ contains two elements. If K is not a closed 1-cell, then it is easy to define a path in K which terminates at a vertex which has at least two free edges. Thus, $\mathfrak{H}(K) \neq 0$.

If $N(K) = 1$, then K contains exactly one circuit. Let S denote the set of vertices which lie on this circuit. Since K has no vertices of degree 2, at each vertex in S there is a tree. If one of these trees is not a free edge, then $\mathfrak{H}(K) \neq 0$.

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If all of these trees are free edges, then any homeomorphism of K onto itself induced by moving the vertices of S into adjacent vertices in S is non-trivial. Therefore, $\mathfrak{H}(K) \neq 0$ whenever $N(K) = 1$.

If $N(K) \geqslant 2$, then let K_1 denote the subgraph of K obtained by Replacement B (cf. §4) and K_2 the cell complex obtained from K_1 by Replacement A (cf. §2).

K_2 cannot contain any loops. For K_1 is homeomorphic to K_2 and K could not, in this case, be reconstructed from K_1 by the adjunction of free edges. In particular, $a_0(K_2) \geqslant 2$.

If $a_0(K_2) = 2$, then K_2 is the cell complex $[(u, v)_1 \ldots (u, v)_i]$ $(i = N(K)+1)$. Here the reconstruction of K necessitates adjoining distinct numbers of free edges at isolated interior points of the 1-cells of K_2 and one free edge at either vertex u or v. Thus, $a_0(K) \geqslant 9$ whenever its associated cell complex K_2 has $a_0(K_2) = 2$. Specifically, if $N(K) = 2$, then the only possible K_2 without loops must be homeomorphic to $[(u, v)_1(u, v)_2(u, v)_3]$. Thus, if $N(K) = 2$, then $a_0(K) \geqslant 9$.

If $a_0(K) = 3$, then K_2 must be of the form

$$[(u, v)_1 \ldots (u, v)_i(v, w)_1 \ldots (v, w)_j(w, u)_1 \ldots (w, u)_k].$$

If one of the subscripts i, j, or k does not appear, then K_2 is the one-point union of two cell complexes of the type considered in the previous paragraphs. Thus, $a_0(K) \geqslant 9$. If the circuit $[(u, v)(v, w)(w, u)]$ appears in K_2, then, since K_2 has no vertices of degrees 2, K_2 must contain at least two simple circuits. Hence, we must adjoin at least three free edges. Thus, $a_0(K) \geqslant 8$ whenever $a_0(K_2) = 3$.

We shall now assume that $4 \leqslant a_0(K) \leqslant 6$. K must have at least one vertex of degree 1. This follows from the result of I. N. Kagno **(3)** that every linear graph K with 6 or less vertices and having no vertices of degree less than 3 must have $A(K) \neq 0$. Thus, $a_0(K_1) \leqslant 5$. Recall that K_1 is a linear graph which has no vertices of degree 1. When we consider K_2 we note that either (1) K_2 is a linear graph (with no vertices of degree 1 or 2), or (2) K_2 is not a linear graph.

In case (1), K_2 is a linear graph tabulated by Kagno **(3)** and it is easy to verify that K cannot be reconstructed from K_2.

In case (2), since $K_2 \neq K_1$, we have a cell complex such that $a_0(K_2) \leqslant 4$. Since we have already considered the cases $N(K_2) < 3$, it remains only to examine those cell complexes such that $a_0(K_2) = 4$ and $N(K_2) \geqslant 3$. These remaining cases also lead to a contradiction of the assumption $a_0(K) \leqslant 6$.

If $N(K) = 3$, then the only possible K_2 without loops which is not a linear graph and such that $a_0(K_2) = 4$ is the complex

$$[(t, u)_1(t, u)_2(u, v)(v, w)_1(v, w)_2(w, t)].$$

This complex has two simple circuits. Hence at least two free edges must be added in the reconstruction of K. Thus, $a_0(K) \geqslant 8$.

If $N(K) = 4$, since K_2 contains seven $(= N(K_2) - 1 + a_0(K_2))$ edges and

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no free edges, then K_2 must contain at least one simple circuit. If K_2 contains one simple circuit, then the complex obtained from K_2 by the removal of one 1-cell of this simple circuit must be a linear graph having six edges and no vertices of degree 1 or 2. Thus, this graph must be the complete 4-point. It is easy to verify that the K_2 in this case cannot be associated with a K such that $a_0(K) \leqslant 6$ and $\mathfrak{H}(K) = 0$. If K_2 contains two or more simple circuits, then $a_0(K_2) \geqslant 8$.

If $N(K) \geqslant 5$ and $a_0(K_2) = 4$, then $a_1(K_2) = N(K_2) - 1 + a_0(K_2) \geqslant 8$. Thus, K_2 must contain at least two simple circuits. Hence $a_0(K) \geqslant 8$.

Therefore, in every case, we have shown that, if $\mathfrak{H}(K) = 0$, then $a_0(K) \geqslant 7$. Furthermore, we have obtained the result $a_1(K) = N(K) - 1 + a_0(K) \geqslant 10$ for all K such that $\mathfrak{H}(K) = 0$.

6. Proof of part (1.2) of the theorem. The following graph M given by I. N. Kagno (**3**, p. 510), is an example of a graph such that $\mathfrak{H}(M) = 0$, $a_0(M) = 7$, and $a_1(M) = 12$:

$$M \equiv [(q, r)(q, s)(q, t)(q, v)(q, w)(r, u)(r, v)(r, w)(s, t)(s, u)(t, w)(u, v)].$$

The graph M^k obtained from M by adjoining k free edges at k isolated interior points of the edge (s, u) is a graph such that $\mathfrak{H}(M^k) = 0$, $a_0(M^k) = 7 + 2k$, and $a_1(M^k) = 12 + 2k$ for all $k \geqslant 0$.

The graph L^k obtained from M^k by adjoining a free edge at the vertex q is a graph such that $H(L^k) = 0$, $a_0(L^k) = 8 + 2k$, and $a_1(L^k) = 13 + 2k$ for all $k \geqslant 0$.

The following graph Q is an example of a graph such that $\mathfrak{H}(Q) = 0$, $a_1(Q) = 10$, and $N(Q) = 2$:

$$Q \equiv [(o, p)(p, q)(q, r)(r, o)(o, s)(p, t)(q, u)(r, v)(s, w)(s, p)].$$

If we add the edge (p, r) to Q we obtain a graph R, such that $\mathfrak{H}(R) = 0$ and $a_1(R) = 11$.

If Q^k denotes the graph obtained from Q by adjoining k 1-cells (o, p) each having a distinct number ($\geqslant 3$) of free edges attached at isolated points of their interiors, then Q^k is a graph such that $\mathfrak{H}(Q^k) = 0$ and $N(Q^k) = 2 + k$ for all $k \geqslant 0$.

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